On Type I blow up for the incompressible Euler equations

Dongho Chae

Chung-Ang University and KIAS, Korea

November 24-27, 2019

Inaugural France-Korea Conference on Algebraic Geometry, Number Theory, and Partial Differential Equations

Institut of Mathematics, University of Bordeaux

Table of contents

1. Introduction

Fluids and Partial Differential Equations

Dynamics of fluids is governed by the Navier-Stokes equations.

C–L. Navier (1785–1836), France

G.G. Stokes (1819–1903), UK

- Incompressible Navier-Stokes equations on $\mathbb{R}^3 \times [0, \infty)$:

$$\boxed{\begin{gathered} \frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p + \nu \Delta v, \\ \operatorname{div} v = 0, \end{gathered}}$$

where

$$\begin{cases} v = (v^1, v^2, v^3) = v(x, t), & \text{velocity} \\ p = p(x, t), & \text{pressure} \\ \nu > 0, & \text{viscosity} \end{cases}$$

- By writing the equations in "dimensionless form" we can replace $\nu \Rightarrow \frac{1}{Re}$, the Reynolds number, which represents the degree of turbulence.
- "Turbulent limit" : $\boxed{Re \to +\infty} \Longleftrightarrow \boxed{\nu \to 0^+}$
- Formally setting $\nu = 0 (Re = +\infty)$:
  "Navier-Stokes equations $\Rightarrow$ Euler equations"

- In this talk we concentrat on the Euler equations.

$$(E)\quad \boxed{\begin{aligned} \frac{\partial v}{\partial t} + (v\cdot\nabla)v &= -\nabla p, \\ \mathrm{div}\, v &= 0, \end{aligned}}$$

Figure: L. Euler (1707-1783), Switzerland

Local existence and blow up of smooth solutions

- For given smooth initial data $v(\cdot, 0) = v_0 \in H^m(\mathbb{R}^3), m > 5/2$, the existence and uniquenes of local in time smooth solution $v \in C([0, T); H^m(\mathbb{R}^3))$ is well-known(e.g. Kato[JFA,'72]).
- We say a local in time smooth solution v blows up at $T < +\infty$ if

$$\limsup_{t \nearrow T} \|v(t)\|_{H^m} = +\infty.$$

Blow up criteria

- The proof of local in time existence by Kato leads to the estimate,

$$\frac{d}{dt}\|v(t)\|_{H^m} \leq c\|\nabla v(t)\|_{L^\infty}\|v(t)\|_{H^m}, \quad m > \frac{5}{2},$$

which, by Gronwall's lemma, provides us with

$$\|v(t)\|_{H^m} \leq \|v_0\|_{H^m} \exp\left(c\int_0^t \|\nabla v(t)\|_{L^\infty} dt\right).$$

- Therefore, we have the following immediate blow up criteria,

$$\text{blow-up at } T \quad \Leftrightarrow \int_0^T \|\nabla v(s)\|_{L^\infty} ds = +\infty.$$

- Beale-Kato-Majda BKM criterion['84]; Using the logarithmic Sobolev inequality, one can replace $\|\nabla v(s)\|_{L^\infty} \to \|\omega(s)\|_{L^\infty}$ in the above criteria to have

$$\text{blow-up at } T \quad \Leftrightarrow \int_0^T \|\omega(s)\|_{L^\infty} ds = +\infty, \qquad \omega = \mathrm{curl}\, \mathrm{v}$$

- Constantin-Fefferman-Majda CFM ['96]; blow up in terms of the direction of vorticity;

$$\text{blow-up at } T \quad \Leftrightarrow \int_0^T \|\nabla \xi(s)\|_{L^\infty}^2 ds = +\infty, \qquad \xi = \frac{\omega}{|\omega|}$$

- In this talk we are trying to answer to the following question.
- Question In the above blow up criteria can we replace

$$\int_0^T \|\nabla v(t)\|_{L^\infty} dt \quad \Rightarrow \quad \sup_{0<t<T} (T-t)\|\nabla v(t)\|_{L^\infty}?$$

- In other words,

$$\boxed{\sup_{0<t<T} (T-t)\|\nabla v(t)\|_{L^\infty} < +\infty \quad \Rightarrow \quad \text{no blow up at } T?}$$

- Note that above quantities are “scaling invariant” as will be explained below.

The scaling property of the Euler system

- The Euler system (E) has scaling property that if (v, p) is a solution, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t)$$

  are also solutions.

- The case $\boxed{\alpha = \frac{3}{2}}$ is important for our analysis, since in this case the energy is scaling invariant.

- Indeed, by the energy conservation we have for $v^{\lambda} = v^{\lambda, \frac{3}{2}}$,

$$\|v^{\lambda}(t)\|_{L^2} = \|v(\lambda^{\frac{5}{2}} t)\|_{L^2} = \|v(t)\|_{L^2}.$$

Type I blow-up

- Hereafter, for convenience we consider (E) in $\mathbb{R}^3 \times (-1, 0)$ and $T = 0$ is the possible first blow-up time, and $t = -1$ is the initial time .
- We say solution v blowing up at $t = 0$ is of **Type I** if
$$\sup_{-1<t<0} (-t)\|\nabla v(t)\|_{L^\infty} < +\infty.$$
- The quantity is independent of scalings with α, λ.
- Self-similar singularity is a special case of Type I blow up.
- If
$$\sup_{-1<t<0} (-t)\|\nabla v(t)\|_{L^\infty} = +\infty,$$
then we say it is of **Type II**.

Exclusion of small Type I blow up in $\mathbb{R}^3$

- Type I blow up under 'global' smallness condition is easily excluded.

Theorem (DC, JFA'07)

Let $v \in C([-1,0); H^m(\mathbb{R}^3))$, $m > 5/2$, be a solution to the Euler equations. Then,

$$\limsup_{t \nearrow 0} (-t)\|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)} < 1$$

$$\Rightarrow \limsup_{t \nearrow 0} \|v(t)\|_{H^m} < +\infty$$

and $t = 0$ is no blow-up time.

Outline of the proof

- Small Type I condition implies there exists $t_0 \in (-1, 0)$ and $0 < c_0 < 1$ such that

$$\sup_{t_0 < s < 0} (-s)\|\nabla v(s)\|_{L^\infty} \leq c_0.$$

- The vorticity form of the Euler equations

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v$$

- This gives us immediately the estimate

$$\begin{aligned}
\|\omega(t)\|_{L^\infty} &\leq \|\omega(t_0)\|_{L^\infty} \exp\left(\int_{t_0}^{t} \|\nabla v(s)\|_{L^\infty} ds\right) \\
&\leq \|\omega(t_0)\|_{L^\infty} \exp\left(c_0 \int_{t_0}^{t} (-s)^{-1} ds\right) \\
&= \|\omega(t_0)\|_{L^\infty} \left(\frac{t_0}{t}\right)^{c_0} \quad \forall t \in (t_0, 0).
\end{aligned}$$

- Since $0 < c_0 < 1$, we have $\int_{t_0}^{0} \|\omega(t)\|_{L^\infty} dt < +\infty$, and by BKM no blow up at 0. ■

Exclusion of local small Type I blow up

- The following is a recent localized version of the above theorem.

Theorem (with J. Wolf, ARMA'18)

Let $v \in L^\infty(-1,0;L^2(B(r))) \cap C([-1,0);W^{2,q}(B(r))), q > 3$ be a solution to the Euler equations with for some $3 < q < +\infty$. Then,

$$\limsup_{t \nearrow 0} (-t)\|\nabla v(t)\|_{L^\infty(B(r))} < 1$$

$$\Rightarrow \limsup_{t \nearrow 0} \|v(t)\|_{W^{2,q}(B(\rho))} < +\infty$$

for all $\rho \in (0,r)$. Namely, local small Type I implies non blow up.

Key idea of the proof:

- In order to have estimates for $\|D^2 v(t)\|_{L^q(B(r))}, q > 3$ we introduce the transform,

$$w(y,t) := v(y + (-t)^\theta y, t), \quad 0 < \theta < 1$$

- Then the new vorticity $\Omega = \nabla \times w$ solves the equation

$$\partial_t \Omega + \frac{\theta(-t)^{\theta-1}}{1+(-t)^\theta}\Omega + \frac{\theta(-t)^{\theta-1}}{1+(-t)^\theta} y \cdot \nabla \Omega + \frac{1}{1+(-t)^\theta}(w \cdot \nabla)\Omega$$
$$= \frac{1}{1+(-t)^\theta}\Omega \cdot \nabla w \quad \text{in} \quad \mathbb{R}^3 \times (-1,0).$$

- Choice of small θ is essential to obtain right sign condition to ignore terms generated by $y \cdot \nabla \Omega$.

2. Energy concentration and Type I blow up

Energy concentration and Type I blow up

- Here we consider possibility of Type I blow up without smallness condition.

- We shall show that under Type I condition the energy concentration in atomic form cannot happen at the blow-up time.

- Energy concentration in atomic form means that there exists an atomic measure μ (i.e. $\mu(\{x\}) > 0$ for some $x \in \mathbb{R}^3$) such that

$$|v(\cdot, t)|^2 dx \rightharpoonup \mu \quad \text{as} \quad t \to 0^-$$

  in the sense of measure.

- Typical example is

$$|v(\cdot, t)|^2 dx \rightharpoonup \sum_{k=1}^{\infty} C_k \delta_{x_k}$$

Motivations for the study of energy concentration

- Self-similar singularity in the energy conserving scale is an example of Type I blow-up with one point energy concentration. Removing this scenario has been open.

- Concentration phenomena in the other equations:
  - Nonlinear Schödinger equations : blow-up with L^2 norm concentration [Merle-Tsutsumi'90, Merle'90]
  - Chemotaxis equations: blow-up with L^1 norm concentration [Herrero-Velázquez'96]

(i) Removing one point energy concentration

- We first remove one point energy concentration under Type I.
- Later, using the blow-up argument we remove general atomic concentration.

Theorem (with J. Wolf, CMP'20)

Let $v \in L^\infty(-1,0; L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([-1,0), W^{1,\infty}(\mathbb{R}^3))$ be a solution to the Euler system, satisfying the following Type I blow-up condition at $t=0$, i.e.

$$\limsup_{t\to 0}(-t)\|\nabla v(t)\|_{L^\infty} < +\infty.$$

Suppose there happens:

$$|v(t)|^2 dx \rightharpoonup c\delta_0 \qquad \text{as} \quad t \to 0,$$

Then $v \equiv 0$ and $c = 0$.

STEP 1 **Decay estimates for the velocity**

Lemma (A)

Let $v \in L^\infty(-1,0; L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([-1,0), W^{1,\infty}(\mathbb{R}^3))$ be a solution to the Euler equations satisfying the Type I condition,

$$\limsup_{t\to 0}(-t)\|\nabla v(t)\|_{L^\infty} < +\infty$$

and the energy concentrates at one point, i.e

$$|v(t)|^2 dx \rightharpoonup c\delta_0 \qquad \text{as} \quad t \to 0.$$

Then for every $0 < \beta < 5$ and $t \in (-1, 0)$ it holds

$$\int_{\mathbb{R}^3} |v(t)|^2 |x|^\beta dx \le c(-t)^{\frac{2\beta}{5}}.$$

STEP 2 **Fast decay estimates for the Helmholtz projection**

Lemma (B)

Let v be a local smooth solution to the Euler equation on $\mathbb{R}^3 \times [-1, 0)$ satisfying Type I condition and the energy concentration at one point $(x, t) = (0, 0)$.
Let $\mathbb{P}_r : L^2(B(r)^c) \mapsto L^2_\sigma(B(r)^c)$ be the Helmholtz projection.
Then, for all $k \in \mathbb{N}$ and for all $r > 0$ there exists $c = c(k)$ such that the following decay estimate holds

$$\boxed{\|\mathbb{P}_r v(t)\|^2_{L^2(B(r)^c)} \le c(k)(-t)^{\frac{2k}{5}} r^{-k}} \quad \forall\, t \in (-1, 0)$$

- The proof is more technical than Lemma (A).

STEP 3 **Assuming Lemma (A), (B) to prove the theorem**

- We choose θ so that $0 < \theta < \frac{1}{5}$.
- For a solution v to the Euler equations we transform: $\boxed{v \mapsto w}$,

$$\boxed{w(x,t) = v((-t)^{\theta} x, t)}$$

- Then, w solves the transformed Euler system,

$$\frac{\partial w}{\partial t} + \theta(-t)^{-1} x \cdot \nabla w + (-t)^{-\theta}(w \cdot \nabla) w = -\nabla \pi,$$
$$\nabla \cdot w = 0.$$

- Using the decay Lemma (A), (B), one can show that there exists $t_0 \in (-1, 0)$ such that

$$\nabla \times w(t) = 0 \quad \text{on} \quad B(1)^c \quad \forall t_0 < t < 0.$$

- Transforming back to the original vorticity, $\boxed{\omega(t) = \nabla \times v(t)}$,

$$\operatorname{supp} \omega(t) \subset B((-t)^{\theta}) \quad \forall t_0 < t < 0.$$

- Since the measure of $\operatorname{supp}\omega(t)$ is preserved due to the Helmholtz formula for the vorticity,

$$\boxed{\omega(X(a,t),t) = \nabla_a X(a,t)\omega_0(a)},$$

we have

$$\mathrm{meas}\{\operatorname{supp}\omega(t_0)\} = \mathrm{meas}\{\operatorname{supp}\omega(t)\} \leq c(-t)^{3\theta} \to 0$$

as $t \to 0$.

- This is possible only if $\omega(t_0) \equiv 0$, and $v(t_0)$ is harmonic.

- Since $v(t_0) \in L^2(\mathbb{R}^3)$, we conclude that $v(t_0) \equiv 0$ by the Liouville theorem for harmonic function, and hence $v \equiv 0$. Namely, one point energy concentration + Type I is impossible! ■

(ii) Exclusion of atomic concentration of energy

- We use the blow-up argument to remove more general form of atomic concentration under local Type I condition.

Theorem

Let $v \in L^\infty(-1,0;L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([-1,0);W^{1,\infty}(\mathbb{R}^3))$ be a solution of the Euler equations satisfying the Type I condition,

$$\boxed{\sup_{t\in(-1,0)} (-t)\|\nabla v(t)\|_{L^\infty} < +\infty}.$$

Suppose there exists $\sigma_0 \in \mathcal{M}(\mathbb{R}^3)$ such that

$$|v(t)|^2 dx \to \sigma_0 \quad \text{as} \quad t \to 0^-.$$

Then, σ_0 is a non-atomic.

Outline of the proof

STEP 1 **Local condition of energy non-concentration**

- We introduce the notion of suitable weak solution (v, p) of (E): a weak solution satisfying the local energy inequality:

$$\int_{\mathbb{R}^3} |v(t)|^2 \phi dx \le \int_{\mathbb{R}^3} |v(s)|^2 \phi dx + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2p) v \cdot \nabla \phi dx d\tau.$$

for all $\phi \in C_c^\infty(\mathbb{R}^3)$ and for a.e. $-1 \le t < s < 0$.

- Below we denote the 'parabolic cylinder' consistent with the energy conserving scale, $Q(R) := B(R) \times (-R^{5/2}, 0)$.

Energy non-concentration criterion

- We first establish the following criterion of energy non-concentration in terms of a Morrey norm.

Lemma (A)

Let $v \in L^\infty(-R^{5/2}, 0; L^2(B(R))) \cap L^3(Q(R))$ be a suitable weak solution to the Euler equations and satisfy

$$\limsup_{r \to 0^+} r^{-1} \|v\|^3_{L^3(Q(r))} < +\infty, \qquad \boxed{\liminf_{r \to 0^+} r^{-1} \|v\|^3_{L^3(Q(r))} = 0}$$

Then, there is no energy concentration of energy at $(0,0)$.

STEP 2 Blow up argument

- We will show by contradiction argument as follows: Assume atomic concentration⇒ Blow-up w.r.t. one atomic point⇒ One-point concentration in $\mathbb{R}^3$ ⇒ contradiction to previous result
- We first note the following interpolation inequality,

$$(I_*) \quad r^{-1}\|v\|^3_{L^3(Q(r))} \le cK_0 r^{-\frac{5}{2}}\|v\|^2_{L^2(Q(r))} + cK_0^{\frac{1}{2}}K_1^{\frac{3}{2}}\left(r^{-\frac{5}{2}}\|v\|^2_{L^2(Q(r))}\right)^{\frac{1}{2}},$$

where we set $K_0 := \|v(t)\|_{L^\infty(-R^{5/2},0);L^2(B(R))}$, $K_1 := \sup_{t\in(-R^{\frac{5}{2}},0)}(-t)\|\nabla v(t)\|_{L^\infty(B(R))}$, which are bounded constants by the hypothesis.

- Note also that $r^{-\frac{5}{2}}\|v\|^2_{L^2(Q(r))} \le \|v\|_{L^\infty(-r^{5/2},0;L^2(B(0,r))} < +\infty$.

- Suppose there exists an atomic concentration, then Lemma (A)-□, combined with the above interpolation inequality (I) implies that there exists $\varepsilon > 0$ and a sequence $r_k \to 0$ such that

$$\liminf_{k\to\infty} r_k^{-\frac{5}{2}} \|v\|^2_{L^2(Q(r_k))} \geq \varepsilon.$$

Otherwise, contradiction to Lemma (A)-□!

- We define a (blow-up) sequence

$$v_k(x,t) = r_k^{\frac{3}{2}} v(r_k x, r_k^{\frac{5}{2}} t).$$

- Using Type I condition and the energy conservation, we can deduce the following uniform bound for $\{v_k\}$,

$$\|v_k\|_{L^\infty(-1,0;L^2_\sigma(\mathbb{R}^3))} + \|v_k\|_{L^3([-1,0);\dot{W}^{\theta,3}(\mathbb{R}^3))} \leq C$$

for all $0 < \theta < \frac{1}{3}$.

- In the above we use the following norm for the fractional derivatives(Sobolev-Slobodeckij semi-norm) in $\mathbb{R}^3$,

$$|f|_{\dot{W}^{\theta,p}} := \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+3}} dxdy\right)^{\frac{1}{p}}.$$

- Taking the limit for a sub-sequence (by compactness lemma), one can construct a non-trivial suitable weak solution to (E),

$$v^* \in L^\infty(-1,0; L^2_\sigma(\mathbb{R}^3)) \cap L^3([-1,0); \dot{W}^{\theta,3}(\mathbb{R}^3)),.$$

satisfying the estimate

$$(*) \quad \boxed{\sup_{r\in(0,R)} \frac{1}{r^{1-3\theta}} \int_{-r^{\frac{5}{2}}}^{0} |v^*(t)|^3_{\dot{W}^{\theta,3}(B(r))} dt < +\infty}$$

- Indeed, we have the following interpolation inequality:

$$\sup_{r\in(0,R)} \frac{1}{r^{1-3\theta}} \int_{-r^{\frac{5}{2}}}^{0} |v(t)|^3_{\dot{W}^{\theta,3}(B(r))} dt$$

$$\leq c \boxed{\sup_{r\in(0,R)} r^{-1} \|v\|^3_{L^3(Q(r))}}$$

$$+c \boxed{\sup_{-R^{\frac{5}{2}}<t<0} (-t)^3 \|\nabla v(t)\|^3_{L^\infty(B(R))}} < +\infty$$

by the inequality (I_*) and Type I condition respectively, which implies (*).

- Moreover, for such limiting solution v^* one can choose a sequence of time $t_k \nearrow 0$ and a positive constant $c_0 > 0$ such that

$$|v^*(t_k)|^2 dx \rightharpoonup c_0\delta_0 \quad \text{as} \quad k \to +\infty$$

in the sense of measure, namely one point concentration in $\mathbb{R}^3$ for the limiting solution!

- Our previous exclusion theorem for one point energy concentration in $\mathbb{R}^3$ with Type I blow-up condition implies $\boxed{c_0 = 0}$, namely no atomic concentration .■

3. On Type I blow up for the axisymmetric solutions

Axisymmetric Euler equations

- We say v is an axisymmetric solution of the Euler equations if it solves the Euler system, and can be written as

$$v = v^r(r, x_3, t)e_r + v^\theta(r, x_3, t)e_\theta + v^3(r, x_3, t)e_3,$$

where $r = \sqrt{x_1^2 + x_2^2}$, and

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (\frac{x_2}{r}, \frac{-x_1}{r}, 0), \quad e_3 = (0, 0, 1),$$

are the basis of the cylindrical coordinate system.

- Our theorem below is an improvement of the BKM theorem off the axis local region.

Theorem (with J. Wolf [ARMA, '19])

Let $v \in C([-1,0); W^{2,q}(\mathbb{R}^3)) \cap L^\infty(-1,0; L^2(\mathbb{R}^3))$, $q > 3$ be an axisymmetric solution to the Euler equations. If the following holds

$$\boxed{\int_{-1}^{0} (-t)\|\omega(t)\|_{L^\infty(B(x_*,R_0))} dt < +\infty}$$

for some ball $B(x_*, R_0)$, which is away from the axis, then there exits no blow-up at $t = 0$ in the torus generated by the rotation of $B(x_*, R_0)$ around the axis.

- As an immediate consequence of this theorem we remove some of Type II as well as Type I singularities in terms of the vorticity blow-up rate off the axis.

Corollary

Let $v \in C([-1,0); W^{2,q}(\mathbb{R}^3)) \cap L^\infty(-1,0; L^2(\mathbb{R}^3))$, $3 < q < +\infty$, be an axisymmetric solution to the Euler equations. Suppose the following vorticity blow-up rate condition holds

$$(*) \quad \sup_{t\in(-1,0)} (-t)^2 \left[\log\left(\frac{1}{-t}\right)\right]^\alpha \|\omega(t)\|_{L^\infty(B(x_*,R_0))} < +\infty$$

for some $\alpha > 1$ and some ball $B(x_*, R_0)$. Then, no singularity at $t = 0$ in the ball.

- Indeed, if $\sup_{t\in(-1,0)}(-t)\|\nabla v(t)\|_{L^\infty(B(x_*,R_0))<+\infty}$(Type I), then (*) is immediate, and no singularity in this case.

The Idea of the Proof:

- The main task of proof is establishing local BKM type criterion for the 2D Boussinesq system of (u, θ):

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \theta e_2 \\ \theta_t + u \cdot \nabla \theta = 0 \\ \nabla \cdot u = 0 \end{cases}$$

- The vorticity form of which is

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \theta_{x_1}, \\ \theta_t + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0. \end{cases}$$

- The 3D axisymmetric Euler system off the axis is equivalent to the 2D Boussinesq system.

$$(\text{Axisym. Euler}) \begin{cases} \partial_t \Omega + \tilde{v} \cdot \tilde{\nabla} \Omega = \dfrac{\partial_3 \Theta}{r^4}, \\ \partial_t \Theta + \tilde{v} \cdot \tilde{\nabla} \Theta = 0, \end{cases}$$

where

$$\Theta = (r v^\theta)^2, \qquad \Omega = \frac{\omega^\theta}{r},$$

and

$$\tilde{v} = v^r e_r + v^3 e_3, \qquad \tilde{\nabla} = e_r \partial_r + e_3 \partial_3.$$

The scaling property of the 2D Boussinesq system

- In the 3D Euler eq the blow-up is controlled by the vorticity ω, which has scaling

$$\omega(x,t) \to \lambda^{\alpha+1}\omega(\lambda x, \lambda^{\alpha+1}t).$$

- In the 2D Boussinesq eq the blow-up is controlled by $\nabla\theta$, which has the scaling

$$\nabla\theta(x,t) \to \lambda^{2\alpha+2}\nabla\theta(\lambda x, \lambda^{\alpha+1}t).$$

- Therefore the local 'scaling invariant' quantity to control the blow-up off the axis region is $\boxed{\nabla\Theta \sim \nabla v^\theta \sim \tilde{\omega}}$ (Recall $\Theta = (rv^\theta)^2$), where

$$\tilde{\omega} = -\partial_3 v^\theta e_r + (\partial_r v^\theta + \frac{v^\theta}{r})e_3.$$

- One can actually observe

$$\int_{-1}^{0}(-t)\|\nabla\Theta(t)\|_{L^\infty(B(x_0,R))}dt \sim \int_{-1}^{0}(-t)\|\nabla v^\theta(t)\|_{L^\infty(B(x_0,R))}dt$$
$$\sim \int_{-1}^{0}(-t)\|\tilde{\omega}(t)\|_{L^\infty(B(x_0,R))}dt,$$

Global BKM-type criterion for the Boussinesq system

- In the simpler case of the whole domain in $\mathbb{R}^2$, we have the following global criterion.

Theorem (with J. Wolf [JNLS '19])

Let $q > 2$, and $(u, \theta) \in C([-1, 0); W^{2,q}(\mathbb{R}^2))$ be a solution to the Boussinesq system. If

$$\boxed{\int_{-1}^{0} (-t)\|\nabla\theta(t)\|_{L^\infty} dt < +\infty.}$$

Then, no blow up at $t = 0$.

Local BKM type criterion for the Boussinesq system

- In order to get the blow-up criterion in the axisymmetric Euler equations off the axis region we need a localized BKM type criterion of the Boussinesq system as follows.

Lemma (Local BKM type criterion for Boussinesq system)

Let $q > 2$, and $(u, \theta) \in C([-1, 0); W^{2,q}(B(1))), \quad u \in C_w([-1, 0]; L^2(B(1)))$ be a local classical solution to the 2D Boussinesq system.

$$\text{If} \quad \boxed{\int_{-1}^{0} (-t) \|\nabla \theta(t)\|_{L^\infty(B(1))} dt} + \int_{-1}^{0} \|u(t)\|_{L^\infty(B(1))} dt < +\infty,$$

then, no blow-up at $t = 0$ on $B(0, r)$ for all $r < 1$.

- The proof uses the above idea, but is more technical.

4. Remarks on the discretely self-similar blow up

- We say $v(x,t)$ is an $(\lambda,\alpha)-$DSS function if $v(x,t)=\lambda^{\alpha}v(\lambda x,\lambda^{\alpha+1}t)$ for all $(x,t)\in\mathbb{R}^3\times(-\infty,0)$.
- For $(\lambda,\alpha)-$DSS function $v(x,t)$ its profile $V(y,s)$ is defined by
$$v(x,t)=(-t)^{-\frac{\alpha}{1+\alpha}}V\Big((-t)^{-\frac{1}{1+\alpha}}x,-\log(-t)\Big).$$
- Previously the scenario of $(\lambda,\alpha)-$DSS singularity is removed for the Euler equations[DC, Math Ann.'15] for $\alpha>-1$ under the assumption,
$$\sup_{s\in\mathbb{R}}|\Omega(y,s)|=o(|y|^{-\alpha-1}),\quad \Omega=\nabla\times V,$$

- For one point DSS singularity the above decay is “almost critical’, since for $(\lambda, \alpha)-$DSS function v at singularity at $(0,0)$, one can show that the curl of the profile satisfies

$$\sup_{s\in\mathbb{R}} |\Omega(y,s)| \leq \frac{C}{(|y|+1)^{\alpha+1}}.$$

- In the previous part of the talk we removed $(\lambda, \frac{3}{2})-$DSS singularity under the condition

$$V \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}; \dot{W}^{1,\infty}(\mathbb{R}^3)).$$

- The following is a substantial improvement of the above results in the case $\alpha \geq 3/2$.

Theorem (with J. Wolf)

Let v be an $(\lambda, \alpha)-$DSS solution of the Euler equations, and let $\alpha \geq \frac{3}{2}$. If V satisfies the sub-linear growth condition,

$$\sup_{\tau \in \mathbb{R}} |V(y,\tau)| = o(|y|) \quad \text{as} \quad |y| \to +\infty.$$

Then $V(y,\tau) = c(\tau)$.

- For the proof of the above theorem we establish new a priori estimate for the solution of the Euler equations as follows.

Lemma

Let $0 < r < +\infty$. Let $v \in C^1(\mathbb{R}^3 \times [t_0, t_1))$ be a solution to the Euler equation. Then, there exists an harmonic function $h_r(\cdot, t)$ on $B(r)$ and a constant c such that for all $t_0 < t < t_1$

$$r^{-5} \|v(t) - \nabla h_r(t)\|^2_{L^2(B(r))}$$

$$\leq c \|v(t_0)(r + |x|)^{-1}\|^2_{L^\infty(\mathbb{R}^3)} \exp\left(c \int_{t_0}^{t} \|v(s)(r + |x|)^{-1}\|_{L^\infty(\mathbb{R}^3)} ds \right).$$

- h_r is the solution of the inhomogeneous Stokes system,

$$\begin{cases} -\Delta u + \nabla h_r = v, \quad \nabla \cdot u = 0 \quad \text{in} \quad B(r) \\ \quad u = 0 \quad \text{on} \quad \partial B(r) \end{cases}$$

- In the case $\alpha > \frac{3}{2}$ the above theorem is an easy consequence of the lemma, combined with the scaling argument.

Thank you!