Stability of the spectral gap for the Boltzmann multi-species operator linearized around non-equilibrium Maxwell distributions

Andrea Bondesan

Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris

Inaugural France-Korea Conference

Bordeaux, November 25 2019

Joint work with

L. Boudin (LJLL, Sorbonne Université)

M. Briant and B. Grec (MAP5, Université Paris Descartes)

Outline of the talk

Outline of the talk

1. Spectral gap for mono-species Boltzmann
2. Boltzmann equation for mixtures
3. Spectral gap for multi-species Boltzmann
   - Linearized operator near the global equilibrium
   - Linearized operator near a local non-equilibrium Maxwellian
4. Elements of proof in the non-equilibrium case
5. Conclusion

Mono-species case

Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F)$$

Thanks to the H theorem:

- time-decreasing entropy $\iint F \log F \, \mathrm{d}v \, \mathrm{d}x$
- F equilibrium iff $Q(F, F) = 0$ iff there exist c, u, T such that

$$F(t, x, v) = \frac{c(t,x)}{(2\pi k T(t,x))^{3/2}} \, e^{-\frac{|v-u(t,x)|^2}{2kT(t,x)}} \qquad \text{(local Maxwellian)}$$

- when $t \to +\infty$, global equilibrium μ, uniform w.r.t. space

$$\mu(v) = \frac{\bar{c}}{(2\pi)^{3/2}} \, e^{-\frac{|v|^2}{2}} \qquad \text{(global Maxwellian)}$$

up to changes of coordinates (temperature $= 1$, velocity $= 0$)

Perturbative method

Write $F = \mu + f$, where we assume $f \ll \mu$
The Boltzmann equation then becomes

$$\partial_t f + v \cdot \nabla_x f = \underbrace{Q(\mu, f) + Q(f, \mu)}_{:= Lf} + \underbrace{Q(f, f)}_{\ll Lf}$$

- Entropy for the linearized operator $H_L(f) = \int f^2 \mu^{-1} \, \mathrm{d}v$ assuming that $f \in L^2_v(\mu^{-1})$
- Multiply the (approx.) eqn on f by $f \mu^{-1}$ and integrate w.r.t. v

$$\partial_t H_L(f) + \nabla_x \cdot \left(\int v f^2 \mu^{-1} \, \mathrm{d}v \right) = \underbrace{\int Lf \, f \mu^{-1} \, \mathrm{d}v}_{:= D_L(f)}$$

$\Rightarrow$ study the entropy production functional $D_L(f)$ associated to L

Coercivity estimate

$$D_L(f) \leq -\lambda_L \|f - \pi_L f\|_{L^2_v(\mu^{-1})}$$

where $\lambda_L > 0$ is the smallest eigenvalue of $-L$ and $\pi_L f$ is the orthogonal projection of f on $\ker L$

- Both Carleman and Grad use the Weyl theorem
  $\Rightarrow \lambda_L$ not explicitly computable
- First explicit estimate by Bobylev in the Maxwell molecule case
- Major contributions by Mouhot and coauthors

In the multi-species case too:

- [Daus, Jüngel, Mouhot, Zamponi], when all the species have the same atomic mass
- [Briant, Daus], in general cases with angular cutoff

Motivation of our work

- **Rigorous** derivation of the Maxwell-Stefan eqns

$$\partial_t c_i + \nabla_x \cdot N_i = 0$$

$$-\nabla_x c_i = \sum_{j \neq i} \frac{c_j N_i - c_i N_j}{\Delta_{ij}}$$

as the diffusive asymptotics $\varepsilon \to 0$ of the Boltzmann eqn for mixtures

$$\varepsilon \partial_t F_i + v \cdot \nabla_x F_i = \frac{1}{\varepsilon} Q_i(F, F)$$

after the formal ones [Boudin, Grec, Salvarani], [Boudin, Grec, Pavan]

- Importance of local Maxwellian functions with **different** velocities for each species

Outline of the talk

Collision operator

- Mixture of N ideal monatomic gases without chemical reactions
- Collision rules parametrized through $\sigma \in \mathbb{S}^2$
  $(m_i, v'), (m_j, v'_*) \mapsto (m_i, v), (m_j, v_*)$

$$
\begin{aligned}
v' &= \frac{1}{m_i + m_j}(m_i v + m_j v_* + m_j |v - v_*| \sigma) \\
v'_* &= \frac{1}{m_i + m_j}(m_i v + m_j v_* - m_i |v - v_*| \sigma)
\end{aligned}
$$

- Define $\theta = \widehat{(v - v_*, \sigma)}$
- Collision operator $Q(F,F) = (Q_1(F,F), \ldots, Q_N(F,F))$

$$
\begin{aligned}
Q_i(F,F) &= \sum_j Q_{ij}(F_i, F_j) \\
&= \sum_j \iint B_{ij}(|v - v_*|, \cos\theta) \left[F'_i \, F'_{j_*} - F_i \, F_{j_*} \right] \mathrm{d}\sigma \, \mathrm{d}v_*
\end{aligned}
$$

Assumptions on the cross sections

(H1) Symmetry $B_{ij} = B_{ji}$

(H2) $B_{ij}(|v - v_*|, \cos\theta) = \Phi_{ij}(|v - v_*|) b_{ij}(\cos\theta)$

(H3) Power-law potential $\Phi_{ij}(|v - v_*|) \propto |v - v_*|^\gamma$, $0 \leq \gamma \leq 1$

(H4) Grad's angular cutoff $0 < b_{ij}(\cos\theta) \leq C|\sin\theta||\cos\theta|$

- Functional setting: weighted L^2_v space

$$\langle F, G\rangle_{L^2_v(W)} = \sum_i \int F_i \, G_i \, W_i \, \mathrm{d}v$$

$$\|F\|^2_{L^2_v(W)} = \sum_i \int {F_i}^2 \, W_i \, \mathrm{d}v$$

if $F, G : \mathbb{R}^3_v \to \mathbb{R}^N$ and $W : \mathbb{R}^3_v \to (\mathbb{R}^*_+)^N$ positive measurable

Properties of Q [Desvillettes, Monaco, Salvarani]

- Weak forms of Q_{ij}
- Collisional invariants $\psi : \mathbb{R}^3 \to \mathbb{R}^N$

$$\psi \in \mathrm{Span}(e^{(1)}, \dots, e^{(N)}, v_1 m, v_2 m, v_3 m, |v|^2 m)$$

  where $e^{(i)} = (\delta_{i,j})_{1 \leq j \leq N}$, $m = (m_1, \dots, m_N)$
- H theorem $\Rightarrow$
  $Q(F,F) = 0$ iff the F_i are local Maxwellians with the **same** velocity
- Stationary solution given, up to change of coordinates, by

$$\mu_i(v) = \bar{c}_i \left(\frac{m_i}{2\pi}\right)^{3/2} \; e^{-m_i \frac{|v|^2}{2}}, \qquad 1 \leq i \leq N$$

- Linearized operator $Lf = (L_1 f, \dots, L_N f)$ with

$$L_i f = \sum_j Q_{ij}(\mu_i, f_j) + Q_{ij}(f_i, \mu_j), \qquad 1 \leq i \leq N$$

Local Maxwellians of interest

$$M_i^\varepsilon(t,x,v) = \bar{c}_i \left(\frac{m_i}{2\pi}\right)^{3/2} e^{-m_i \frac{|v-\varepsilon u_i(t,x)|^2}{2}}, \qquad 1 \le i \le N$$

- Small macroscopic velocities $O(\varepsilon)$ of each species, so that $M^\varepsilon \simeq \mu$
- Local in both time and space Maxwellian M^ε
- **Not a local equilibrium** for Q since $u_i \neq u_j \Rightarrow Q(M^\varepsilon, M^\varepsilon) \neq 0$
- Writing $F = M^\varepsilon + f$ implies

  $$\partial_t M^\varepsilon + v \cdot \nabla_x M^\varepsilon + \partial_t f + v \cdot \nabla_x f = Q(M^\varepsilon, M^\varepsilon) + L^\varepsilon f + Q(f,f)$$

  which allows to introduce the operator L^ε linearized around M^ε

  $$L_i^\varepsilon f = \sum_j Q_{ij}(M_i^\varepsilon, f_j) + Q_{ij}(f_i, M_j^\varepsilon), \qquad 1 \le i \le N$$

- L^ε is **not self-adjoint** since $M_i^{\varepsilon\prime} M_{j\,*}^{\varepsilon\prime} \neq M_i^\varepsilon M_{j\,*}^\varepsilon$

Outline of the talk

Spectral gap for L (1)

Theorem (Daus, Jüngel, Mouhot, Zamponi)

Denoting $\langle v \rangle = (1 + |v|^2)^{1/2}$ and $\mathscr{H} = L^2_v(\langle v \rangle^\gamma \mu^{-1})$, there exists $\lambda_L > 0$, which can be explicitly computed, such that, for any $f \in L^2_v(\mu^{-1})$,

$$\langle f, Lf \rangle_{L^2_v(\mu^{-1})} \leq -\lambda_L \| f - \pi_L f \|^2_{\mathscr{H}}.$$

- Decompose L into the sum of its mono-species and bi-species parts

$$L^m_i f = 2Q_{ii}(f_i, \mu_i), \quad L^b_i f = \sum_{j \neq i} Q_{ij}(\mu_i, f_j) + Q_{ij}(f_i, \mu_j), \quad 1 \leq i \leq N$$

- Five-step proof:
  1. Coercivity estimate on the mono-species operator L^m
  2. Take into account the cross-effects on $g = f - \pi_m f$
  3. "Coercivity estimate" on the bi-species operator L^b
  4. Link the previous estimate to g and $f - \pi_L f$
  5. Gather all the estimates to conclude

Spectral gap for L (2)

- Coercivity for L^m

$$\begin{aligned}\langle f, Lf\rangle_{L^2_v(\mu^{-1})} &= \overbrace{\langle f, L^b f\rangle_{L^2_v(\mu^{-1})}}^{\leq 0} + \langle f, L^m f\rangle_{L^2_v(\mu^{-1})} \\ &\leq -\lambda_m \| \underbrace{f - \pi_m f}_{\perp \ker L^m} \|^2_{\mathscr{H}}\end{aligned}$$

If we stop here, we neglect all the cross effects between the species.

- $\langle f, Lf\rangle_{L^2_v(\mu^{-1})} \leq (8\eta - \lambda_m)\|g\|^2_{\mathscr{H}} + \eta\langle \pi_m f, L^b \pi_m f\rangle_{L^2_v(\mu^{-1})}$ for some $\eta > 0$
- Coercivity for L^b on $\ker L^m$: since $(\pi_m f)_i = \alpha_i + u_i \cdot v + e_i|v|^2$,

$$\langle \pi_m f, L^b \pi_m f\rangle_{L^2_v(\mu^{-1})} \leq -C\sum_{i,j} |u_i - u_j|^2 + |e_i - e_j|^2 := -C\mathscr{P}(u, e)$$

- Then $\mathscr{P}(u, e) \geq C\|f - \pi_L f\|^2_{\mathscr{H}} - 2C\|f - \pi_m f\|^2_{\mathscr{H}}$
- Simple algebraic computations to conclude

"Spectral gap" for L^ε

Set $\mathscr{R}^\varepsilon_{i,\delta}(t,x) = 1 + |u_i(t,x)| \exp\left(\frac{4m_i}{1-\delta}\varepsilon^2|u_i(t,x)|^2\right)$
and $\mathscr{S}^\varepsilon_\delta(t,x) = \max_i \left(\bar{c}_i^{1-\delta}|u_i(t,x)|\mathscr{R}^\varepsilon_{i,\delta}(t,x)\right)$.

Theorem (AB, Boudin, Briant, Grec)

There exists $C > 0$, which can be explicitly computed, such that, for any $f \in L^2_v(\mu^{-1})$, and for some $\delta \in (0,1)$ adequately chosen,

$$\langle f, L^\varepsilon f\rangle_{L^2_v(\mu^{-1})} \leq -\left(\lambda_L - \varepsilon C \mathscr{S}^\varepsilon_\delta(t,x)\right) \|f - \pi_L f\|^2_{\mathscr{H}} + \varepsilon C \mathscr{S}^\varepsilon_\delta(t,x)\|\pi_L f\|^2_{\mathscr{H}}$$

- Main idea: $L^\varepsilon = L + (L^\varepsilon - L)$
- Spectral gap for L
- How can we handle the correction term in $(L - L^\varepsilon)$?
  - $M^\varepsilon = \mu + O(\varepsilon)$
  - $L - L^\varepsilon =$ multiplicative operator $+$ integral operator in a kernel form

Outline of the talk

Estimate on $M^\varepsilon - \mu$ pointwise

Lemma

For any $\delta \in (0,1)$, there exists an explicit constant $C_\delta > 0$ s.t., for any v,

$$|M_i^\varepsilon(v) - \mu_i(v)| \leq C_\delta \, \bar{c}_i^{1-\delta} \, |u_i| \, \mathscr{R}_{i,\delta}^\varepsilon \, \mu_i(v)^\delta \, \varepsilon$$

$$\begin{aligned}
|M_i^\varepsilon(v) - \mu_i(v)| &= \varepsilon \left| \int_0^1 ((v - \varepsilon s u_i) \cdot u_i) \, \mu_i(v - \varepsilon s u_i) \, \mathrm{d}s \right| \\
&\leq \varepsilon \, |u_i| \, \mu_i(v)^\delta \int_0^1 (|v| + \varepsilon |u_i|) \, \mu_i(v - \varepsilon s u_i) \, \mu_i(v)^{-\delta} \, \mathrm{d}s \\
&\leq \varepsilon \, |u_i| \, \bar{c}_i^{1-\delta} \, \left(\frac{m_i}{2\pi} \right)^{3/2} \, \mu_i(v)^\delta \, (|v| + \varepsilon |u_i|) \\
&\qquad \exp\left(-(1-\delta) \frac{m_i}{2} |v|^2 + \varepsilon m_i |u_i| |v| \right)
\end{aligned}$$

Then discuss whether $|v| \geq \dfrac{4\varepsilon |u_i|}{1-\delta}$ or not

Writing the correction term

$$\begin{aligned}\langle f, L^\varepsilon f - Lf\rangle_{L^2_v(\mu^{-1})} \;=\; & \sum_{i,j}\iiint B_{ij}(M_i^{\varepsilon\prime} - \mu_i')\, f_{j*}'\, f_i\, \mu_i^{-1}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ & + \sum_{i,j}\iiint B_{ij}(M_{j\,*}^{\varepsilon\prime} - \mu_{j*}')\, f_i'\, f_i\, \mu_i^{-1}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ & - \sum_{i,j}\iiint B_{ij}(M_i^{\varepsilon} - \mu_i)\, f_{j*}\, f_i\, \mu_i^{-1}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ & - \sum_{i,j}\iiint B_{ij}(M_{j\,*}^{\varepsilon} - \mu_{j*})\, f_i^2\, \mu_i^{-1}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\end{aligned}$$

$\longrightarrow$ 4 terms $\mathscr{T}_1^\varepsilon(f)$, $\mathscr{T}_2^\varepsilon(f)$, $\mathscr{T}_3^\varepsilon(f)$, $\mathscr{T}_4^\varepsilon(f)$

The last terms $\mathscr{T}_3^\varepsilon(f)$ and $\mathscr{T}_4^\varepsilon(f)$ are the simplest ones:
use the previous lemma to obtain the estimate.

Carleman representations [BRIANT, DAUS]

In order to deal with $\mathscr{T}_1^\varepsilon(f)$ and $\mathscr{T}_2^\varepsilon(f)$, we need to rewrite them thanks to

$$\iint B_{ij} f_{i*}' f_j' \,\mathrm{d}\sigma\,\mathrm{d}v_* = \int \left(\frac{C_{ij}}{|v-v_*|} \int_{\tilde{E}_{vv_*}^{ij}} \frac{B_{ij}}{|u-v_*|} f_i(u)\,\mathrm{d}\tilde{E}(u) \right) f_j^*\,\mathrm{d}v_*$$

$$\iint B_{ij} f_i' f_{j*}' \,\mathrm{d}\sigma\,\mathrm{d}v_* = \int \left(\frac{C_{ji}}{|v-v_*|} \int_{E_{vv_*}^{ij}} \frac{B_{ij}}{|u-v_*|} f_j(u)\,\mathrm{d}E(u) \right) f_i^*\,\mathrm{d}v_*$$

where $\mathrm{d}E$ is the Lebesgue measure on some hyperplane $E_{vv_*}^{ij}$ orthogonal to $v-v_*$and $\mathrm{d}\tilde{E}$ the Lebesgue measure on the set $\tilde{E}_{vv_*}^{ij}$ which is $E_{vv_*}^{ij}$ if $m_i = m_j$, and a sphere of radius and center depending on v, v_*, if $m_i \neq m_j$

We focus on $\mathscr{T}_2^\varepsilon(f)$, because $\mathscr{T}_1^\varepsilon(f)$ can be treated in the same way, but with the additional difficulty of dealing with different masses in the Maxwellians

Auxiliary function I_{ji}^{δ}

Define $I_{ji}^{\delta}(v,v_*) = \dfrac{C_{ji}}{|v-v_*|}\displaystyle\int_{E_{vv_*}^{ij}} \frac{B_{ij}}{|u-v_*|}\mu_j(u)^{\delta}\,\mathrm{d}E(u)$

Then, in the same way as in [Guo, CPAM '02], there exists $\bar\delta \in (0,1)$ such that, for any $\delta \in (\bar\delta, 1)$, there exists $C_{ji}(\delta) > 0$ such that

$$\int I_{ji}^{\delta}(v,v_*)\sqrt{\frac{\mu_i(v_*)}{\mu_i(v)}}\,\mathrm{d}v \leq C_{ji}(\delta), \qquad \int I_{ji}^{\delta}(v,v_*)\sqrt{\frac{\mu_i(v_*)}{\mu_i(v)}}\,\mathrm{d}v_* \leq C_{ji}(\delta)$$

Hence, setting $h_i = f_i\mu_i^{-1/2}$,

$$\begin{aligned}
\mathscr{T}_2^{\varepsilon}(f) &\leq \varepsilon C_\delta \mathscr{S}_\delta^{\varepsilon} \sum \iiint B_{ij}{\mu_j}'^{\delta}_{*}|f_i'||f_i|\mu_i^{-1}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v && \text{(lemma)}\\
&\leq \varepsilon C_\delta \mathscr{S}_\delta^{\varepsilon} \sum \iint I_{ji}^{\delta}(v,v_*)\sqrt{\frac{\mu_{i*}}{\mu_i}}|h_{i*}||h_i|\,\mathrm{d}v_*\,\mathrm{d}v && \text{(+ Young)}\\
&\leq \varepsilon \tilde C_\delta \mathscr{S}_\delta^{\varepsilon}\|f\|^2_{L^2_v(\langle v\rangle^{\gamma}\mu^{-1})}
\end{aligned}$$

Outline of the talk

Conclusions

- Spectral gap on L, near a global Maxwellian equilibrium
- Spectral gap on L^{ε}, near a local non-equilibrium Maxwellian
- Rigorous convergence, in the diffusion asymptotics, of the solutions to the Boltzmann eqn for mixtures towards the solutions to the Maxwell-Stefan equations (AB, Briant, submitted 2019)

Thank you for your attention!