On an Initial-Boundary Value Problem for the Landau Equation

Jin Woo Jang (IBS - Center for Geometry and Physics)

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Institute of Mathematics, University of Bordeaux

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The Boltzmann equation (1872)

- Electrically neutral, single-species, and rarefied gases.
- The Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \;\; x \in \Omega, \;\; v \in \mathbb{R}^3, \;\; t \geq 0$$

- $F = F(t, x, v)$: probability density in (position,velocity)
- $\Omega \subset \mathbb{R}^3$: domain in space
- $v \cdot \nabla_x F$: free transport term
- $Q(F, F)$: collision operator, local in (t,x), quadratic integral operator

$$Q(F, G)(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(v - v_*, \sigma) \times [F(v_*')G(v') - F(v_*)G(v)].$$

A binary collision

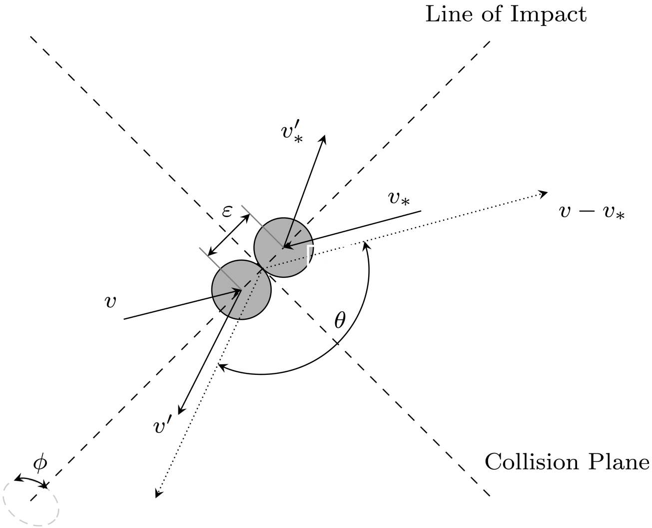

Figure: A binary collision of particles (special thanks to Hyunwoo Kwon, Sogang Univ.)

Derivation of the Boltzmann equation

- The Liouville equation (microscopic dynamics, time-reversible)
- Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy
- Boltzmann-Grad limit
- The Boltzmann equation (time-irreversible)
- Poincaré Recurrence Theorem?

Boltzmann's H-functional and H-Theorem

- The Entropy functional is defined as

$$S(f) = -H(f) \stackrel{\text{def}}{=} -\int_{\Omega\times\mathbb{R}^3} F(x,v)\log F(x,v)dvdx.$$

- Boltzmann identifies S with the entropy of the gas and proves that S can only increase in time. More precisely, we have

$$\frac{d}{dt}S(t) \geq 0.$$

- Entropy production and the time-irreversibility.
- Any rigorous proof?

Global Maxwellian steady-state

- These equlibrium states are characterized as a particle distribution which maximizes the entropy subject to constant mass density ρ, mean velocity u, and the temperature T, given by

$$\mu(v) = \rho \frac{e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{3/2}},$$

- $\rho = T = 1$ and $u = 0$ gives

$$\mu(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}.$$

The aftermath of a binary collision

A complete description of the aftermath of collisions requires more precise physics.

- Hard spheres: Here the particles are assumed to be literally billiard balls of vanishingly small diameter in the continuum limit.
- Inverse power-law interactions $\phi(r) = \frac{1}{r^{p-1}}$: Here the particles are assumed to interact pairwise through inverse power law potentials, $\phi(r) = \frac{1}{r^{p-1}}$ for some $p > 2$. In this case, the preferred consequence of a collision is a glancing or grazing collision.
- The Coulomb potential $\phi(r) = \frac{1}{r}$: this requires some additional limiting arguments and leads to what is called the Landau equation. Good physical model for the dynamics of plasma.

Special cases of collisional kernels

- For the inverse power law $\phi(r) = \frac{1}{r^{p-1}}$ (Maxwell 1867):

$$B(|v - v_*|, \sigma) = C|v - v_*|^{\gamma}\theta^{-\gamma'}b_0(\theta),$$

  where $s > 1$, $\gamma = \frac{p-5}{p-1}$, $\gamma' = \frac{2p}{p-1}$, b_0 is bounded, and

$$\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma,$$

  with $0 < \theta \leq \frac{\pi}{2}$.

- For the hard spheres in the limit $p \to \infty$:

$$B(|v - v_*|, \sigma) = C|v - v_*|\frac{b_0(\theta)}{\theta^2},$$

  where b_0 is bounded.

Mathematical difficulties: lack of a priori estimates

- Even if we do not take into account the integrability of $B(|v - v_*|, \sigma)$, the only simple bound one can expect on Q is

$$|Q(F,F)|_{L^1(\mathbb{R}^n_v)} \leq C|F|^2_{L^1(\mathbb{R}^n_v)}.$$

- Therefore, it is reasonable to ask for an esimate of the following form

$$F \in L^2_{loc}((0,\infty) \times \mathbb{R}^n_x; L^1(\mathbb{R}^n_v)),$$

and such an estimate does not seem to be available in general.

- This lack of estimates has been the major obstruction to a complete understanding of the *Boltzmann equation*.

Plasma Physics and the kinetic theory

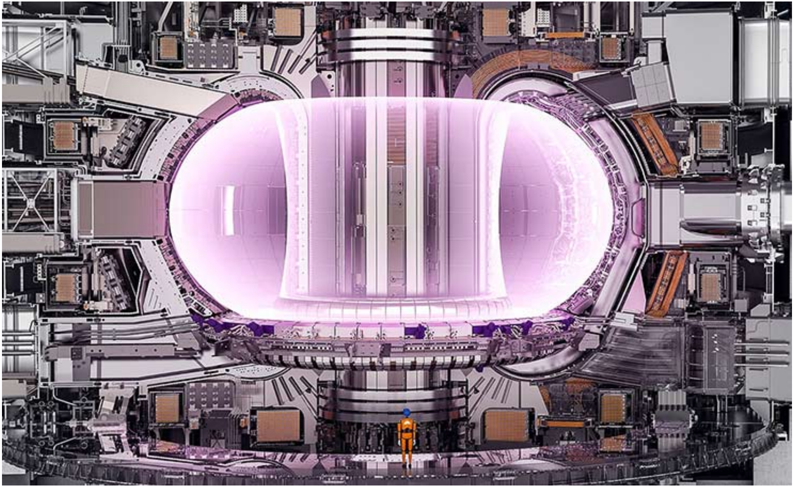

Figure: ITER Tokamak, picture from American Institute of Physics.

A Remedy: screening and the Debye potential

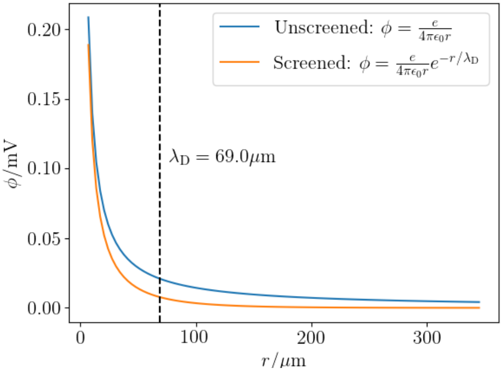

Figure: Coulomb potential vs. Debye potential

A Remedy: screening and the Debye potential

- To ease the strong angular singularity from the slow decay of Coulomb potential $1/r$, we instead consider so-called Debye (or Yukawa) potential:

$$\phi(r) = \frac{e^{-r/\lambda_D}}{4\pi\epsilon_0 r},$$

where λ_D is the *Debye length* (i.e., screening distance) and ϵ_0 is the permittivity of vacuum.

- In the classical theory of plasmas,

$$\lambda_D = \sqrt{\frac{kT}{\rho}},$$

where k is Boltzmann's constant, T is the temperature, ρ is its mean density.

Formal limit

- In most of the cases of interest,

$$r_0 \ll \rho^{-1/3} \ll \lambda_D,$$

where

$$r_0 = \frac{1}{4\pi kT}$$

is the *Landau length* for collisions.

- By a formal procedure, Landau (in 1936) showed that, as the ratio

$$\Lambda = 2\frac{\lambda_D}{r_0} \to \infty,$$

the Boltzmann collision operator for Debye potential behaves as

$$\frac{\log \Lambda}{2\pi\Lambda} Q_L(f, f),$$

where the Q_L is the so-called *Landau collision operator*.

The Landau equation (1936)

The equation takes the form of

$$\partial_t F + v \cdot \nabla_x F = Q_L(F, F),$$

where $F = F(t, x, v)$ for $x \in \Omega \subset \mathbb{R}^3$, $v \in \mathbb{R}^3$ is the probability density function, and the *Landau collision operator* Q_L is defined as

$$Q_L(F, G)(v) \stackrel{\text{def}}{=} \nabla_v \cdot \int_{\mathbb{R}^3} \phi(v - v')\big[F(v')\nabla_v G(v) - G(v)\nabla_v F(v')\big]\, dv',$$

and the collision kernel

$$\phi(z) \stackrel{\text{def}}{=} \left\{ I - \frac{z}{|z|} \otimes \frac{z}{|z|} \right\} \cdot |z|^{-1}$$

is a symmetric and non-negative matrix such that $\phi_{ij}(z) z_i z_j = 0$.

The global equilibrium and the H-theorem

- The global equilibrium $\mu(v) = \frac{1}{4\pi} e^{-\frac{|v|^2}{2}}$.
- H-functional $H(f) = \int F \log F dx dv$.
- Entropy production

$$\frac{d}{dt} S(f) = -\frac{d}{dt} H(f) = -\int Q(F, F) \log F \geq 0.$$

First wellposedness for the Boltzmann and the Landau equations in the perturbation framework

- The global-in-time existence theory in the perturbation framework for the Landau equation has been well established for suitably small initial data in smooth Sobolev spaces.
- Landau equation in a periodic box with $\gamma \geq -3$: Guo (CMP, 2002), $f(t,x,v) \in L^\infty(0,\infty; H^8_{x,v})$.

Any L^∞ solutions?

- Collision operators for the Landau equation: the velocity diffusion property
- It is very difficult to apply the characteristic approach to obtain the $L^\infty_{x,v}$ bounds for the solutions.

$L^\infty_{t,x,v}$ solutions to the Landau equation in a periodic box

- Kim-Guo-Hwang (Peking Math. Journal, 2019) developed an L^2 to L^∞ approach to the Landau equation in the torus domain
- The initial data are required to be small in $L^\infty_{x,v}$ but additionally belong to $H^1_{x,v}$.
- Doesn't use the Sobolev embedding
- A key point of the proof is to control the L^∞ bound by the L^2 estimates via De Giorgi-Nash-Moser's method, and also to control the velocity derivatives to ensure uniqueness by the Hölder estimates again via De Giorgi's method.

Initial-boudary value problem for kinetic equations

- Boltzmann equation with Grad's angular cutoff: Shizuta and Asano (Proc. Japan Acad., 1977), Guo (ARMA, 2010), Kim (CMP, 2011), Guo, Kim, Tonon, Trescases (Invent. Math. 2016, ARMA 2016), Kim and Lee (CPAM 2018, ARMA 2019), etc.
- Vlasov(-Poisson)-Fokker-Planck equation: Bonilla-Carrillo-Soler (JFA, 1993), Carrillo (Math. Methods Appl. Sci., 1998), Hwang-Jang-Velazquez (ARMA 2014, ARMA 2019), Cesbron (CMP, 2018), etc.
- Boltzmann equation without angular cutoff: Duan-Liu-Sakamoto-Strain (arXiv, 2019) in a *finite channel* $\Omega = (-1, 1) \times \mathbb{T}^2$.
- Landau equation: Duan-Liu-Sakamoto-Strain (arXiv, 2019) in a *finite channel* $\Omega = (-1, 1) \times \mathbb{T}^2$.

Nash-Moser iteration

Lemma (Imbert-Mouhot (2016))

Let h be a nonnegative periodic function in x satisfying $(\partial_t + v \cdot \nabla_x - A)h \leq 0$. *Then h satisfies*

$$\int_{Q_1} |\nabla_v h|^2 \leq C \int_{Q_0} h^2$$

$$\|h\|^2_{L^2_t L^2_x L^q_v(Q_1)} \leq C \int_{Q_0} h^2$$

$$\|h\|^2_{L^\infty_t L^2_x L^2_v(Q_1)} \leq C \int_{Q_0} h^2.$$

for some $q > 2$ *and* $C = \bar{C}(R_0)\left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2}\right)$, $\bar{C}(R_0) = C'(1 + R_0)^3$.

Nash-Moser iteration

Lemma (Imbert-Mouhot (2016))

$$\|D_x^{1/3}h\|^2_{L^2(Q_1)} \leq C\|h\|^2_{L^2(Q_0)}$$
$$\|D_t^{1/3}h\|^2_{L^2(Q_1)} \leq C\|h\|^2_{L^2(Q_0)}$$

for some $C = \bar{C}(R_0)\left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2}\right)$.

Nash-Moser iteration

Lemma (Imbert-Mouhot (2016))

There exists $p > 2$ such that

$$\|h\|^2_{L^2_t L^p_x L^2_v(Q_1)} \leq C\|h\|^2_{L^2(Q_0)}.$$

Lemma (Imbert-Mouhot (2016))

We have

$$\|h\|_{H^s_{x,v,t}(Q_1)} \leq C\|h\|_{L^2(Q_0)}$$

where $s = 1/3$.

Lemma (Imbert-Mouhot (2016))

There exists $q > 2$ such that

$$\|h\|^2_{L^q(Q_1)} \leq C\|h\|^2_{L^2(Q_0)}$$

De Giorgi lemma

Lemma (Golse et al. (2016))

Let $\hat{Q} := Q_{1/4}(0,0,-1)$. *For any (universal) constants* $\delta_1 \in (0,1)$ *and* $\delta_2 \in (0,1)$ *there exist* $\nu > 0$ *and* $\vartheta \in (0,1)$ *(both universal) such that for any solution* f *in* Q_2 *with* $|f| \leq 1$ *and*

$$|\{f \geq 1 - \vartheta\} \cap Q_{1/4}| \geq \delta_1 |Q_{1/4}|,$$
$$|\{f \leq 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}|,$$

we have

$$|\{0 < f < 1 - \vartheta\} \cap B_1 \times B_1 \times (-2,0]| \geq \nu.$$

De Giorgi Lemma

Lemma (Golse et al. (2016))

Let $\hat{Q} := Q_{1/4}(0,0,-1)$ *and* f *be a weak solution in* Q_2 *with* $|f| \leq 1$. *If*

$$|\{f \leq 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}|,$$

then

$$\sup_{Q_{1/8}} f \leq 1 - \lambda$$

for some $\lambda \in (0,1)$, *depending only on dimension and the eigenvalue of* σ.

S^p estimates

Theorem (Bramanti et al. (1994))

Let Ω be a bounded open set in $\mathbb{R}^7$ and let f be a strong solution in Ω to the equation

$$\sum_{i,j=1}^{3} \sigma^{ij}(t,x,v)\partial_{v_i,v_j} f + Yf = h,$$

where $Y = -\partial_t - v \cdot \nabla_x$. Suppose that σ is uniformly elliptic, $\|\sigma^{ij}\|_{C^\alpha(\Omega)} \leq C$, and $f, h \in L^p$.
Then $\partial_{v_i,v_j} f \in L^p_{loc}$, $Yf \in L^p_{loc}$ and for every open set $\Omega' \subset\subset \Omega$ there exists a positive constant c_1 depending only on $p, \Omega', \Omega, \alpha, C$ and elliptic constant of σ such that

$$\|\partial_{v_i,v_j} f\|_{L^p(\Omega')} \leq c_1(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}),$$
$$\|Yf\|_{L^p(\Omega')} \leq c_1(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}).$$

Specular reflection boundary conditions

- Denote the phase boundary of $\Omega \times \mathbb{R}^3$ as $\gamma \stackrel{\text{def}}{=} \partial\Omega \times \mathbb{R}^3$.
- Then we now split this boundary into an outgoing boundary γ_+, an incoming boundary γ_-, and a singular boundary γ_0 for grazing velocities, defined as

$$\gamma_+ \stackrel{\text{def}}{=} \{(x,v) \in \Omega \times \mathbb{R}^3 : n_x \cdot v > 0\},$$
$$\gamma_- \stackrel{\text{def}}{=} \{(x,v) \in \Omega \times \mathbb{R}^3 : n_x \cdot v < 0\},$$
$$\gamma_0 \stackrel{\text{def}}{=} \{(x,v) \in \Omega \times \mathbb{R}^3 : n_x \cdot v = 0\}.$$

- In terms of the probability density function F, we formulate the *specular reflection boundary condition* as

$$F(t,x,v)|_{\gamma_-} = F(t,x,v - 2n_x(n_x \cdot v)) = F(t,x,R_x v),$$

for all $x \in \partial\Omega$ where

$$R_x v \stackrel{\text{def}}{=} v - 2n_x(n_x \cdot v).$$

Main theorem

Theorem (Guo-Hwang-J.-Ouyang, 2019)

There exist ϑ' *and* $0 < \varepsilon_0 \ll 1$ *such that for some* $\vartheta \geq \vartheta'$ *if* f_0 *satisfies* $\|f_0\|_{\infty,\vartheta} \leq \varepsilon_0, \quad \|f_{0t}\|_{\infty,\vartheta} + \|D_v f_0\|_{\infty,\vartheta} < \infty$, *where* $f_{0t} \overset{def}{=} -v \cdot \nabla_x f_0 + \bar{A}_{f_0} f_0$.

- *(Existence and Uniqueness) Then there exists a unique weak solution* f *to the initial-boundary value problem on* $(0,\infty) \times \Omega \times \mathbb{R}^3$.
- *(Decay of solutions in* L^2 *and* L^∞*) Moreover, for any* $t > 0$, $\vartheta_0 \in \mathbb{N}$, *and* $\vartheta \geq \vartheta'$, *there exist* $C_{\vartheta,\vartheta_0} > 0$ *and* $l_0(\vartheta_0) > 0$ *such that* f *satisfies*

$$\|f(t)\|_{2,\vartheta} \leq C_{\vartheta,\vartheta_0} \mathcal{E}_{\vartheta+\vartheta_0/2}(0)^{1/2} \left(1 + \frac{t}{\vartheta_0}\right)^{-\vartheta_0/2} \quad \text{and}$$

$$\|f(t)\|_{\infty,\vartheta} \leq C_{\vartheta,\vartheta_0}(1+t)^{-\vartheta_0} \|f_0\|_{\infty,\vartheta+l_0}.$$

Main theorem

Theorem (Guo-Hwang-J.-Ouyang, arXiv 2019)

- *(Boundedness in $C^{0,\alpha}$ and $W^{1,\infty}$) In addition, there exist $C > 0$ and $0 < \alpha < 1$ such that f satisfies*

$$\|f\|_{C^{0,\alpha}\left((0,\infty)\times\Omega\times\mathbb{R}^3\right)} \leq C\left(\|f_{0t}\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta}\right),$$

and

$$\|D_v f\|_{L^\infty\left((0,\infty)\times\Omega\times\mathbb{R}^3\right)} \leq C\left(\|f_{0t}\|_{\infty,\vartheta} + \|D_v f_0\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta}\right).$$

- *(Positivity) Let $F(t,x,v) = \mu(v) + \sqrt{\mu}(v) f(t,x,v)$. If $F(0) \geq 0$, then $F(t) \geq 0$ for every $t \geq 0$.*

Plan of the whole proof

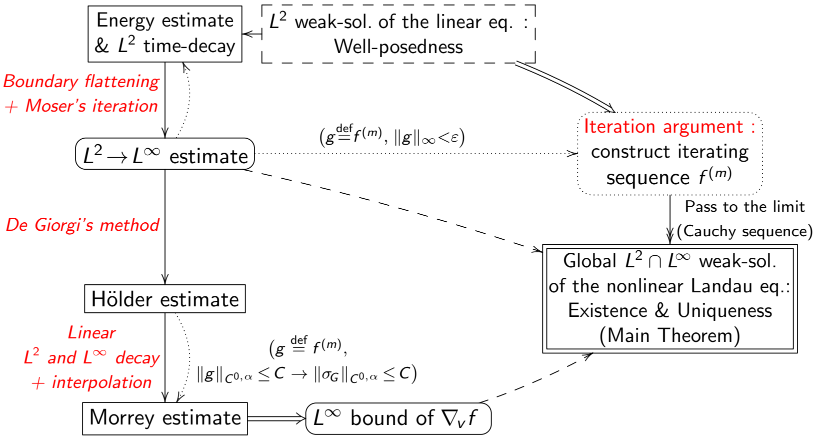

Figure: $L^2 \to L^\infty$ approach.

Main strategy: Step 1. Linear problem

- Write the linearized equation as

$$\partial_t f + v \cdot \nabla_x f = \bar{A}_g f + \bar{K}_g f,$$

  for some given function g, where $\bar{A}_g f$ consists of the terms which contain at least one momentum-derivative of f while $\bar{K}_g f$ consists of the rest.
- Then, we first consider constructing weak solutions for the linearized equation

$$\partial_t f + v \cdot \nabla_x f = \bar{A}_g f,$$

  where the wellposedness is then obtained via regularizing the problem, constructing approximate solutions, and showing L^1 and L^∞ estimates for the adjoint problem to the approximate problem.

Step 1. Linear problem

- Once we are equipped the wellposedness of the reduced equation, we may associate a continuous semigroup of linear and bounded operators $U(t)$ such that $f(t) = U(t)f_0$ is the unique weak solution of the reduced equation.
- Then by the Duhamel principle, the solution $\bar{f}$ of the whole linearized equation can further be written as

$$\bar{f}(t) = U(t)\bar{f}_0 + \int_0^t U(t-s)\bar{K}_g \bar{f}(s)ds.$$

Step 2. L^2 decay estimates: Hypocoercivity

- Key point: L has a null space and is coercive:

$$\langle Lf, f\rangle_v \gtrsim |(I-P)f|_\sigma^2.$$

- Macro-micro decomposition and the dissipation.
- The following polynomial decay follows from the energy inequality:

$$\|f(t)\|_{2,\vartheta} \leq C_{\vartheta,\vartheta_0}\mathcal{E}_{\vartheta+\vartheta_0/2}(0)^{1/2}\left(1+\frac{t}{\vartheta_0}\right)^{-\vartheta_0/2}.$$

Step 3. *boundary-flattening* transformation

- Let

$$\begin{array}{rcl} \vec{\Phi}: \ \overline{\Omega} \times \mathbb{R}^3 & \to & \overline{\mathbb{H}}_- \times \mathbb{R}^3 \\ (x, v) & \mapsto & (y, w) \stackrel{\text{def}}{=} (\vec{\phi}(x), Av) \end{array}$$

be the (local) transformation that flattens the boundary, where

$$A \stackrel{\text{def}}{=} \left[\frac{\partial y}{\partial x}\right] = D\vec{\phi}$$

is a non-degenerate 3×3 Jacobian matrix, and the explicit definition of $y = \vec{\phi}(x)$ will be given below. Let

$$\tilde{f}(t, y, w) \stackrel{\text{def}}{=} f\left(t, \vec{\Phi}^{-1}(y, w)\right) = f\left(t, \vec{\phi}^{-1}(y), A^{-1}w\right) = f(t, x, v).$$

denote the solution under the new coordinates.

Step 3. *boundary-flattening* transformation

- Suppose the boundary $\partial\Omega$ is (locally) given by the graph $x_3 = \rho(x_1, x_2)$, and $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < \rho(x_1, x_2)\right\} \subseteq \Omega$.
- Define $y = \vec{\phi}(x)$ explicitly as follows:

$$\begin{aligned}
\vec{\phi}^{-1} : \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &\mapsto \vec{\eta}(y_1, y_2) + y_3 \cdot \vec{n}(y_1, y_2) \\
&\stackrel{\text{def}}{=} \begin{pmatrix} y_1 \\ y_2 \\ \rho(y_1, y_2) \end{pmatrix} + y_3 \cdot \begin{pmatrix} -\rho_1 \\ -\rho_2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} y_1 - y_3 \cdot \rho_1 \\ y_2 - y_3 \cdot \rho_2 \\ \rho + y_3 \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned}$$

Step 3. *boundary-flattening* transformation

$$\vec{\eta}(y_1, y_2) \stackrel{\text{def}}{=} \big(y_1, y_2, \rho(y_1, y_2)\big) \in \partial\Omega$$
$$\partial_1 \vec{\eta} \stackrel{\text{def}}{=} \frac{\partial \vec{\eta}}{\partial y_1} = \langle 1, 0, \rho_1 \rangle$$
$$\partial_2 \vec{\eta} \stackrel{\text{def}}{=} \frac{\partial \vec{\eta}}{\partial y_2} = \langle 0, 1, \rho_2 \rangle,$$

then the (outward) normal vector at the point $\vec{\eta}(y_1, y_2) \in \partial\Omega$ is chosen to be

$$\vec{n}(y_1, y_2) \stackrel{\text{def}}{=} \partial_1 \vec{\eta} \times \partial_2 \vec{\eta} = \langle -\rho_1, -\rho_2, 1 \rangle.$$

We also remark that the map is locally well-defined and is a smooth homeomorphism in a tubular neighborhood of the boundary.

Step 3. *boundary-flattening* transformation

Directly we compute the Jacobian matrix

$$A^{-1} = D\vec{\phi}^{-1} = \left[\frac{\partial x}{\partial y}\right] = \left[\partial_1\vec{\eta} + y_3\cdot\partial_1\vec{n}; \partial_2\vec{\eta} + y_3\cdot\partial_2\vec{n}; \vec{n}\right]$$

$$= \begin{pmatrix} 1-y_3\cdot\rho_{11} & -y_3\cdot\rho_{12} & -\rho_1 \\ -y_3\cdot\rho_{12} & 1-y_3\cdot\rho_{22} & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{on } \partial\Omega:\ y_3=0} \left[\partial_1\vec{\eta}; \partial_2\vec{\eta}; \vec{n}\right]$$

$$= \begin{pmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}.$$

Step 3. *boundary-flattening* transformation

So we can write out $\vec{\Phi}^{-1}$ as

$$\vec{\Phi}^{-1}: \ (y,w) \ \mapsto \ (x,v) \stackrel{\text{def}}{=} (\vec{\phi}^{-1}(y), A^{-1}w)$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \mapsto \begin{pmatrix} 1-y_3\cdot\rho_{11} & -y_3\cdot\rho_{12} & -\rho_1 \\ -y_3\cdot\rho_{12} & 1-y_3\cdot\rho_{22} & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$= \begin{pmatrix} (1-y_3\rho_{11})\cdot w_1 - y_3\rho_{12}\cdot w_2 - \rho_1\cdot w_3 \\ -y_3\rho_{12}\cdot w_1 + (1-y_3\rho_{22})\cdot w_2 - \rho_2\cdot w_3 \\ \rho_1\cdot w_1 + \rho_2\cdot w_2 + w_3 \end{pmatrix} =: \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Step 3. *boundary-flattening* transformation

Restricted on the boundary $\partial\Omega$ i.e.,$\{y_3 = 0\}$, the map becomes

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = w_1 \cdot \partial_1 \vec{\eta} + w_2 \cdot \partial_2 \vec{\eta} + w_3 \cdot \vec{n}$$

$$= \begin{pmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 - \rho_1 \cdot w_3 \\ w_2 - \rho_2 \cdot w_3 \\ \rho_1 \cdot w_1 + \rho_2 \cdot w_2 + w_3 \end{pmatrix}$$

Step 3. *boundary-flattening* transformation

- Key feature: $\vec{\Phi}$ preserves the "specular symmetry" on the boundary: it sends any two points $(x, v), (x, R_x v)$ on the phase boundary $\gamma = \partial\Omega \times \mathbb{R}^3$ with specular-reflection relation to two points on $\{y_3 = 0\} \times \mathbb{R}^3$ which are also specular-symmetric to each other.
- In other words, we have the following commutative diagram (when $x \in \partial\Omega$ i.e.,$y_3 = 0$):

$$\begin{array}{ccc} (y, w) & \xrightarrow{\vec{\Phi}^{-1}} & (x, v) \\ R_y \downarrow & & \downarrow R_x \\ (y, R_y w) & \xrightarrow{\vec{\Phi}^{-1}} & (x, R_x v) \end{array}$$

using the definition $R_x v = v - 2(n_x \cdot v) n_x$.

Step 4: Mirror extension

- Having this property, the specular reflection boundary condition on the solutions is also preserved:

$$\tilde{f}(t, y, w) = \tilde{f}(t, y, Rw), \quad \text{on } \{y_3 = 0\},$$

where $R \stackrel{\text{def}}{=} \mathrm{diag}\{1, 1, -1\}$, which allows us to construct the mirror extension (as in the next subsection) that is consistent with this restriction (and thus is automatically satisfied).

- After flattening the boundary, we then "flip over" $\tilde{f}$ to the upper half space by setting

$$\bar{f}(t, y', w') \stackrel{\text{def}}{=} \begin{cases} \tilde{f}(t, y', w'), & \text{if } y' \in \overline{\mathbb{H}}_- \\ \tilde{f}(t, Ry', Rw'), & \text{if } y' \in \overline{\mathbb{H}}_+ \end{cases},$$

where $R \stackrel{\text{def}}{=} \mathrm{diag}\{1, 1, -1\}$. Combined with the corresponding partition of unity, we are able to define our solutions in the whole space.

Step 4: Mirror extension

Summing up the above computations, we now obtain that $\bar{f}$ satisfies (pointwisely) the following equation(s) in the lower and upper space, respectively:

$$\partial_t \bar{f} + w' \cdot \nabla_{y'} \bar{f} = \nabla_{w'} \cdot \left(\mathbb{A} \nabla_{w'} \bar{f} \right) + \mathbb{B} \cdot \nabla_{w'} \bar{f} + \mathbb{C} \bar{f},$$

where the coefficients $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are piecewise-defined and $\mathbb{A}$ is further Hölder continuous across the boundary.

Plan of the whole proof

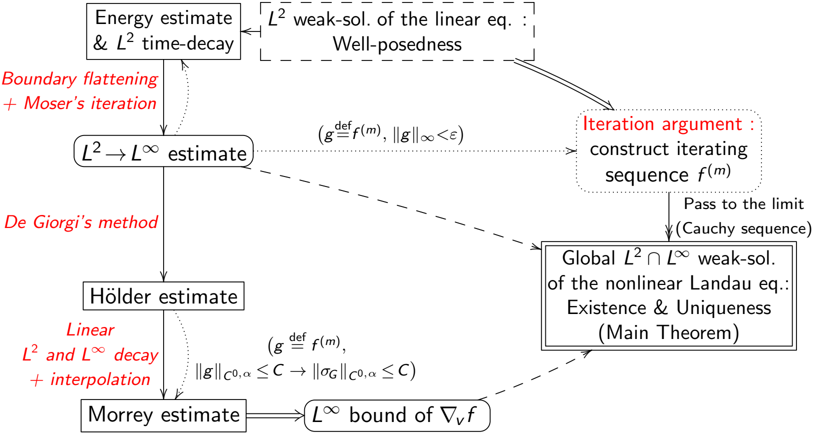

Figure: $L^2 \to L^\infty$ approach.

Thank you for your attention.