Lectures given at
IMCA _ 17-22 march
Lyapunov exponents

I Notions of a dynamical system and Lyapunov exponents

II Abstract ergodic Theory

III Lyapunov exponents for 2x2 matrices

IV The general theory (Mañé - Ruelle approach)

V Some applications to twist maps.

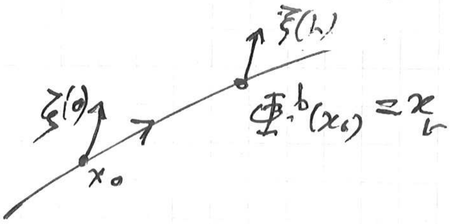
I: Introduction: dynamical system and Lyapunov exponents.

Some simple stability results for vector field on the plan.

$\dot{X} = V(X)$ ODE $X(0) = x_0$ $V: \mathbb{R}^2 \to \mathbb{R}^2$ is at least C^2

assume $V(0) = 0$ $\Rightarrow$ $0 = x_0$ is an equilibrium point

and we want to understand the local behavior of the flow $\Phi^t(x)$.



$x \mapsto \Phi^t(x)$ is C^2, we differential

$$\xi(t) = D\Phi^t(x_0).\xi_0 = \frac{\partial \phi^t(x)}{\partial x}\Big|_{x=x_0} . \xi_0$$

$$\dot{\xi}(t) = \frac{\partial}{\partial x}\left(\frac{\partial \phi^t(x)}{\partial t}\right)\Big|_{x=x_0} . \xi_0$$

$$= \frac{\partial}{\partial x}\left(V \circ \Phi^t(x)\right)\Big|_{x=x_0} . \xi_0$$

$$= DV(x_t) . \frac{\partial \phi^t(x)}{\partial x}\Big|_{x=x_0} . \xi_0$$

$$= DV(x_t) . \xi(t)$$

The infinitesimal vector $\xi(t)$ satisfies an ODE

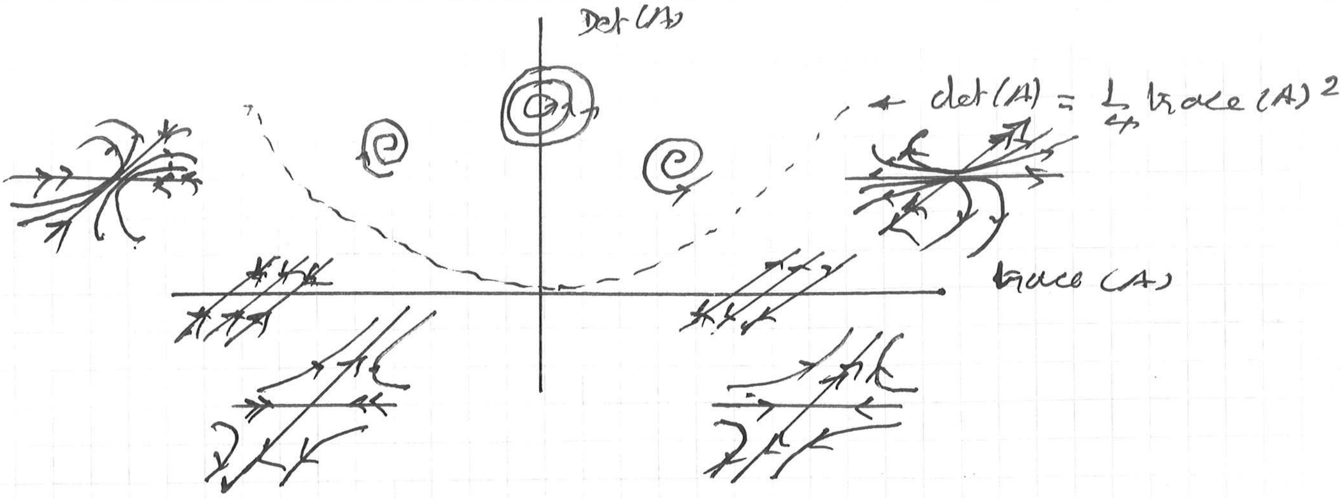
$\dot{\xi} = A(t)\xi$ $A(t) = DV(x_t)$ $\dot{x}_t = V(x_t)$

In the case of an equilibrium point $A(t) = A(0) = DV(x_0)$

$\dot{\xi} = A\xi$ $\to$ 3 possibilities for A up to conjugacy $A = PBP^{-1}$

$\zeta = P\xi$ $\dot{\zeta} = B\zeta$

Depending on $(\mathrm{trace}(B), \det(B))$, the flow has several different behavior



We solve some special cases:

$\star$ $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ $\dot{\xi}_t = \lambda \xi \to \xi(t) = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} \xi_0$

λ_1 and λ_2 are called the Lyapunov exponents of the flow:

In other words, for some initial vector ξ_0

$$\frac{1}{t} \ln \| D\phi^t(x_0) \cdot \xi_0 \| \to \lambda_1 \text{ or } \lambda_2$$

$\star$ $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ $\begin{cases} \dot{\xi}_1 = 0 \\ \dot{\xi}_2 = a \xi_1 \end{cases}$ $\begin{cases} \xi_1 = \xi_1(0) \\ \xi_2 = a\, \xi_1(0) t + \xi_2(0) \end{cases}$

$$\frac{1}{t} \ln \| D\phi^t(x_0) \cdot \xi_0 \| \to 0 \quad \text{for all initial conditions } \xi_0$$

here the two Lyapunov exponent are equal to zero

$\star$ $A = \begin{bmatrix} 0 & +\theta \\ -\theta & 0 \end{bmatrix}$. $\begin{cases} \dot{\xi}_1 = +\theta \xi_2 \\ \dot{\xi}_2 = -\theta \xi_1 \end{cases}$ $\ddot{\xi}_1 = -\theta^2 \xi_1$

$$\begin{cases} \xi_1 = \xi_1(0) \cos(\theta t) + \xi_2(0) \sin(\theta t) \\ \xi_2 = - \xi_1(0) \sin(\theta t) + \xi_2(0) \cos(\theta t) \end{cases}$$

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{bmatrix} \begin{bmatrix} \xi_1(0) \\ \xi_2(0) \end{bmatrix}$$

Again the infinitesimal vector $\xi(t)$ is bounded.

$$\frac{1}{t} \ln \| D\phi^t(x_0) \cdot \xi_0 \| \to 0 \quad \text{for any conditions}$$

the two Lyapunov exponents ($d = 2$) are equal to zero.

Lyapunov exponents = | eigenvalue (A) |

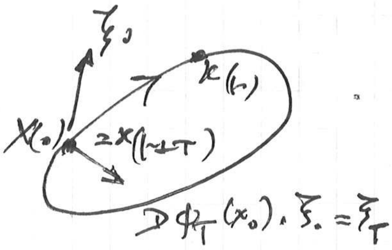
<u>First definition</u>

Consider the flow of the vector field $\dot{x} = V(x)$. Assume that a semi trajectory $x(t)|_{t \geq 0}$ exists. Consider the linearized equation $\dot{\xi} = DV(x_t) \cdot \xi$. Then a Lyapunov exponent is any

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \ln \| D\phi^t(x_0) \cdot \xi_0 \| \quad , \text{for some } \xi_0$$

Floquet theory.

We again considere a flow (autonomous) $\dot{x} = V(x)$. We assume that $x(t+T) = x(t)$ is a periodic solution. We linearized



$$\begin{cases} \dot{\xi} = A(t)\,\xi \\ A(t) = DV(x_t) \quad ; \quad A(t+T) = A(t) \end{cases}$$

The linearized equation is not any more autonomous, but periodic

$$\dot{M}(t) = A(t) M \qquad M(0) = Id$$

$M(t)$ fundament matrix. Assume we can write

$$M(T) = e^{TB} \quad \text{for some fixed matrix } B.$$

Let $N(t) = M(t+T)\, e^{-(t+T)B}$, $N(0) = Id$, satisfies

$$\dot{N} = A(t+T)\, M(t+T)\, e^{-(t+T)B} - M(t+T)\, e^{-(t+T)B} B$$

$$= A(t)\, N - N B$$

Similarly $N(t) = M(t)\, e^{-tB}$, $N(0)$, satifies

$$\dot{N} = A N - N B$$

By uniqueness of the solutions

$$M(t)\, e^{-tB} = M(t+T)\, e^{-(t+T)B} = Z(t)$$

is a periodic matrix

$$M(t) = Z(t)\, e^{tB}$$

$$\xi(t) = M(t)\,\xi_0 = Z(t)\,\zeta(t)$$

$$\zeta(t) = e^{tB}\,\xi_0 \qquad \dot{\zeta} = B\zeta.$$

We have solve the original equation in two step: $\xi(t) = Z(t)\zeta(t)$

$$\dot{\xi} = A(t)\,\xi$$

$$\dot{\zeta} = B\,\zeta \quad \text{(autonomous)}$$

$$Z(t+T) = Z(t) \quad \text{and solve} \quad \begin{cases} \dot{Z} = A(t) Z - Z B \\ Z(0) = Id. \end{cases}$$

All Lyapunov exponents can be computed using $t = nT$, $n \in \mathbb{N}$

$$\frac{1}{nT} \ln \|\xi(nT)\| \to \lambda$$

$$\xi(nT) = \zeta(nT) = e^{nTB}$$

the Lyapunov exponents are equal to all eigenvalue of B

Lyapunov exponents = | eigenvalue of B |.

<u>The very beginning of quasiperiodic ODE</u>

In Floquet theory: $A(t+T) = A(t)$. Assume that $A(t)$ is written in the form

$$A(t) = \tilde{A}(\omega t + \theta_0)$$

$$\tilde{A} : \mathbb{T}^d \to \mathbb{R} \text{ 1-periodic in } d\text{-variables.}$$

$$\omega = (\omega_1 \ldots \omega_d) \in \mathbb{R}^d, \quad \theta_0 \in \mathbb{T}^d$$

In Floquet theory $\omega_1 = \ldots = \omega_d = \frac{1}{T}$. A quasiperiodic equation is given by any choice of ω; a "true quasiperiodic" is when $(\omega_1 - \omega_d)$ are rationally independent.

It is also interesting to write an autonomous system equivalent

$$\begin{cases} \dot{\xi} = \tilde{A}(\omega t + \theta)\xi, & \xi(0) = \xi_0 \in \mathbb{R}^m \\ \dot{\theta} = \omega, & \theta(0) = \theta_0 \in \mathbb{T}^d. \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{\xi} = \tilde{A}(\theta)\xi \\ \dot{\theta} = \omega \end{cases} \quad \begin{cases} \xi(0) = \xi_0 \\ \theta(0) = \theta_0. \end{cases}$$

The flow of the associated non autonomous system is described in terms of Vector bundle over a dynamical system:

- Dynamical systems: $X = \mathbb{T}^d$

  $(f^t)_{t \in \mathbb{R}}$ flow $\quad f^t(\theta) = \theta + \omega t.$

- vector bundle, $\quad X \times \mathbb{R}^m$

  $(\Phi^t)_{t \in \mathbb{R}}$ flow $\quad \Phi^t(\theta_0, \xi_0) = (f^t(\theta_0), \Phi^t_{\theta_0}(\xi_0))$

where $\Phi^t_{\theta_0}(\xi_0) = \xi(t)$ is the solution of $\dot{\xi}(t) = \tilde{A}(\omega t + \theta_0)\xi$

By linearity $\xi(t) = M^t_{\theta_0} \cdot \xi_0$ where $M^t_{\theta_0}$ is the fundamental solution: We write $M(\theta_0, t) = M^t_{\theta_0}$,

$$\Phi^t(\theta_0, \xi_0) = (f^t(\theta_0), M(\theta_0, t) \cdot \xi_0)$$

We show that M^t_θ satisfies a cocycle property (equivalent to the multiplicative structure in a group).

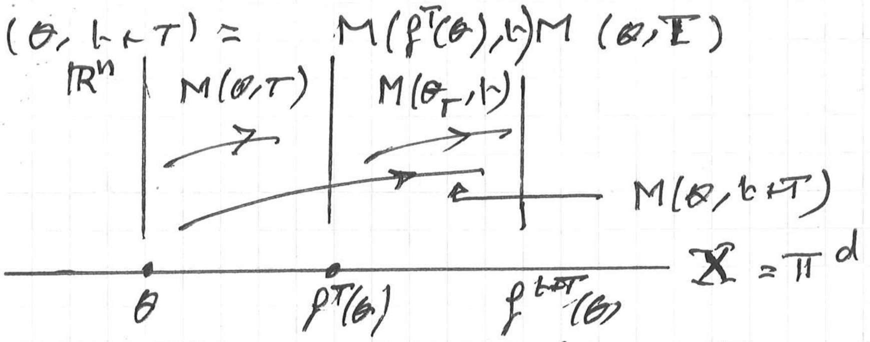
$Z = M^t_{\theta_0}$ satisfies $\quad \dot{Z} = A\circ f^t(\theta_0)\cdot Z \quad Z(0) = Id$

$Z = M^{t+T}_{\theta_0}(M^T_{\theta_0})^{-1}$ $\quad$ " $\quad \dot{Z} = A\circ f^t(\theta_T)\cdot Z \quad Z(0) = Id$

with $\theta_T = f^T(\theta_0)$. Thus

$$M^{t+T}_{\theta_0}(M^T_{\theta_0})^{-1} = M^t_{\theta_T}$$

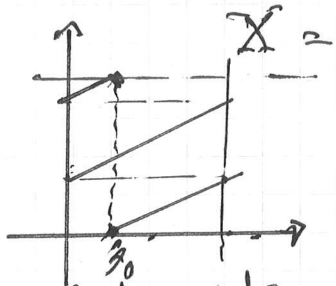
or $\quad M(\theta, t+T) = M(f^T(\theta), t)\,M(\theta, T)$



General framework: (X, f^t) dynamical system, $X\times\mathbb{R}^n$ a vector bundle, $M: X\times\mathbb{R}^n \to GL(n)$ a matrix cocycle $M(x,t+s) = M(f^s(x),t)\,M(x,s)$

The base dynamical system is $(X, (f^t)_{t\in\mathbb{R}})$

$X = \mathbb{T}^d \qquad f^t(\theta) = \theta + \omega t$



example of periodic
Not true quasiperiodic

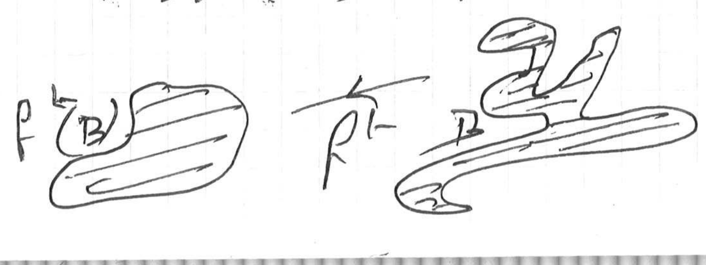
This system is very basic in terms of recurrence property and ergodic theory.

Proposition If $(\omega_1,\dots,\omega_d)$ are rationally independant then the dynamical system $f^t(\theta) = t\omega + \theta$ on $X = \mathbb{T}^d$ is minimal and uniquely ergodic

Definition * (X, f^t) is said minimal if all orbits $(f^t(\theta_0))_{t\in\mathbb{R}}$ are dense: $X = \text{closure}\{f^t(\theta) : t\in\mathbb{R}\}$.

* (X, f^t) is said to be uniquely ergodic if there exists a unique probability measure μ which is invariant by the flow, that is, satisfies

$$\Leftrightarrow \begin{cases} (f^t_*)\mu = \mu & \forall t\in\mathbb{R} \\ \mu[f^t(B)] = \mu[B] & \forall B \text{ Borel set}, \forall t\in\mathbb{R} \end{cases}$$



Actually we will show a stronger property which implies both minimality and unique ergodicity.

Proposition The flow $\theta \mapsto \omega t + \theta$ on $\mathbb{T}^d$ is equidistributed:
$\forall \varphi \in \mathcal{C}^0(\mathbb{T}^d)$ $\quad \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\omega k + \theta) \to \int_{\mathbb{T}^d} \varphi(x)\, dx$
uniformly in θ.

Proof of minimality

$\theta_0 \in \mathbb{T}^d$, $Y =$ closure $\{ f^t(\theta_0) \, t \in \mathbb{R}\}$. Assume $Y \subsetneq X$, then $X \setminus Y$ is a non empty open set. Let $\varphi \in \mathcal{C}^0(\mathbb{T}^d)$ with supp$(\varphi) \subset X \setminus Y$, $\varphi \geq 0$ and non equal to zero. By invariance of Y. $(f^t(Y) = Y)$

$$\varphi(f^t(\theta_0)) = 0 \quad \forall t \in \mathbb{R}.$$

The previous proposition would imply $\int_{\mathbb{T}^d} \varphi(x)\,dx = 0$, which is a contradiction.

Proof of unique ergodicity. Let μ be an invariant probability measure:
$$\mu[f^t(B)] = \mu[B] \quad \forall B \text{ Borel set}, \forall t \in \mathbb{R}$$
$$\iff \int \mathbb{1}_B \circ f^t \, d\mu = \int \mathbb{1}_B \, d\mu$$
$$\iff \forall \varphi \in \mathcal{L}^\infty \quad \int \varphi \circ f^t d\mu = \int \varphi \, d\mu.$$

But the previous proposition implies
$$\int \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k \, d\mu \to \int \varphi \text{ Leb.}$$

Thus $\mu =$ Leb is the normalized Lebesgue measure on $\mathbb{T}^d$.

Proof of the equidistribution property

We want to prove that, for any continuous function $\varphi : \mathbb{T}^d \to \mathbb{R}$
$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(\theta) \to \int \varphi \text{ Leb.}$$

A trigonometric polynomials $\varphi = \sum_{p \in \mathbb{Z}^d} \varphi_p \exp(i 2\pi \langle p | \theta \rangle)$ is a sum of $\exp(i 2\pi \langle p, \theta \rangle)$ with almost all $\varphi_p = 0$. Here $\langle p ; \theta \rangle = \sum_{j=1}^d p_j \theta_j$

Since trigonometric polynomials are dense in $\mathcal{C}^0(\mathbb{T}^d)$. It is enough to show that the convergence exists for $\varphi = \exp(i 2\pi \langle p, \theta \rangle)$

$$\frac{1}{n}\sum_{k=0}^{n-1} \exp(i2\pi \langle p, \omega k+\theta\rangle)$$

$$* = \exp i2\pi\langle p,\theta\rangle \ \frac{1}{n}\sum_{k=0}^{n-1} \left(\exp(i2\pi \langle p,\omega\rangle)\right)^k$$

since $\omega_1 - \omega_d$ are rationally indépendant $\exp(i2\pi\langle p,\omega\rangle) \neq 1$

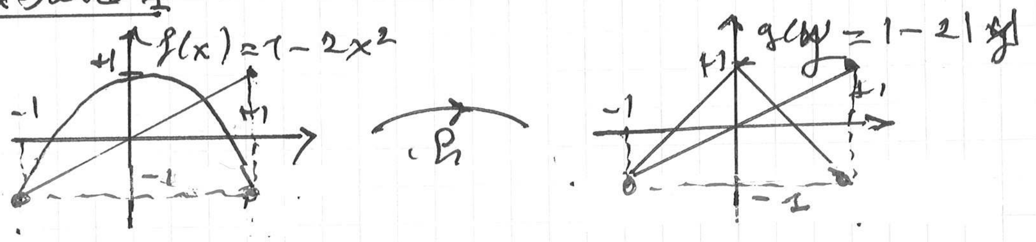
$$(p\neq 0) \Rightarrow * = \exp(i2\pi\langle p,\theta\rangle)\ \frac{1-\exp(i2\pi n\langle p,\omega\rangle)}{n\,(1-\exp(i2\pi\langle p,\omega\rangle))} \to 0.$$

$$(p=0) \Rightarrow * = 1$$

Thus $\quad \frac{1}{n}\sum_{k=0}^{n-1} \varphi\circ f^k(\theta) \to \varphi_0 = \int_{\mathbb{T}^d} \varphi(x)\,dx.$ ∎

Other example of dynamical systems

Exercice 1



logistic map $\qquad$ tent map

we consider the diffeomorphism $y=h(x)=\sin\left(\frac{\pi}{2}x\right)$

1) Show that h conjugates the two dynamical systems

$$\begin{array}{ccc} [-1,1] & \xrightarrow{h} & [-1,1] \\ f\downarrow & & \downarrow g \\ [-1,1] & \xrightarrow{h} & [-1,1] \end{array} \qquad g\circ h = h\circ f.$$

2) Show that g preserves Lebesgue

$* \ \mathrm{Leb}(g^{-1}(B)) = \mathrm{Leb}(B) \quad \forall B$ Borel set

(be carefull, it is very different from $\mathrm{Leb}(g(B)) \neq \mathrm{Leb}(B)$) when g is not invertible, then $*$ is the correct definition of invariance of Leb.

3) Deduce from 2) that $\dfrac{dx}{\pi\sqrt{1-x^2}}$ is invariant by f.

4) show that f admits another invariant measure.

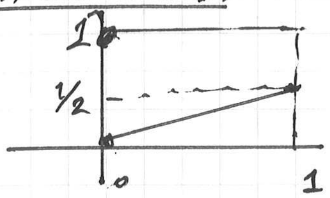
Exercice 2 1) Consider a flow of a vector field of divergence null. Assume the vector field has compact support (in order to garanty that the flow is complete). Show that leb is preserved.

2) Show that the Hamiltonian flow

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

(where $H = H(q,p)$ is a smooth function which is constant at sot) preserve the Lebesgue measure.

<u>Exercice 3.</u>



Let $f(0) = 1$

$f(x) = \frac{1}{2} x$ for $x \in]0,1]$

* 1) show that f admits no finite invariant measure.

2) Is there an infinite (σ-finite) invariant measure?

<u>II Ergodic Theory: the basic theorems</u>

<u>Definition</u> A measurable dynamical system: (X, f, μ)
X is a good space (with respect to probability theory), X is a Polish space, $f : X \to X$ is Borel (not necessarily invertible) and μ is a probability measure on the Borel sets of X.

<u>Example 1</u> (Tails and Heads game)

(or $\mathbb{R}^{\mathbb{N}}$) (with $\nu^{\otimes \mathbb{N}}$)

* $X = \{ x = (x_0, x_1, \dots) : x_i = 0 \text{ or } x_i = 1 \} = \{0,1\}^{\mathbb{N}}$
* the shift map $\sigma(x) = (x_1, x_2, \dots)$ is the map f, the corresponding discrete semi-flow $(\sigma^n)_{n \geq 0}$.
* $\mathcal{A}$ = algebra of cylinders: a cylinder of size n is a Borel set C_n where the first n symbols of $x \in C_n$ are fixed:

$$C_n[\bar{x}_0 \dots \bar{x}_{n-1}] = \{ x \in X : x = (x_0, x_1, \dots) \quad x_i = \bar{x}_i \ \forall i = 0 \dots n-1 \}$$

Let p = be the probability that "tails" happens. Let μ be a finite measure on $\mathcal{A}$ defined by

$$\mu[C_n] = p^{\# \text{tails}} (1-p)^{\# \text{Heads}}$$

$$\mu[C_n[\bar{x}_0 \dots \bar{x}_{n-1}]] = p^{\sum_{k=0}^{n-1} \bar{x}_k} (1-p)^{\sum_{k=0}^{n-1} (1-\bar{x}_k)}.$$

What is important to notice that, μ has been so that

(I)* μ is consistent on cylinders

$$\sum_{y=0/1} \mu\left(C_{n+1}[\bar{x}_0 \dots \bar{x}_{n-1}\, y]\right) = \mu\left(C_n[\bar{x}_0 \dots \bar{x}_{n-1}]\right)$$

(II)* μ is invariant by the shift

$$\sum_{y=0/1} \mu\left(C_{n+1}[y, \bar{x}_1, \dots, \bar{x}_n]\right) = \mu\left(C_n[\bar{x}_1 \dots \bar{x}_n]\right)$$

notice that $\sigma^{-1}\left(C_n[\bar{x}_1 \dots \bar{x}_n]\right) = \bigcup_{y=0,1} C_{n+1}[y, \bar{x}_1, \dots, \bar{x}_n]$

Then Kolmogoroff's theorem says that (I) is enough to garanty the existence of measure μ which is σ-additive on the Borel set et extends μ defined previously on $\mathcal{A}$.

Moreover (II) implies that μ is σ-invariant:

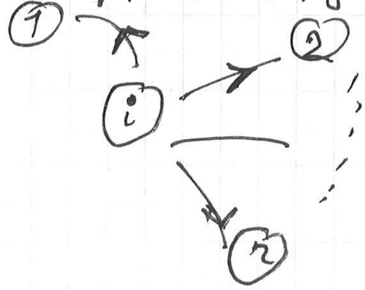
$$\mu(\sigma^{-1}B) = \mu(B) \quad \forall B \in \mathcal{A} \Rightarrow \mu(\sigma^{-1}B) = \mu(B) \quad \forall B \in \mathcal{B}_X.$$

<u>Conclusion</u> ($X = \{0,1\}^{\mathbb{N}}$, $\sigma: X \to X$ shift map, $\mu = ((1-p)\delta_0 + p\delta_1)^{\otimes \mathbb{N}}$) is a measurable dynamical system. Let $\pi_n : X \to \{0,1\}$ the nth projection: then $(\pi_0, \pi_1, \dots)$ is just a sequence of independant identically distributed random variables called the Bernoulli process.

<u>Example 2</u> Markov process: $X = \{1, 2, \dots, r\}^{\mathbb{N}}$, $\sigma: X \to X$, the shift map, and μ is built using a transition matrix and an initial distribution ν:

$\nu = \sum_{k=1}^{r} \nu_k \delta_{(x_0 = k)}$ is a probability on $\{1, 2, \dots, r\}$

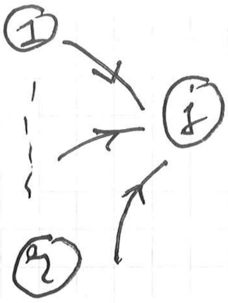
$P = [p(i,j)]_{1 \le i,j \le r}$ is a matrix with non negative entries



* $p(i,j)$ denotes the probability that a member of a population in site (i) at time 0 jumps to the site (j) at time 1

* in particular $\begin{cases} p_{ij} \ge 0 \\ \sum_{j=1}^{r} p_{ij} = 1 \end{cases}$

If ν denotes the distribution of the population at time 0, $(\nu p)(j) = \sum_{i=1}^{r} \nu(i)\, p(i,j)$



$(\nu p)_j$ denotes the distribution at time 1
We assume that ν has been choosen so that the distribution is stationary

$$\nu p = \nu$$

The invariant measure by σ is defined as before: for a cylinder $C_n[\bar{x}_0 \dots \bar{x}_{n-1}]$ of size n, $\bar{x}_i \in \{1, 2, \dots, r\}$ by

$$\mu(C_n[\bar{x}_0 \dots \bar{x}_{n-1}]) = \nu(\bar{x}_0)\, p(\bar{x}_0, \bar{x}_1)\, p(\bar{x}_1, \bar{x}_2) \cdots p(\bar{x}_{n-2}, \bar{x}_{n-1})$$

Again μ is consistent on the algebra of cylinders and σ-invariant. By Kolmogoroff's theorem, μ extends to a σ-additive probability measure.

Conclusion: $(X = \{1 \dots r\}^{\mathbb{N}}, \sigma: X \to X, \mu = \text{Markov chain})$ is a measurable dynamical system. The sequence of projections $(\pi_0, \pi_1, \pi_2, \dots)$ is the Markov process associated to (ν, p).

The following definition of ergodicity in the measurable setting is similar to minimality in the topological setting

Definition A dynamical system (X, f, μ) is ergodic if whenever $Y \subset X$ is an invariant Borel subset of X (i.e. $\mu(\sigma^{-1}(Y)) = \mu(Y)$), then $\mu(Y) = 0$ or $\mu(Y) = 1$.

In otherwords, there is no invariant decomposition nontrivial

$$X = X_1 \cup X_2 \qquad \sigma^{-1}(X_i) = X_i$$

Example 1 Bernoulli processes are ergodic. Actually they are mixing in the following sense

$$\left\{ \begin{array}{l} \mu(A \cap \sigma^{-n} B) \to \mu(A)\mu(B) \\ \text{for any Borel sets } A \text{ and } B \end{array} \right.$$

Mixing implies obviously ergodicity. Let be Y σ-invariant

$$\sigma^{-1}(Y) = Y \Rightarrow \sigma^{-n}(Y) = Y$$

$$\mu(Y \cap \sigma^{-n} Y) = \mu(Y) = \mu(Y)^2 \Rightarrow \mu(Y) = 0 \text{ or } 1.$$

(This is the zero-one law in probability). To prove mixing property, it is enough to prove

$$\mu(A \cap \sigma^{-n} B) \to \mu(A)\,\mu(B)$$

for a dense family of subset A' and B' $\in \mathcal{A}$: dense in the sense

$\forall \varepsilon > 0$, $\exists\, A' \in \mathcal{A}$ such that $\mu(A \,\Delta\, A') = \int |1\!\!1_A - 1\!\!1_{A'}|\, d\mu < \varepsilon$

The algebra of cylinder is dense in $\mathcal{B}_X$, the Borel sets. Let

$A = C_m[\bar{x}_0 \ldots \bar{x}_{m-1}]$ a cylinder

$B = C_n[\bar{y}_0 \ldots \bar{y}_{n-1}]$ another cylinder.

Then if $k > m$

$$A \cap \sigma^{-k} B = \{ x = (x_0, x_1, \ldots) : x_0 = \bar{x}_0 \ldots x_{m-1} = \bar{x}_{m-1},\ x_k = \bar{y}_0 \ldots x_{k+n-1} = \bar{y}_{n-1} \}$$

Since there is no intersection in the two sets of indices

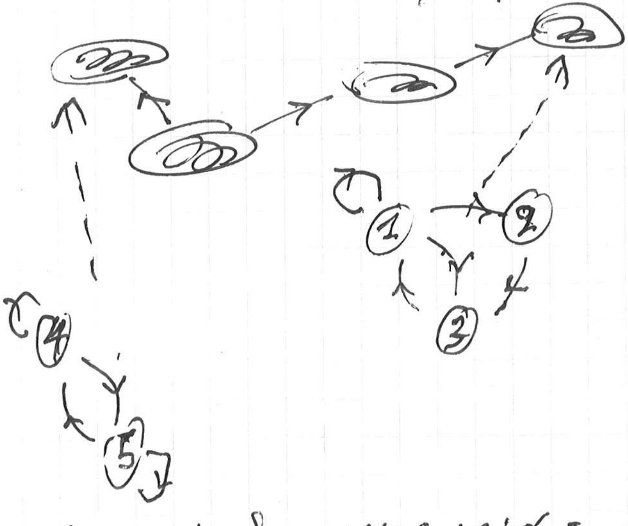
$$\{0, 1, \ldots, m-1\} \text{ and } \{k, k+1, \ldots, k+n-1\}$$

The two sets A and B are independant

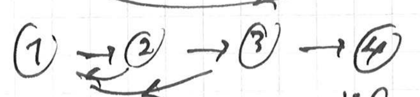
$$\mu(A \cap \sigma^{-k} B) = \mu(A)\,\mu(\sigma^{-k} B) = \mu(A)\,\mu(B)$$

is constant for large k.

<u>Example 2</u> For a Markov chain (with finite numbers of states) ergodicity depends on the numbers of charged terminal irreducible sub graphs



* subgraph : $S \subset \{1, 2, \ldots, r\}$
* irreducible : $\forall i, j \in S$ $\exists$ path $i_0 = i, i_1, \ldots, i_n = j$ s.b. $p(i_k, i_{k+1}) > 0$ $\forall k = 0, 1, \ldots, n-1$



YES

NON

* terminal : $\forall i \in S\ \forall j \notin S$ $p(i,j) = 0$ (no arrow with positive

probability going out of S)

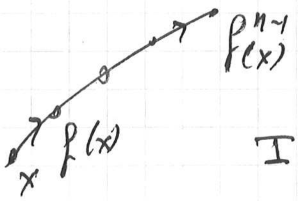
* charged: if ν is an invariant initial law $\nu p = \nu$ (or $\nu(j) = \sum_i \nu(i) p(i,j)$) then supp$(\nu)$ belongs to the union of terminal irreducible subgraphs. Conversely each terminal irreducible subgraph admits an invariant initial law: we say that this subgraph is charged.

Then the Markov chain is ergodic iff only one terminal irreducible subgraph is charged.

Pointwise Birkhoff's ergodic theorem Let (X, f, μ) be a measurable dynamical system. Let $\varphi : X \to \mathbb{R}$ in $L^1(\mu)$. Then

$$\begin{cases} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) \to \bar{\varphi}(x) & \text{exists } \mu\text{.a.e.} \\ \int_B \bar{\varphi}\, d\mu = \int_B \varphi\, d\mu & \forall B \ f\text{-invariant} \end{cases} \Leftarrow \bar{\varphi} \text{ is } f\text{-inv}$$

(B is f invariant if $f^{-1}(B) = B$)



If (X, f, μ) is ergodic then $\bar{\varphi}$ is constant. and $\bar{\varphi} = \int \varphi\, d\mu$

(In the ergodic case, space average equal time average).

Remark. For Bernoulli process $X = \mathbb{R}^{\mathbb{N}}$, σ = shift, $\mu = \nu^{\otimes \mathbb{N}}$

If $\varphi_0(x) = x_0$ (the first coordinate of x), then $\varphi_0 \circ \sigma^n(x) = x_n = \varphi_n(x)$ and $(\varphi_n)_{n \geq 0}$ is an iid process with distribution ν.

The strong law of large numbers says

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi_k \longrightarrow \mathbb{E}[\varphi] = \int \varphi\, d\mu = \int_{\mathbb{R}} x\, \nu(dx)$$

Birkhoff's ergodic theorem is a generalization of the strong law of large numbers for non independent processes but for processes only stationary.

L^2 - Birkhoff's ergodic theorem If $\varphi \in L^2(X, f, \mu)$, then

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) \to \bar{\varphi}(x) \quad \text{in } L^2$$

where $\bar{\varphi}$ orthogonal projection of φ on the space of L^2 inv. funct.

Proof ① We first prove that $L^2(\mu) = \overline{\mathcal{C}_\mu} \overset{\perp}{\oplus} \mathcal{I}_\mu$ where

$\mathcal{C}_\mu = \{ \varphi\circ f - \varphi : \varphi \in L^2 \}$ = the space of coboundaries

$\mathcal{I}_\mu = \{ \varphi \in L^2 : \varphi\circ f = \varphi \}$ = the space of invariant funct.

We prove actually that $\mathcal{I}_\mu = \mathcal{C}_\mu^\perp$. Let $\varphi \perp (\psi\circ f - \psi)$, we want to prove that φ is f-invariant. Let $U_T : L^2 \to L^2$ defined by

$$U_T \cdot \varphi = \varphi\circ f.$$

U_T is an isometric but not unitary. We have

1) $\mathrm{Im}(U_T)$ closed

2) $U^*U = Id \qquad (\langle U^*U\varphi | \psi\rangle = \langle U\varphi | U\psi\rangle = \langle \varphi, \psi\rangle)$

3) $U^* \equiv 0$ on $(\mathrm{Im}\, U)^\perp \qquad (\langle U^*\varphi | \psi\rangle = \langle \varphi | U\psi\rangle = 0)$

From $\varphi \perp (\psi\circ f - \psi)$ $\forall \psi$ we obtain

$$\langle \varphi | U\psi\rangle = \langle \varphi | \psi\rangle = \langle U^*\varphi | \psi\rangle$$

$$\Leftrightarrow U^*\varphi = \varphi$$

φ can be decomposed into $\varphi = U\alpha + \beta$ with $\beta \perp \mathrm{Im}(U)$

Then $U^*\varphi = \varphi$ emplies $\varphi = \alpha$. Moreove

$$\|\varphi\|^2 = \|\alpha\|^2 = \|U\alpha\|^2 + \|\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$$

Thus $\beta = 0$ and $\varphi \in \mathrm{Im}\, U$, and $\varphi = U\varphi$: φ is f-invariant.

We have proved $\mathcal{C}_\mu^\perp \subseteq \mathcal{I}_\mu$. The converse is easy.

$$U\varphi = \varphi \Rightarrow \langle \varphi | \psi\circ f - \psi\rangle = \langle U\varphi | U\psi\rangle - \langle \varphi | \psi\rangle = 0$$

Thus $\mathcal{C}_\mu^\perp = \mathcal{I}_\mu$.

② We prove the Birkhoff's ergodic theorem

For $\varphi \in \mathcal{I}_\mu$, $U\varphi = \varphi$, $\varphi\circ f = \varphi$,

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi\circ f^k = \varphi$$

For $\varphi \in \mathcal{C}_\mu$, $\varphi = \psi\circ f - \psi$.

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi\circ f^k = \frac{1}{n}(\psi\circ f^n - \psi)$$

For $\varphi_i \to \varphi$ in L^2 and $\varphi_i \in \mathcal{C}_\mu$

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi\circ f^k = \frac{1}{n}\sum_{k=0}^{n-1} (\varphi - \varphi_i)\circ f^k + \frac{1}{n}\sum_{k=0}^{n-1} \varphi_i\circ f^k$$

$$\Big\| \frac{1}{n}\sum_{k=0}^{n-1} \varphi\circ f^k \Big\| \leq \|\varphi - \varphi_i\| + \Big\| \frac{1}{n}\sum_{k=0}^{n-1} \varphi_i\circ f^k \Big\|$$

The first terms on the right hand side tends to zero uniformly in n. The second term tends to zero for each fixed φ_i.

L^1 – Birkhoff's ergodic theorem If f $\varphi \in L^1$ then
$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k \to \overline{\varphi} \quad \text{in } L^1$$
where $\overline{\varphi}$ is f-invariant and
$$\int_B \overline{\varphi}\, d\mu = \int_B \varphi\, d\mu \qquad \forall B \text{ } f\text{-invariant}$$

Proof ① We prove that the orthogonal projection $P_\mu : L^2_{(\mu)} \to \mathcal{I}_\mu$ extends to a unique operator $P_\mu : L^1(\mu) \to \mathcal{I}_\mu$ where P_μ is characterized by
$$\begin{cases} P_\mu \varphi \in \mathcal{I}_\mu \\ \int_B P_\mu \varphi\, d\mu = \int_B \varphi\, d\mu \quad \forall B \text{ } f\text{-invariant} \end{cases}$$
Indeed for $\varphi, \psi \in L^2(\mu)$ we have

a) $\varphi \geq 0 \Rightarrow P_\mu \varphi \geq 0$

b) $\varphi \geq \psi \Rightarrow P_\mu \varphi \geq P_\mu \psi$

c) $|\varphi| \geq \varphi \geq -|\varphi| \Rightarrow |P_\mu \varphi| \leq P_\mu |\varphi|$

d) $\| P_\mu \varphi \|_{L^1(\mu)} \leq \|\varphi\|_{L^1(\mu)}$

Assume that $\varphi \in L^1(\mu)$ and let $\varphi_i \to \varphi$ in L^1 with $\varphi_i \in L^2(\mu)$. Then $(P_\mu \varphi_i)_{i\geq 0}$ is a Cauchy sequence in $\mathcal{I}_\mu$, thus converges to some $P_\mu \varphi$ in $\mathcal{I}_\mu$. And
$$\int_B P_\mu \varphi_i\, d\mu = \int_B \varphi_i\, d\mu \Rightarrow \int_B P_\mu \varphi\, d\mu = \int_B \varphi\, d\mu$$

② We prove Birkhoff's ergodic theorem. Let $\varphi \in L^1(\mu)$ and $\varphi_i \to \varphi$ in $L^1(\mu)$, $\varphi_i \in L^2(\mu)$.
$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k - P_\mu \varphi \right\|$$
$$\leq 2\,\|\varphi - \varphi_i\| + \left\| \frac{1}{n}\sum_{k=0}^{n-1} \varphi_i \circ f^k - P_\mu \varphi_i \right\|.$$
The first term of the right hand side is small uniformly in n. The second term tends to zero as $n \to +\infty$ for fixed φ_i. ∎

The maximal ergodic lemma Let $\varphi \in L^1(\mu)$. For any $\alpha > 0$ let

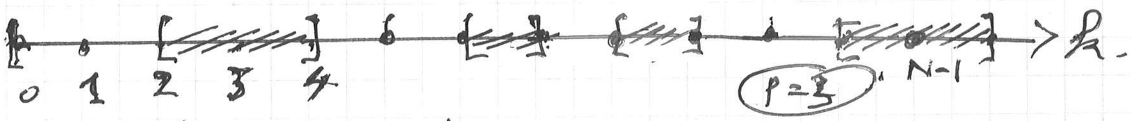
$$B_\alpha = \{x \in X : \exists n \geq 1 \quad S_n\varphi(x) > n\alpha\} \quad \text{where } S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi\circ f^k.$$

Then $\quad \int|\varphi|\,d\mu \geq \alpha\,\mu(B_\alpha) \quad$ or $\int_A |\varphi|\,d\mu \geq \alpha\,\mu(A\cap B_\alpha) \quad \forall A \in \mathcal{B}_\mu$

Proof Let $p \geq 1$ and $B_{\alpha,p} = \{x \in X : \exists 1 \leq n \leq p \quad S_n\varphi(x) > n\alpha\}$

We want to prove that $\int_{B_{\alpha,p}} \varphi\,d\mu \geq \alpha\,\mu(B_{\alpha,p})$. Instead we prefer to compare for large N

$$\sum_{k=0}^{N-1} |\varphi|\circ f^k(x) \quad \text{and} \quad \alpha \sum_{k=0}^{N-1} \mathbb{1}_{B_{\alpha,p}} \circ f^k(x)$$



We fix x and the set of indices

$$E_{\alpha,p,N,x} = \{k \in [0, N-1] : f^k(x) \in B_{\alpha,p}\}$$

Each time $k \in E$, there exists $n = n(k)$ such that

$$(S_{n(k)}|\varphi|)\circ f^k(x) \geq \alpha n(k) \geq \alpha\left(S_{n(k)} \mathbb{1}_{B_{\alpha,p}}\right)\circ f^k(x)$$

If $k \notin E$, we choose $n(k) = 1$, and

$$(S_{n(k)}|\varphi|)\circ f^k(x) = |\varphi|\circ f^k(x) \geq \alpha\, \underbrace{S_{n(k)} \mathbb{1}_{B_{\alpha,p}}\circ f^k(x)}_{=0}$$

We have to take into account the last terms of size at most p. By summing all these inequality

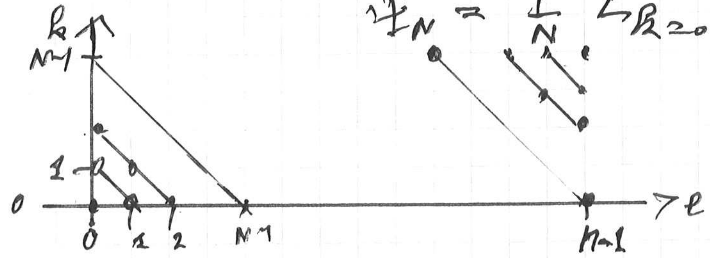
$$\sum_{k=0}^{N-1} |\varphi|\circ f^k(x) \geq \alpha \sum_{k=0}^{N-p} \mathbb{1}_{B_{\alpha,p}} \circ f^k(x)$$

We integrate, divide by N and let $N \to +\infty$.

Proof the pointwise Birkhoff ergodic theorem

① We first assume $\varphi \in L^\infty(\mu)$. We may assume $\bar{\varphi} = 0$ by taking $\varphi - \bar{\varphi}$ instead of φ. Let

$$\Phi_N = \frac{1}{N}\sum_{k=0}^{N-1} \varphi\circ f^k.$$



$$\frac{1}{n}\sum_{l=0}^{n-1} \Phi_N \circ f^l$$

$$= \frac{1}{nN} \sum_{l=0}^{n-1} \sum_{k=0}^{N-1} \varphi\circ f^{k+l}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \varphi\circ f^k + \frac{2N^2}{nN}\|\varphi\|_{\infty}$$

Therefore $\quad \varphi^* := \limsup_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi\circ f^k = \limsup_{n\to+\infty} \frac{1}{n}\sum_{l=0}^{n-1} \Phi_N\circ f^l$

and φ_* defined similarly with liminf.

The only advantage is that $\Phi_N \to 0$ in L^1 ($P_\mu \psi_N = 0$). We fixe N

Let $\alpha > 0$ and $B_\alpha^* = \{\varphi^* > \alpha\}$. Then $B_\alpha^* \subset B_{\alpha,N}$ where

$$B_{\alpha,N} = \{x : \exists n \geq 1 \quad \frac{1}{n} S_n \Phi_N > \alpha\}$$

Then from the maximal ergodic lemma

$$\int_A |\Phi_N| \, d\mu \geq \alpha \mu(A \cap B_{\alpha,N}) \geq \alpha \mu(A \cap B_\alpha^*)$$

Since $\Phi_N \to 0$ in $L^1(\mu)$, we have $\mu(A \cap B_\alpha^*) = 0 \ \forall A \in \mathcal{I}_\mu \ \forall \alpha > 0$

We thus have proved, since φ^* is f-invariant

$$\varphi^* \leq 0 \quad \mu.e.$$

Changing φ to $-\varphi$ we obtain $\varphi_* \geq 0$. therefore $\varphi_* = \varphi^* = 0$

② We prove Birkhoff's ergodic theorem for $\varphi \in L^1$.

Approximate $\varphi_i \to \varphi$. From the maximal ergodic lemma

$$\alpha \mu(\{\limsup_{n \to +\infty} \frac{1}{n} S_n(\varphi - \varphi_i) > \alpha\}) \leq \int |\varphi - \varphi_i| \, d\mu$$

Since $\lim_{n \to +\infty} \frac{1}{n} S_n \varphi_i \to P_\mu \varphi_i$ a.e.

$$\alpha \mu(\{\limsup_{n \to +\infty} \frac{1}{n} S_n \varphi > P_\mu \varphi_i + \alpha\}) \leq \int |\varphi - \varphi_i| \, d\mu$$

We have also

$$\alpha \mu(|P_\mu \varphi - P_\mu \varphi_i| > \alpha) \leq \int |P_\mu \varphi - P_\mu \varphi_i| \, d\mu$$
$$\leq \int |\varphi - \varphi_i| \, d\mu$$

Then

$$\alpha \mu(\{\limsup_{n \to +\infty} \frac{1}{n} S_n \varphi > P_\mu \varphi + 2\alpha\}) \leq 2 \|\varphi - \varphi_i\|_{L^1(\mu)}$$

By taking $\varphi_i \to \varphi$, and then $\alpha \to 0$, we obtain

$$\limsup \frac{1}{n} S_n \varphi \leq P_\mu \varphi \quad \text{a.e.}$$

Again by taking $-\varphi$ instead φ we have

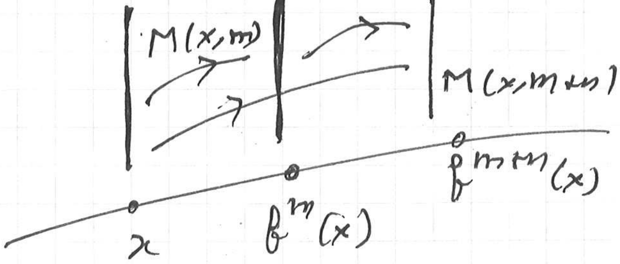
$$\lim \frac{1}{n} S_n \varphi = P_\mu \varphi \quad \text{a.e.}$$

III Lyapunov exponents

<u>Notations</u> We consider a matrix cocycle over an abstract dynamical system:

- (X, f, μ) a measurable dynamical system
- $X \times \mathbb{R}^n$ a vector bundle over X
- $M : X \times \mathbb{N} \to Mat(\mathbb{R}^d)$ a measurable matrix cocycle

$$M(x, m+n) = M(f^m(x), n)\, M(x, m)$$



It is actually enough to introduce

$$M(x, 1) =: M(x)$$

Then $M(x,n) = M(f^{n-1}(x)) \cdots M(fx) M(x)$. We are thus computing the product of n matrices along the trajectory starting at x

<u>Example</u> Let X be a compact manifold, $f : X \to X$ smooth map and $T_x f$ the tangent. Then

$$T_x f^n = T_{f^{n-1}(x)} f \cdots T_{f(x)} f \cdot T_x f$$

$T_x f^n$ is a cocycle in the tangent bundle. It is always possible to trivialize the bundle measurably. In the new bundle $T_x f^n = M(x,n)$ is a matrix cocycle described as before.

<u>Definition</u> Let $M \in Mat(\mathbb{R}^d)$, the singular values of M are the eigenvalues of $\sqrt{M^* M}$. The polar decomposition

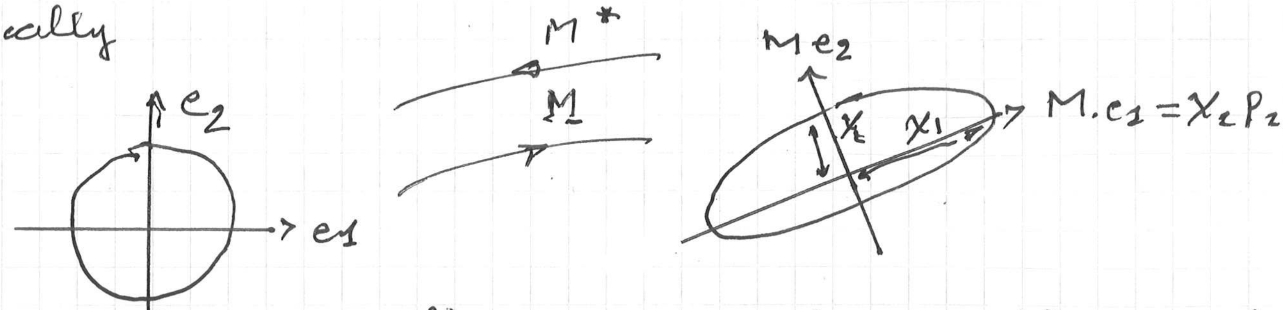
$$M = R\sqrt{M^* M}$$

says that R can be chosen in $O(d)$, the orthogonal group. When M is non singular, R is unique. Let $(e_1 \dots e_d)$ be a orthonormal basis which diagonalizes $\sqrt{M^* M}$:

$$\sqrt{M^* M} \cdot e_i = \chi_i e_i$$

Assume $\chi_1 \geq \chi_2 \geq \cdots \geq \chi_d$: χ_i are called the singular values of M.

Geometrically



M sends a circle to an ellipse : $\exists (f_1 \dots f_d)$ an orthonormal basis such that

$$\begin{cases} M e_i = \chi_i f_i \\ M^* f_i = \chi_i e_i \end{cases}$$

(the base $(f_1 \dots f_d)$ is unique if M is invertible).

<u>Notations</u> For a cocycle $M(x,n) = M(f^{n-1}(x)) \cdots M(f(x)) M(x)$, we denote by $\chi_1(x,n) \geq \cdots \geq \chi_d(x,n)$ the singular values of $M(x,n)$ and by $\{e_1(x,n) \dots e_d(x,n)\}$ the corresponding orthonormal basis.

<u>Oseledet's theorem (Part I)</u> Let (X, f, μ) be an abstract dynamical system. $M : X \to Mat(\mathbb{R}^d)$ measurable satisfying $\ln^+ \|M\| \in L^1(\mu)$, $\quad (\ln^+(u) = \max(\ln u, 0))$. Then

$$\lambda_i(x) = \lim_{n \to +\infty} \frac{1}{n} \ln \chi_i(x,n) \qquad \text{exists a.e.}$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ are called the Lyapunov exponent. They satisfy in addition : $\lambda_i \circ f(x) = \lambda_i(x)$.

<u>Remark</u> Let $M \in Mat(\mathbb{R}^d)$ with singular values $\chi_1 \geq \cdots \geq \chi_d$

$$\|M\|^2 = \sup_{|e|=1} \langle Me | Me \rangle = \sup_{|e|=1} \langle M^* M e | e \rangle$$
$$= \sup_{\|e\| = \|f\| = 1} \langle M^* M e | f \rangle = \|M^* M\| = \chi_1^2$$

Therefore $\chi_1 = \|M\|$. More generally $\chi_2 \dots \chi_d$ are related to a norm of a matrix.

Define.

$$\mathrm{Gram}(u_1 \dots u_r) := \sqrt{\det [\langle u_i, u_j \rangle_{1 \le i, j \le r}]}$$

$$\|\Lambda^r M\| := \sup_{\|u_i\|=1} \mathrm{Gram}(Mu_1, \dots, Mu_r)$$

Then $\|\Lambda^r M\| = \chi_1(M) \dots \chi_r(M)$ where $\chi_i(M)$ are singular value.

In particular

$$[\chi_1(MN) \dots \chi_r(MN)] \le [\chi_1(M) - \chi_r(M)]\,[\chi_1(N) - \chi_r(N)]$$

Proof of Oseledet's theorem (part I)

Let $\varphi_n(x) = \ln \|M(x,n)\|$. From the cocycle property

$$M(x, m+n) = M(f^m(x), n)\, M(x, m)$$

we obtain $\varphi_{m+n}(x) \le \varphi_m(x) + \varphi_n \circ f^m(x)$.

Recall that $\varphi_n(x) = \ln \chi_1(x,n)$. More generally, let

$$\varphi_n(x) = \ln \|\Lambda^r M(x,n)\| = \ln \chi_1(x,n) - \chi_r(x,n)$$

Then $\varphi_{m+n}(x) \le \varphi_m(x) + \varphi_n \circ f^m(x)$.

Moreover $\varphi_n^+(x) \in L^1(\mu)$

Then $\lim \frac{1}{n} \varphi_n(x)$ exists almost everywhere from Kingman's theorem.

$$\lim \tfrac{1}{n} \ln \chi_1(x,n) \to \lambda_1$$

$$\lim \tfrac{1}{n} \ln [\chi_1(x,n)\, \chi_2(x,n)] \to \lambda_1 + \lambda_2$$

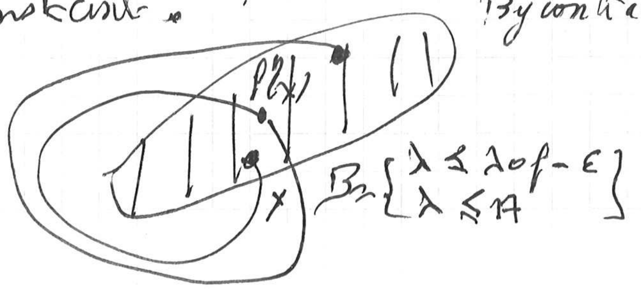
Invariance comes from the following observation.

$$M(x, n+1) = M(f(x), n)\, M(x)$$

$$\|M(x, n+1)\| \le \|M(f(x), n)\| \quad \|M(x)\|$$

$$\lambda_1(x) \le \lambda_1 f(x).$$

But any function measurable satisfying $\lambda \le \lambda \circ f$ has to be constant.



By contradiction, choose $\varepsilon > 0$, $A > 0$ large, so that

$$\mu(\lambda < \lambda \circ f - \varepsilon,\ \lambda \le A) > 0$$

But a.e. point in B returns infinitely often in B: contradiction.

Kingman's theorem Let (φ_n) be a subadditive sequence

$$\varphi_{m+n} \leq \varphi_m + \varphi_n \circ f^n$$

Assume $\varphi_1^+ \in L^1(\mu)$. Then

$$\frac{1}{n} \varphi_n(x) \to \overline{\varphi} \quad \text{a.e.}$$

where $\overline{\varphi} = \inf_{n \geq 1} \frac{1}{n} \int P_\mu \varphi_n \, d\mu$ ($P_\mu \varphi$: extended orthogonal projection onto the space of invariant functions).

Remark Kingman extends Birkhoff. If $\varphi_n = \sum_{k=0}^{n} \varphi \circ f^k$, then

$$\varphi_{m+n} = \varphi_m + \varphi_n \circ f^m \quad \text{(is additive)}.$$

Notations Let $\overline{\lambda}_1 > \cdots > \overline{\lambda}_r$ the r distinct Lyapunov exponents.

If $r = 1$, there is no part II. Let

$$H_i(x,n) = \text{span}\{ e_s(x,n) : \lambda_s = \overline{\lambda}_i \}$$

Then $\mathbb{R}^d = H_1 \oplus^\perp \cdots \oplus^\perp H_r$.

Oseledets's theorem - Part II Assume $\ln^+ \|M(x)\| \in L^1(\mu)$.

Then almost everywhere

1) $[M(x,n)^* M(x,n)]^{\frac{1}{2n}} \to \exp \Lambda(x)$

where $\Lambda(x)$ symmetric with eigenvalue $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$

2) $H_i(x,n) \to H_i(x)$ = eigenspace of $\Lambda(x)$ for $\overline{\lambda}_i(x)$.

Moreover $M(x) H_i(x) = H_i(f(x))$.

3) Let $V_1(x) = \mathbb{R}^d = H_1 \oplus \cdots \oplus H_r$

$$V_2(x) = H_2 \oplus \cdots \oplus H_r$$

— — — — — — — — —

$$V_r(x) = H_r(x)$$

Then $\forall v \in V_i \setminus V_{i+1}$ $\lim \frac{1}{n} \ln \|M(x,n) \cdot v\| = \overline{\lambda}_i$

Another ergodic lemma Assume $\varphi^+ \in L^1(\mu)$, Then.

$$\limsup_{n \to +\infty} \frac{1}{n} \varphi^+ \circ f^n(x) \leq 0. \quad \text{a.e.}$$

proof $\varphi \leq \varphi^+$: it is enough to show that

$$\lim_{n \to +\infty} \frac{1}{n} \varphi^+ \circ f^n(x) = 0 \quad \text{a.e.}$$

but

$$\frac{1}{n}\left(\varphi^{+}\circ f^{n}-\varphi^{+}\right)=\frac{1}{n}\sum_{k=0}^{n-1}\left(\varphi^{+}\circ f-\varphi^{+}\right)\circ f^{k}\to\bar{\varphi}$$

By Birkhoff. $\int_B \bar{\varphi} = \int_B (\varphi^{+}\circ f - \varphi^{+})\,d\mu = 0 \Rightarrow \bar{\varphi} = 0$.

Oseledets Theorem. Part III: the deterministic case

Let be $(A_k)_{k\geq 0}$ a sequence of matrices. Denote

$$M(n) = A_{n-1}A_{n-2}\cdots A_1 A_0$$

Denote also $\chi_1(n) \leq \chi_2(n) \leq \dots \leq \chi_r(n)$, the singular values of $M(n)$. Assume

- $\lim\limits_{n\to+\infty} \frac{1}{n}\chi_i(n) = \lambda_i$ exists
- $\limsup\limits_{n\to+\infty} \frac{1}{n}\|A_n\| \leq 0$

Let $\bar{\lambda}_1 < \dots < \bar{\lambda}_r$ the distinct Lyapunov exponents. Let $H_i(n)$ the eigenspace of $\sqrt{M^{*}(n)M(n)}$ for the eigenvalues $\chi_j(n)$ so that $\lambda_j = \bar{\lambda}_i$. Then

1) $[M^{*}(n)M(n)]^{\frac{1}{2n}} \longrightarrow \exp(\Lambda)$ where Λ is a symmetric matrice and $\Lambda|H_i = \bar{\lambda}_i \, Id|H_i$

2) $H_i(n) \to H_i$

3) Let $V_1 = H_1$, $V_2 = H_1\oplus H_2, \dots, V_r = H_1\oplus\dots\oplus H_r = \mathbb{R}^d$.

$\forall v \in V_{i+1}\setminus V_i \qquad \frac{1}{n}\ln\|M(n)\,v\| \longrightarrow \bar{\lambda}_{i+1}$

~~**Remark** For the proof, in order to simplify, we assume $r=d$; there are d distincts Lyapunov exponents.~~

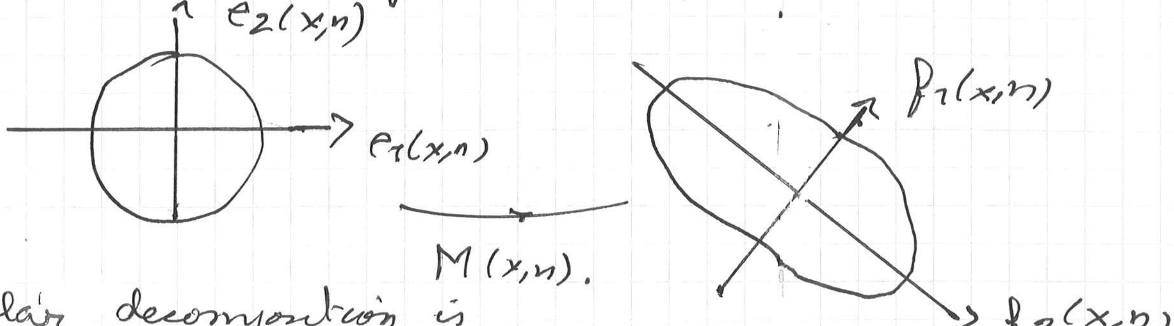
IV Examples of computation of Lyapunov exponents.

<u>Notations</u> The dynamical system is now invertible and the linear cocycle is also supposed to be invertible : $M(x,n) \in GL(\mathbb{R}^d)$.

In this case, the assumption is

$$\begin{cases} \ln \|M(x)\| \in L^1(\mu) \\ \ln \|M(x)^{-1}\| \in L^1(\mu) \end{cases}$$

Let $\chi_1(x,n)$ the singular values of $M(x,n)$.



The polar decomposition is

$$M(x,n) = U(x,n)\,|M(x,n)|.$$

$$M(x,n)\, e_i(x,n) = \chi_i(x,n)\, f_i(x,n)$$

$$M(x,n)^{-1} f_i(x,n) = \chi_i(x,n)^{-1} e_i(x,n).$$

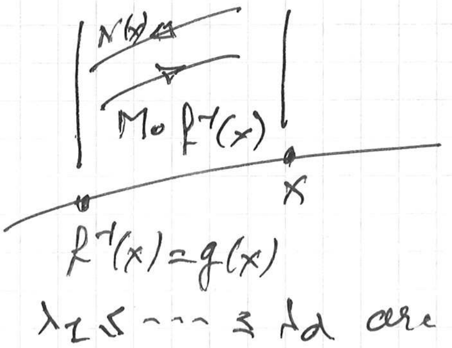
The polar decomposition for $M(x,n)^{-1}$ is

$$M(x,n)^{-1} = U^{-1}(x,n)\,|M(x,n)^{-1}|$$

where $U(x,n)\cdot f_i(x,n) = e_i(x,n)$. The singular value of $M(x,n)^{-1}$ are $\chi_d(x,n)^{-1} \leq \chi_{d-1}(x,n)^{-1} \leq \dots \leq \chi_1(x,n)^{-1}$

<u>Definition</u> In the invertible case, we introduce the reverse cocycle : $g = f^{-1}$, $N(x,n) = N(g^{n-1}(x)) \cdots N(g(x)) N(x)$, where

$$N(x) = (M(f^{-1}(x)))^{-1}$$



Thus

$$N(x,n) = (M(f^{-n}(x), n))^{-1}$$

<u>Lemma</u> If $\lambda_1 \leq \dots \leq \lambda_d$ are the Lyapunov exponents for $M(x,n)$ The the Lyapunov exponents for the reverse cocycle $N(x,n)$ is

$$-\lambda_d \leq \dots \leq -\lambda_1$$

proof From Kingman

$$\frac{1}{n} \ln \chi_i(x,n) \to \lambda_i(x) \quad \text{a.e. and in } L^1$$

Assume that $\lim_{n\to+\infty} \frac{1}{n} \ln \chi_i(x,n) = \lim_{n\to+\infty} \frac{1}{n} \ln \chi_i(f^{-n}x, n)$

Since the singular value of $M(x,n)$ are

$$\chi_d(f^{-n}(x), n)^{-1} \leq \dots \leq \chi_1(f^{-n}(x), n)^{-1}$$

The Lyapunov exponents are $\quad -\lambda_d(x) \leq \dots \leq -\lambda_1(x)$.

We prove the claim for $i = d$: $\chi_d(x,n) = \|M(x,n)\|$. Define

$$C_\varepsilon(x) = \sup_{n\geq 0} \{ \|M(x,n)\| \exp -n(\lambda_d(x) + \varepsilon) \} < +\infty$$

since $\quad \|M(x,n+1)\| \geq \|M(x)^{-1}\|^{-1} \|M(f x, n)\|$

$$C_\varepsilon(x) \geq C_\varepsilon \circ f(x) \, \|M(x)^{-1}\|^{-1} \exp(-(\lambda_d(x) + \varepsilon))$$

$$\ln C_\varepsilon(x) - \ln C_\varepsilon \circ f(x) \geq -\ln \|M(x)^{-1}\| - \lambda_d(x) - \varepsilon$$

Then we use the ergodic result that.

$$(\varphi - \varphi \circ f)^- \in L^1(\mu) \implies \begin{cases} \varphi - \varphi \circ f \in L^1(\mu) \\ \frac{1}{n} C_\varepsilon \circ f^n, \ \frac{1}{n} C_\varepsilon \circ f^{-n} \to 0. \end{cases}$$

From the definition of $C_\varepsilon(x)$.

$$\|M(x,n)\| \leq C_\varepsilon(x) \exp n(\lambda_d(x) + \varepsilon)$$

$$\|M(f^{-n}(x), n)\| \leq C_\varepsilon \circ f^{-n}(x) \exp n(\lambda_d(x) + \varepsilon)$$

$$\overline{\lim_{n\to+\infty}} \frac{1}{n} \ln \chi_d(f^{-n}(x), n) \leq \lambda_d(x) + \varepsilon$$

$$\overline{\lim_{n\to+\infty}} \frac{1}{n} \ln \chi_d(f^{-n}(x), n) \leq \lambda_d(x).$$

We do the same analysis with

$$C_\varepsilon(x) = \inf_{n\geq 0} \{ \|M(x,n)\| \exp -n(\lambda_d(x) - \varepsilon) \}$$

$$\|M(f(x), n)\| \cdot \|M(x)\| \geq \|M(x, n+1)\|$$

$$C_\varepsilon \circ f(x) \, \|M(x)\| \exp -(\lambda_d - \varepsilon) \geq C_\varepsilon(x).$$

$$\ln C_\varepsilon(x) - \ln C_\varepsilon(f(x)) \leq \ln \|M(x)\| - \lambda_d(x) + \varepsilon.$$

$$\|M(x,n)\| \geq C_\varepsilon(x) \exp n(\lambda_d(x) - \varepsilon)$$

$$\|M(f^{-n}(x), n)\| \geq C_\varepsilon \circ f^{-n}(x) \exp n(\lambda_d(x) - \varepsilon)$$

$$\underline{\lim} \frac{1}{n} \ln \|M(f^{-n}(x), n)\| \geq \lambda_d(x) - \varepsilon$$

$$\underline{\lim} \frac{1}{n} \ln \chi_d(f^{-n}(x), n) \geq \lambda_d(x)$$

Oseledets theorem in the invertible case

(X, f, μ) invertible dynamical system, μ f-invariant, $M : X \to GL(d)$ measurable such that $\ln \|M(x)\| \in L^1(\mu)$ and $\ln \|M(x)^{-1}\| \in L^1(\mu)$.

Let $\bar{\lambda}_1 < \cdots < \bar{\lambda}_r$ be the distinct Lyapunov sequence of $M(x,n)$, with multiplicity $m_1 + \cdots + m_r = d$. Let

$$V_i^+(x) = \{ v \in \mathbb{R}^d : \limsup_{n \to +\infty} \frac{1}{n} \ln \| M(x,n) . v \| \leq \bar{\lambda}_i(x) \}$$

$$V_i^-(x) = \{ v \in \mathbb{R}^d : \limsup_{n \to +\infty} \frac{1}{n} \ln \| M(f^{-n}x, n)^{-1} v \| \leq -\bar{\lambda}_{r-i+1} \}$$

Define $E_i(x) = V_i^+(x) \cap V_{r-i+1}^-(x)$. Then a.e.

1) $\mathbb{R}^d = E_1(x) \oplus \cdots \oplus E_r(x)$

2) $E_i \circ f(x) = M(x) . E_i(x)$

3) $\forall v \in E_i(x)$, $v \neq 0$, $\lim_{n \to +\infty} \frac{1}{n} \ln \| M(x,n) . v \| = \bar{\lambda}_i(x)$

$$\lim_{n \to +\infty} \frac{1}{n} \ln \| M(f^{-n}x, n)^{-1} . v \| = -\bar{\lambda}_i$$

Proof. We first notice

$$\limsup_{n \to +\infty} \frac{1}{n} \ln \| M(x,n) | V_i^+(x) \| \leq \bar{\lambda}_i(x)$$

$$\limsup_{n \to +\infty} \frac{1}{n} \ln \| M(f^{-n}x, n)^{-1} | V_i^-(x) \| \leq -\bar{\lambda}_{r-i+1}(x).$$

Using the same lemma as before, a.e.

$$\limsup_{n \to +\infty} \frac{1}{n} \ln \| M(x,n)^{-1} | V_i^-(f^n(x)) \| \leq -\bar{\lambda}_{r-i+1}(x)$$

We have shown

$$V_1^+(x) \subset V_2^+(x) \subset \cdots \subset V_r^+$$

$$V_r^-(x) \supset V_{r-1}^-(x) \supset \cdots \supset V_1^-$$

We show $V_1^+ \oplus V_{r-1}^-(x) = \mathbb{R}^d$

$$V_2^+(x) \oplus V_{r-2}^-(x) = \mathbb{R}^d$$

$$\cdots\cdots$$

It is enough, because of the dimension to check, say, $V_1^+ \cap V_{r-1}^- = \{0\}$

$$v \in V_1^+ \cap V_{r-1}^-$$

$$v = M(x,n)^{-1} M(x,n) v$$

$$\|v\| \leq \| M(x,n)^{-1} | V_{r-1}^- \| \| M(x,n) v \|$$

with $\lim \frac{1}{n} \ln \| M(x,n)^{-1} | V_{r-1}^-(x) \| \leq -\bar{\lambda}_2$

and $\lim \frac{1}{n} \ln \| \Pi(x_n) . v \| \leq \lambda_1$. Therefore $\|v\| = 0$.

We want to show

$$\mathbb{R}^d = V_1^+(x) \oplus V_2^+(x) \cap V_{s-1}^-(x) \oplus V_{s-2}^-(x).$$

It is enough to show that $\dim (V_2^+ \cap V_{s-1}^-) = m_2$. Indeed

$$\dim(V_2^+ \cap V_{s-1}^-) = \dim(V_2^+) + \dim(V_{s-1}^-) - \dim(V_2^+ + V_{s-1}^-)$$
$$= (m_1 + m_2) + (d - m_1) - d = m_2.$$

V How to remove zero Lyapunov exponent

An example of Shub-Wilkinson (2000).

Notation

Let M be a 3 dimensional compact manifold, μ a Riemannian Lebesgue measure, $\mathrm{Diff}^2_\mu(M)$ the space of $\mathcal{C}^2$ diffeomorphism preserving μ

Question Is it true that for a generic $f \in \mathrm{Diff}^2_\mu(M)$, either all Lyapunov exponents are equal to zero, or none of the Lyapunov exponent are equal to zero?

By Herman we cannot exclude the case where all Lyapunov are equal to zero

Theorem (Herman) For any manifold of dim ≥ 2, for any r sufficiently large, there exists $\mathcal{U} \subset \mathrm{Diff}^r_\mu(M)$ open such that for all $f \in \mathcal{U}$, M contains a union of positive Lebesgue measure of smooth invariant circles where the dynamics is conjugated to a rotation (in particular it has at least one zero Lyapunov exponent)

In the $\mathcal{C}^1$ topologie and $\dim(M) = 2$, the answer is Yes.

Theorem (Mañé) $\dim M = 2$, there is a generic set in $\mathrm{Diff}^1_\mu(M)$ such that either all Lyapunov exponents are zero, or f is Anosov.

<u>Definition</u>: $f \in \mathrm{Diff}^2_\mu(M^3)$ is said to be PH partially hyperbolic if there exists a continuous decomposition

$$T_x M = E^u_x \oplus E^c_x \oplus E^s_x$$

$$T_x f \cdot E^\alpha_x = E_{f(x)} \qquad \alpha = u, c, s. \qquad \dim(E^\alpha_x) = 1$$

$$\forall v \in E^u_x \qquad \|T_x f^n \cdot v\| \geq C \exp(n\gamma^u)\|v\|$$

$$\forall v \in E^s_x \qquad \|T_x f^n \cdot v\| \leq C \exp(n\gamma^s)\|v\|$$

$$\forall v \in E_s{}^c \qquad C'\|v\| \exp(-n\gamma^c) \leq \|T_x f^n v\| \leq C \exp(n\gamma^c)\|v\|$$

with $\qquad \gamma^s < -\gamma^c \leq 0 \leq \gamma^c < \gamma^u$

Because of the fact that $\dim(E^\alpha_x) = 1$, the 3 Lyapunov exponents can be computed using the formula (when f is ergodic)

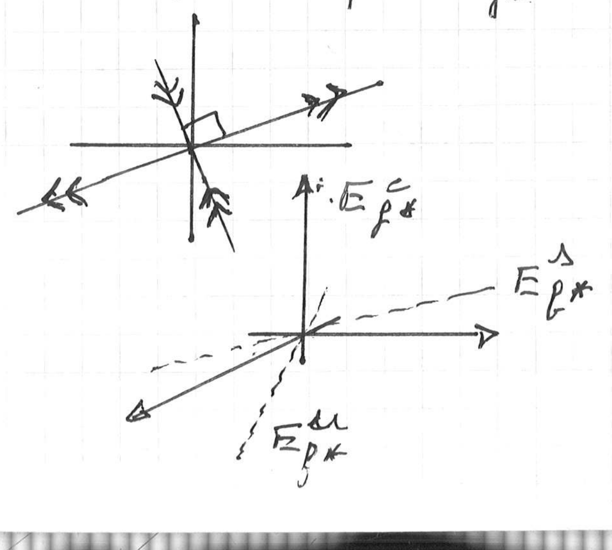
$$\chi^\alpha_f = \int \ln \|T_x f \,|\, E^\alpha_x\| \, d\mu(x) \qquad \alpha = u, c, s.$$

<u>Theorem (Shub–Wilkinson)</u> $\quad M = \mathbb{T}^3 \quad f_* = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Diff}^\infty_\mu(M)$

1) f_* is non ergodique – partially hyperbolique with a center Lyapunov exponent equals to zero.

2) $\exists\, \mathcal{U} \subset \mathrm{Diff}^\infty(M)$ such that $f_* \in \overline{\mathcal{U}}$ and any $g \in \mathcal{U}$ is (stably) ergodic – partially hyperbolic – and has no zero Lyapunov exponents.

<u>Remark.</u> The map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is a model of an Anosov map. More generally $A_* \in SL(2, \mathbb{Z})$ define modulo 1 an invertible map $A_* : \mathbb{T}^d \to \mathbb{T}^d$ and A is ergodic iff the eigenvalues are not root of unity.



Here $A_* = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is Anosov, the eigenvalues

$$X^2 - 3X + 1 = 0 \qquad X = \frac{3 \pm \sqrt{5}}{2}$$

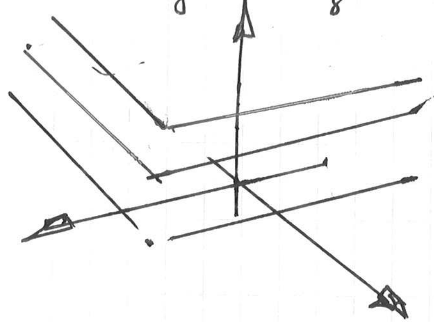
The eigenvectors of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ are orthonormal

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \text{ and } \begin{bmatrix} v_0 \\ -u_0 \end{bmatrix}$$

$$\mathbb{R}^2 = T_x \mathbb{T}^2 = \underset{\mathbb{R}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}}{\overset{\|}{E^u_x}} \oplus \underset{\mathbb{R}\begin{bmatrix} v_0 \\ -u_0 \end{bmatrix}}{\overset{\|}{E^s_x}}$$

$f_*(x,y,z) = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, z \right) = \left(A_* \begin{bmatrix} x \\ y \end{bmatrix}, z \right)$

Every horizontal plan is invariant $\rightarrow$ non ergodic



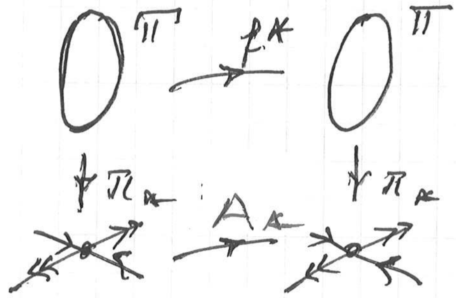
$$\mathbb{R}^3 = E^u_x(f_*) \oplus E^c_x(f_*) \oplus E^s_x(f_*)$$

$$\begin{matrix} \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{R}\begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix} & & \mathbb{R}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & & \mathbb{R}\begin{bmatrix} v_0 \\ -u_0 \\ 0 \end{bmatrix} \end{matrix}$$

The rates of expansion are uniform

$$\lambda^u(f^*) = \ln \chi, \quad \lambda^c(f^*) = 0, \quad \lambda^s(f^*) = -\ln \chi$$

f_* is a fibration over an Anosov map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$: we say normally hyperbolic



$$\pi : \begin{cases} \mathbb{T}^3 \longrightarrow \mathbb{T}^2 \\ (x,y,z) \longrightarrow (x,y) \end{cases}$$

$\left\{ \begin{array}{l} \pi^{-1}(x,y) \text{ are circle invariant by} \\ \text{the base dynamical system is Anosov} \end{array} \right.$

Theorem (Hirsch – Pugh – Shub) If f_* is a normally hyperbolic fibration with compact leaves, then any $f \in \mathrm{Diff}^2_\mu(M)$, C^1 close to f_* is also a normally hyperbolic fibration.

$$\begin{array}{ccc} \mathbb{T}^3 & \xrightarrow{\ f\ } & \mathbb{T}^3 \\ \downarrow \pi \overset{C^0}{\sim} \pi_* & & \downarrow \pi \\ \mathbb{T}^2 & \xrightarrow{\ A_*\ } & \mathbb{T}^2 \end{array}$$

- $\exists\ \pi : \mathbb{T}^3 \to \mathbb{T}^2$ C^0 close to π_*
- $\pi f = A_* \pi$
- $\pi^{-1}(x)$ are C^2 circles. (periodic circle are dense).

Perturbation model: Part One

$$f_b = f_* \circ \Phi_b \qquad \Phi_b \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z + b\varphi(x,y) \end{bmatrix} \qquad \varphi(x,y) = \sin(2\pi x)$$

The purpose of Φ_b is to obtain ergodicity.

Lemma Let $f : \begin{cases} \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2 \times \mathbb{T} \\ (x,y,z) \mapsto (A_*(x,y), z + \varphi(x,y)) \end{cases}$ with A_* Anosov, $\varphi \in C^0(\mathbb{T}^2, \mathbb{R})$

Then f is stably ergodic $\Longleftrightarrow$ φ is not cohomologous to a constant (no solution θ to $\varphi = \theta \circ A_* - \theta + c$)

Lemma $\forall b \neq 0$ $b \sin(2\pi x)$ is not cohomologous to a constant above the Anosov $\begin{cases} \mathbb{T}^2 \to \mathbb{T}^2 \\ (x,y) \mapsto A^*(x,y). \end{cases}$

Perturbation : part IV

$$f_{a,b} = \Psi_a \begin{pmatrix} 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} f_* \Phi_b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\Psi_a \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + a\,\Psi(z)\,u_0 \\ y + a\,\psi(z)\,v_0 \\ z \end{bmatrix}$$ perturbation in the direction of E^u

$$f_{a,b} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y + a\,\psi(z+\varphi_b(x,y))\,u_0 \\ x+y + a\,\psi(z+\varphi_b(x,y))\,v_0 \\ z + \varphi_b(x,y) \end{bmatrix}$$

$\varphi_b(x,y) = x+y + b\varphi(x,y)$ ($x+y$ is a coboundary).

$$\mathrm{Jac} = \begin{bmatrix} 2 + a\psi'\frac{\partial \varphi_b}{\partial x} u_0 & 1 + a\psi'\frac{\partial \varphi_b}{\partial y} u_0 & a\psi' u_0 \\ 1 + a\psi'\frac{\partial \varphi_b}{\partial x} v_0 & 1 + a\psi'\frac{\partial \varphi_b}{\partial y} v_0 & a\psi' v_0 \\ \frac{\partial \varphi_b}{\partial x} & \frac{\partial \varphi_b}{\partial y} & 1 \end{bmatrix}$$ $\boxed{\psi' \circ f_{a,b}}$

Lemma $f_{a,b}$ preserves Leb $\Rightarrow \lambda^u_{ab} + \lambda^c_{ab} + \lambda^s_{ab} = 0$

proof all maps have $\det(\mathrm{Jac}) = 1$.

Lemma $\mathbb{R}\begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix} + \mathbb{R}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is invariant by $f_{a,b}$ and the induced map has constant jacobian $\chi = \frac{3+\sqrt{5}}{2}$

proof $$Df_{ab}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a\psi' u_0 \\ a\psi' v_0 \\ 1 \end{bmatrix} = a\psi'\begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Df_{a,b}\begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix} = \chi\begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix} + \langle \nabla\varphi_b \mid \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\rangle \left\{ a\psi'\begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathrm{Mat}\left(Df_{a,b}, \begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \chi + a\psi'\langle\cdot|\cdot\rangle & a\psi' \\ \langle\cdot|\cdot\rangle & 1 \end{bmatrix}$$

$\det(\mathrm{Jac}\, f_{a,b}) = \chi$

Corollary $\lambda^u_{ab} + \lambda^c_{ab} = \ln\chi \Rightarrow \lambda^s_{a,b} = -\ln\chi$

Notation The perturbed unstable direction depends C^∞ on a,b

Let $E^u_{a,b}(x,y,z) = \mathbb{R}\begin{bmatrix} u_0 \\ v_0 \\ \gamma(x,y,z) \end{bmatrix}$ ($E^u_{a,b}$ is a graph over $\mathbb{R}\begin{bmatrix} u_0 \\ v_0 \\ u_0 \end{bmatrix}$ for small perturbation). The invariance of $E^u \Rightarrow$ cocycle relation for $\gamma_{a,b}$

$\boxed{E^u_{00} = \mathbb{R}\begin{bmatrix} u_0 \\ v_0 \\ u_0 \end{bmatrix}}$

Lemma $$\lambda^u(a,b) = \int \ln\left[\frac{\chi}{1 - a\psi'(z)\,\gamma_{a,b}(x,y,z)}\right] d\mu$$

By invariance $Df_{a,b} \cdot E^{\mu}_{a,b}(x,y,z) = E^{\mu}_{a,b} \circ f_{a,b}(x,y,z)$

$$Df_{a,b} \begin{bmatrix} u_0 \\ v_0 \\ \gamma \end{bmatrix} = \chi^{\mu}_{a,b} \begin{bmatrix} u_0 \\ v_0 \\ \gamma \circ f_{a,b} \end{bmatrix}$$

$$\begin{bmatrix} \chi + a\psi' \circ f \langle \cdot | \cdot \rangle & a\psi' \circ f \\ \langle \cdot | \cdot \rangle & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \gamma_{a,b} \end{bmatrix} = \chi^{\mu}_{a,b} \begin{bmatrix} 1 \\ \gamma_{a,b} \circ f_{a,b} \end{bmatrix}$$

$$\chi^{\mu}_{a,b} = [\chi + a\psi' \circ f \langle \cdot | \cdot \rangle] + a\psi' \gamma_{a,b}$$

$$\chi^{\mu}_{a,b} \gamma_{a,b} \circ f_{a,b} = \langle \cdot | \cdot \rangle + \gamma_{a,b}$$

$$\chi^{\mu}_{a,b} = \chi + a\psi' \circ f \chi^{\mu}_{a,b} \gamma_{a,b} \circ f_{ab}.$$

$$\chi^{\mu}_{a,b} = \frac{\chi}{1 - a\psi' \circ f_{a,b} \gamma_{a,b} \circ f_{ab}}$$

$$\lambda^{\mu}(a,b) = \int \ln(\chi^{\mu}_{a,b})\, d\mu = \int \ln\left(\frac{\chi}{1 - a\psi' \gamma_{a,b}}\right) d\mu$$

[**Lemma** $\frac{\partial \lambda^{\mu}_{a,b}}{\partial a} = 0 \quad \forall a$

proof $\lambda^{\mu}(a,b) = \int \ln\left(\frac{\chi}{1 - a\psi'(z)\gamma_{a,b}}\right) d\mu$

$$\frac{\partial \lambda^{\mu}}{\partial a}(a,b) = \int \frac{\psi' \gamma_{a,b} + a\psi'(z)\frac{\partial \gamma}{\partial a}}{1 - a\psi'\gamma} d\mu$$

$$\frac{\partial \lambda^{\mu}}{\partial a}(0,b) = \int \psi'(z)\gamma_{0,b}(x,y,z)\, d\mu$$

$$E_{0,b} = \mathbb{R} \begin{bmatrix} u_0 \\ v_0 \\ \gamma_{0,b}(x,y,z) \end{bmatrix} \qquad f_{0,b}(x,y,z) = \begin{bmatrix} 2x+y \\ x+y \\ z + \varphi_b(x,y) \end{bmatrix}$$

Since $f_{0,b}$ commutes with $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z+c \end{bmatrix}$, $E^{\alpha}_{0,b}(x,y,z)$ is independent of z.

Therefore $\gamma_{0,b}(x,y,z)$ is independent on z, $\gamma_{0,b}(x,y)$.

$$\frac{\partial \lambda^{\mu}}{\partial a}(0,b) = \int \psi'(z)\gamma_{a,b}(x,y)\, d\mu = 0.$$

[**Lemma** $\frac{\partial^2 \lambda}{\partial a^2}(0,0) = -u_0^2 \int (\psi'(z))^2\, d\mu < 0$

proof $\frac{\partial^2 \lambda}{\partial a^2}(a,b) = \int \left[\left(\frac{\psi'\gamma + a\psi'\frac{\partial \gamma}{\partial a}}{1 - a\psi'\gamma} \right)^2 + \frac{2\psi'\frac{\partial \gamma}{\partial a} + a\psi'\frac{\partial^2 \gamma}{\partial a^2}}{1 - a\psi'\gamma} \right] d\mu$

$$\frac{\partial^2 \lambda}{\partial a^2}(0,b) = \int \left[(\psi'\gamma)^2 + 2\psi'\frac{\partial \gamma}{\partial a} \right] d\mu$$

$\gamma_{a,b}$ solution de

$$\gamma\circ f = \frac{\nabla\varphi_b\cdot\begin{bmatrix}u_0\\v_0\end{bmatrix}+\gamma}{\chi + a\psi'\circ f\,(\nabla\varphi_b\begin{bmatrix}u_0\\v_0\end{bmatrix}+\gamma)}$$

$b=0$ $\quad \varphi_b(x,y) = x+y$.

Define $\Gamma(a,z,\gamma) = \dfrac{u_0+v_0+\gamma}{\chi + a\psi'(z)(u_0+v_0+\gamma)}$

$$\frac{\partial\Gamma}{\partial a}\Big|_{a=0} = -\psi'(z)\frac{(v_0+v_0+\gamma)^2}{\chi^2}$$

$$\frac{\partial\Gamma}{\partial z}\Big|_{a=0} = 0$$

$$\frac{\partial\Gamma}{\partial\gamma}\Big|_{a=0} = \frac{1}{\chi}$$

γ is a solution of.

$$\gamma_a(x,y,z) = \Gamma(a,z,\gamma_a\circ f_a^{-1}(x,y,z))$$

$$\gamma_0(x,y,z) = u_0. \qquad 2u_0+v_0 = \chi u_0$$

$$\frac{\partial\gamma}{\partial a}\Big|_{a=0}(x,y,z) = -\psi'(z)\,u_0^2 + \frac{1}{\chi}\frac{\partial\gamma}{\partial a}\Big|_{a=0}\circ f_0^{-1}(x,y,z)$$

More generally, by iterating the previous relation $x=(x,y,z)$

$$\frac{\partial\gamma}{\partial a}\Big|_{a=0}(x) = -\sum_{j\geq 0}\psi'(x_{-j})\,u_0^2$$

$$\frac{\partial^2\lambda}{\partial a^2}\Big|_{a=b=0} = \int(u_0\psi'(z))^2\,d\mu - 2\sum_{j\geq 0}\frac{u_0^2}{m^j}\int\psi'(x)\psi'(x_{-j})\,d\mu$$

$$\int\psi'(x)\psi'(x_{-j})\,d\mu = \int\psi'(x)\,\psi'\circ f_{0,0}^{-j}(x)\,d\mu$$
$$= \int\psi'\circ f_{0,0}^{j}(x)\,\psi(x)\,d\mu$$

$$f_{00}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2&1&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} \qquad f_{0,0}^{j}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}2&1\\1&0\end{bmatrix}^j\begin{bmatrix}x\\y\end{bmatrix}\\ z+a_jx+b_jy\end{bmatrix}$$

$$\int_0^1\psi'(z+a_jx+b_jy)\,dx = |a_j|\int_0^1\psi'(z)\,dz \quad \text{if } |a_j|\neq 0$$

So there is unique non zero term.

$$-2\sum_{j\geq 0}\frac{u_0^2}{m^j}\cdot\int\psi'(x)\,\psi'\circ f_{00}^{-j}(x)\,d\mu = -2u_0^2\int[\psi'(z)]^2\,d\mu(z).$$

$$\frac{\partial^2\lambda}{\partial a^2}\Big|_{a=b=0} = -u_0^2\int(\psi'(z))^2\,dz.$$