Asymptotic analysis for Vlasov-Fokker-Planck/compressible Navier-Stokes equations with a density-dependent viscosity

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(Joint work with Prof. Y.-P. Choi)

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Outline

Outline

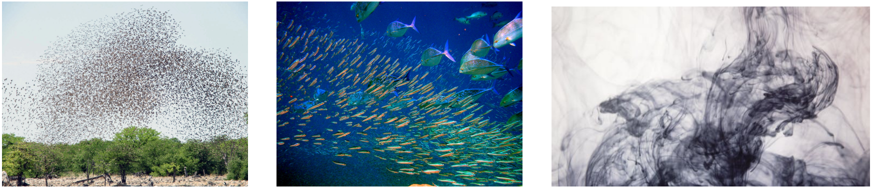

Figure 1: Particles/Agents in fluid

: The dynamics of particles/agents is affected by the fluid that contains them.

P. by Alastair Rae, Bruno de Giusti and zavarykin from the left

Kinetic equations with fluid equations

- Examples of applications
  - analysis for **sedimentation phenomenon** with applications in medicine, chemical engineering or waste water treatment
  - modeling of **aerosols and sprays** with applications (ex. the study of Diesel engines)

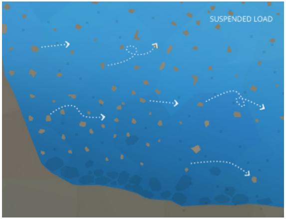

Figure 2: Examples of applications

Our system of interest

$$
\begin{aligned}
&\partial_t f + \xi \cdot \nabla_x f + \nabla_\xi \cdot ((v-\xi) f) = \nabla_\xi \cdot (\nabla_\xi f - (u-\xi) f), \\
&\partial_t n + \nabla_x \cdot (nv) = 0, \\
&\partial_t (nv) + \nabla_x \cdot (nv \otimes v) + \nabla_x p - 2\nabla_x \cdot (\nu(n)\mathbb{D}v) \\
&\qquad = -\int_{\mathbb{R}^d} (v-\xi) f \, d\xi.
\end{aligned}
\tag{VFPNS}
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&\qquad = -\int_{\mathbb{R}^d} (v-\xi) f \, d\xi.
\end{aligned} \qquad \text{(VFPNS)}$$

- $f = f(x,\xi,t)$: the number density function on $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ at $t \in \mathbb{R}_+$,
- $n = n(x,t)$, $v = v(x,t)$: the local mass density and the bulk velocity.
- (Boundary conditions) $f(x,\xi,t) \to 0$, $n(x,t) \to n_\infty \in \mathbb{R}_+$, $v(x,t) \to 0$, sufficiently fast as $|x|$, $|\xi| \to \infty$.
- $\rho(x,t) := \int_{\mathbb{R}^d} f(x,\xi,t)\, d\xi$ and $(\rho u)(x,t) := \int_{\mathbb{R}^d} \xi f(x,\xi,t)\, d\xi$.
- $p = p(n) = n^\gamma$ $(\gamma \geq 1)$ and $\mathbb{D}v := (\nabla v + \nabla v^T)/2$.

Purpose of the work

: the asymptotic regime corresponding to a strong local alignment and a strong Brownian motion.

$$
\begin{aligned}
&\partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon + \nabla_\xi \cdot ((v^\varepsilon - \xi) f^\varepsilon) = \frac{1}{\varepsilon} \nabla_\xi \cdot (\nabla_\xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon), \\
&\partial_t n^\varepsilon + \nabla_x \cdot (n^\varepsilon v^\varepsilon) = 0, \\
&\partial_t (n^\varepsilon v^\varepsilon) + \nabla_x \cdot (n^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \nabla_x p(n^\varepsilon) - 2\nabla_x \cdot (\nu(n^\varepsilon) \mathbb{D} v^\varepsilon) \\
&\qquad = -\rho^\varepsilon (v^\varepsilon - u^\varepsilon).
\end{aligned}
\tag{VFPNS-ε}
$$

Here, we assume the far-field behavior $n^\varepsilon \to n_\infty$ as $|x| \to \infty$ for all $\varepsilon \geq 0$.

Main purpose is to investigate the **convergence** of **weak solutions** $(f^\varepsilon, n^\varepsilon, v^\varepsilon)$ of the system (VFPNS-ε) to the **strong solutions** (ρ, u, n, v) to the following system of fluid equations:

$$\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0,\\
&\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \rho = \rho(v - u),\\
&\partial_t n + \nabla_x \cdot (nv) = 0,\\
&\partial_t(nv) + \nabla_x \cdot (nv \otimes v) + \nabla_x p(n) - 2\nabla_x \cdot (\nu(n)\mathbb{D}v)\\
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\end{aligned} \tag{ENS}$$

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Main strategy: the **relative entropy argument** with *entropy inequalities*.

Outline

Weak solutions to VFP/compressible NS

Definition

For $T \in (0, \infty)$, we say a triplet (f, n, v) is a weak solution to the system (VFPNS) if the following conditions are satisfied:

1. $f \in L^\infty(0, T; (L^1_+ \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d))$,
$(|x|^2 + |\xi|^2) f \in L^\infty(0, T; L^1(\mathbb{R}^d \times \mathbb{R}^d))$.

2. $n - n_\infty \in L^\infty(0, T; (L^1_+ \cap L^\gamma)(\mathbb{R}^d)), \quad n|v|^2 \in L^\infty(0, T; L^1(\mathbb{R}^d))$,
$\sqrt{\nu(n)} \nabla_x v \in L^2(0, T; L^2(\mathbb{R}^d))$.

3. (f, n, v) satisfies (VFPNS) in a distributional sense.

Strong solutions to compressible Euler-compressible NS

Definition

Let $s > d/2 + 2$. For $T \in (0, \infty)$, (ρ, u, n, v) :is a strong solution to (ENS) on $[0, T]$ if

1. It satisfies the system (ENS) in the sense of distributions.
2. It satisfies the following regularity conditions:

$$\rho, n \in \mathcal{C}([0, T]; H^s(\mathbb{R}^d)), \quad u, v \in \mathcal{C}([0, T]; [H^s(\mathbb{R}^d)]^d).$$

Main result

Assumptions

- $d > 2$, $\gamma \in [1, 2]$, $(f^\varepsilon, n^\varepsilon, v^\varepsilon)$ are weak solutions to (VFPNS-ε) up to $T > 0$ corresponding to $(f_0^\varepsilon, n_0^\varepsilon, v_0^\varepsilon)$ satisfying
  - $f_0^\varepsilon \in (L_+^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d)$, $(|x|^2 + |\xi|^2) f_0^\varepsilon \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$.
  - $n_0^\varepsilon - n_\infty \in (L_+^1 \cap L^\gamma)(\mathbb{R}^d)$, $n_0^\varepsilon |v_0^\varepsilon|^2 \in L^1(\mathbb{R}^d)$, $\sqrt{\nu(n_0^\varepsilon)} \nabla_x v_0^\varepsilon \in L^2(\mathbb{R}^d)$.

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- $s > d/2 + 2$ and (ρ, u, n, v) be a strong solution to (ENS) up to $T > 0$ corresponding to (ρ_0, u_0, n_0, v_0) satisfying
  - $\rho_0 > 0$ in $\mathbb{R}^d$, $\inf_{x \in \mathbb{R}^d} n_0(x) > 0$.
  - $\rho_0, n_0 \in H^s(\mathbb{R}^d)$, $u_0, v_0 \in [H^s(\mathbb{R}^d)]^d$.

- The viscosity coefficient $\nu \in \mathcal{C}^1(\mathbb{R}_+)$ is Lipschitz continuous satisfying

$$|\nu(x)-\nu(y)| \leq \nu_{\mathrm{Lip}}|x-y|, \quad \nu(x) \geq \nu_* > 0, \quad \text{and} \quad x^2 \leq c_0\nu(x)p(x),$$

for all $x, y \in \mathbb{R}_+$, where ν_{Lip}, ν_*, and c_0 are positive constants.

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for all $x, y \in \mathbb{R}_+$, where ν_{Lip}, ν_*, and c_0 are positive constants.

- The initial data $(f_0^\varepsilon, n_0^\varepsilon, v_0^\varepsilon)$ and (ρ_0, u_0, n_0, v_0) are well-prepared such that

(H1): Difference between initial entropy for weak solutions and strong solutions goes to 0 as $\varepsilon \to 0$.

(H2): Initial relative entropy goes to 0 as $\varepsilon \to 0$.

Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2\,dx + \int_{\mathbb{R}^d} n^\varepsilon |v^\varepsilon - v|^2\,dx + \int_{\mathbb{R}^d}\int_\rho^{\rho^\varepsilon} \frac{\rho^\varepsilon - z}{z}\,dzdx \\
&\quad + \int_{\mathbb{R}^d}\left(n^\varepsilon \int_n^{n^\varepsilon} \frac{p(z)}{z^2}\,dz - \frac{p(n)}{n}(n^\varepsilon - n)\right) dx \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \nu(n^\varepsilon)|\mathbb{D}(v - v^\varepsilon)|^2\,dxds + \int_0^t\int_{\mathbb{R}^d} \rho^\varepsilon |(u^\varepsilon - v^\varepsilon) - (u - v)|^2\,dxds \\
&\qquad \le C\sqrt{\varepsilon},
\end{aligned}$$

where C is independent of ε.

As a consequence, we have the following strong convergences:

$$
\begin{aligned}
&(\rho^\varepsilon, n^\varepsilon) \to (\rho, n) \text{ a.e. and} \\
&\qquad \text{in} \quad L^1_{loc}(0,T; L^1(\mathbb{R}^d)) \times L^1_{loc}(0,T; L^p_{loc}(\mathbb{R}^d)) \ \forall p \in [1,\gamma], \\
&(\rho^\varepsilon u^\varepsilon, n^\varepsilon v^\varepsilon) \to (\rho u, nv) \text{ a.e. and} \\
&\qquad \text{in} \quad L^1_{loc}(0,T; L^1(\mathbb{R}^d)) \times L^1_{loc}(0,T; L^1_{loc}(\mathbb{R}^d)), \\
&(\rho^\varepsilon |u^\varepsilon|^2, n^\varepsilon |v^\varepsilon|^2) \to (\rho |u|^2, n|v|^2) \text{ a.e. and} \\
&\qquad \text{in} \quad L^1_{loc}(0,T; L^1(\mathbb{R}^d)) \times L^1_{loc}(0,T; L^1_{loc}(\mathbb{R}^d)),
\end{aligned}
$$

as $\varepsilon \to 0$.

Outline

Uniform-in-ε estimates

$$\mathcal{F}(f^\varepsilon, n^\varepsilon, v^\varepsilon) := \int_{\mathbb{R}^{2d}} f^\varepsilon \left[\log f^\varepsilon + \frac{|\xi|^2}{2}\right] dx d\xi + \int_{\mathbb{R}^d} \frac{1}{2} n^\varepsilon |v^\varepsilon|^2 \, dx + \int_{\mathbb{R}^d} H(n^\varepsilon)\, dx,$$

$$D_1(f^\varepsilon) := \int_{\mathbb{R}^{2d}} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx d\xi,$$

$$D_2(f^\varepsilon, n^\varepsilon, v^\varepsilon) := \int_{\mathbb{R}^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dx d\xi + \int_{\mathbb{R}^d} \nu(n^\varepsilon) |\mathbb{D} v^\varepsilon|^2 \, dx,$$

where $H = H(n)$ is given by

$$H(n) := K(n) - K'(n_\infty)(n - n_\infty), \quad K(n) := n \int_{n_\infty}^n \frac{p(z)}{z^2}\, dz.$$

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D_2(f^\varepsilon, n^\varepsilon, v^\varepsilon) &:= \int_{\mathbb{R}^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dxd\xi + \int_{\mathbb{R}^d} \nu(n^\varepsilon) |\mathbb{D} v^\varepsilon|^2 \, dx,
\end{aligned}
$$

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$$
H(n) := K(n) - K'(n_\infty)(n - n_\infty), \quad K(n) := n \int_{n_\infty}^n \frac{p(z)}{z^2}\, dz.
$$

Then we can easily find the following entropy inequality:

$$
\mathcal{F}(f^\varepsilon, n^\varepsilon, v^\varepsilon) + \frac{1}{\varepsilon} \int_0^t D_1(f^\varepsilon)\, ds + \int_0^t D_2(f^\varepsilon, n^\varepsilon, v^\varepsilon)\, ds \le \mathcal{F}(f_0^\varepsilon, n_0^\varepsilon, v_0^\varepsilon) + dt
$$

for $t \geq 0$.

Then, we can get a **uniform-in-ε** upper bound estimates:

$$\int_{\mathbb{R}^{2d}} f^\varepsilon \left[1 + |\log f^\varepsilon| + \frac{1}{4}(|x|^2 + |\xi|^2)\right] dx d\xi + \frac{1}{2}\int_{\mathbb{R}^d} n^\varepsilon |v^\varepsilon|^2 \, dx$$
$$+ \int_{\mathbb{R}^d} H(n^\varepsilon)\, dx + \frac{1}{\varepsilon}\int_0^t D_1(f^\varepsilon)\, ds + \int_0^t D_2(f^\varepsilon, n^\varepsilon, v^\varepsilon)\, ds$$
$$\leq C(T) + \mathcal{O}(\sqrt{\varepsilon}).$$

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$$\leq C(T) + \mathcal{O}(\sqrt{\varepsilon}).$$

Based on this estimate, we obtain

$$\mathcal{F}(f^\varepsilon, n^\varepsilon, v^\varepsilon) + \frac{1}{2\varepsilon}\int_0^t D_1(f^\varepsilon)\, ds + \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - v^\varepsilon|^2 dxds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \nu(n^\varepsilon)|\mathbb{D}v^\varepsilon|^2 \, dxds \qquad (\star)$$
$$\leq \mathcal{F}(f_0^\varepsilon, n_0^\varepsilon, v_0^\varepsilon) + C(T)\varepsilon.$$

Relative entropy estimates

We introduce

$$U = \begin{pmatrix} \rho \\ m \\ n \\ w \end{pmatrix}, \quad A(U) := \begin{pmatrix} m & 0 & 0 & 0 \\ (m \otimes m)/\rho & \rho \mathbb{I}_d & 0 & 0 \\ w & 0 & 0 & 0 \\ (w \otimes w)/n & n^\gamma \mathbb{I}_d & 0 & 0 \end{pmatrix},$$

and

$$F(U) = \begin{pmatrix} 0 \\ \rho(v-u) \\ 0 \\ -\rho(v-u) + 2\nabla_x \cdot (\nu(n)\mathbb{D}v) \end{pmatrix},$$

where $\mathbb{I}_d$ denotes the $d \times d$ identity matrix, $m := \rho u$, and $w := nv$, and then we rewrite (ENS) in the form of **conservation laws**:

$$U_t + \nabla_x \cdot A(U) = F(U).$$

The corresponding **macroscopic entropy** $E(U)$ to (ENS):

$$E(U) := \frac{m^2}{2\rho} + \frac{w^2}{2n} + \rho \log \rho + H(n),$$

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The **relative entropy and flux** functionals $\mathcal{H}$ and $A(V|U)$:

$$\mathcal{H}(V|U) := E(V) - E(U) - DE(U)(V-U), \qquad V = \begin{pmatrix} \bar{\rho} \\ \bar{m} \\ \bar{n} \\ \bar{w} \end{pmatrix},$$

$$A(V|U) := A(V) - A(U) - DA(U)(V-U).$$

Now, let

$$U := \begin{pmatrix} \rho \\ \rho u \\ n \\ nv \end{pmatrix} \quad \text{and} \quad U^\varepsilon := \begin{pmatrix} \rho^\varepsilon \\ \rho^\varepsilon u^\varepsilon \\ n^\varepsilon \\ n^\varepsilon v^\varepsilon \end{pmatrix},$$

- $(f^\varepsilon, n^\varepsilon, v^\varepsilon)$: weak solutions to (VFPNS-ε),
- (ρ, u, n, v): a unique strong solution to (ENS).

From direct estimates, we get

$$\int_{\mathbb{R}^d} \mathcal{H}(U^\varepsilon|U)\,dx + \int_0^t\int_{\mathbb{R}^d} \nu(n^\varepsilon)|\mathbb{D}(v-v^\varepsilon)|^2\,dxds + \int_0^t\int_{\mathbb{R}^d} \rho^\varepsilon|(u^\varepsilon - v^\varepsilon)-(u-v)|^2\,dxds$$

$$= \int_{\mathbb{R}^d} \mathcal{H}(U_0^\varepsilon|U_0)\,dx$$

$$+ \int_0^t\int_{\mathbb{R}^d} \partial_s E(U^\varepsilon)\,dxds + \int_0^t\int_{\mathbb{R}^d} \nu(n^\varepsilon)|\mathbb{D}v^\varepsilon|^2\,dxds + \int_0^t\int_{\mathbb{R}^d} \rho^\varepsilon|u^\varepsilon - v^\varepsilon|^2\,dxds$$

$$- \int_0^t\int_{\mathbb{R}^d} DE(U)(\partial_s U^\varepsilon + \nabla\cdot A(U^\varepsilon) - F(U^\varepsilon))\,dxds$$

$$- \int_0^t\int_{\mathbb{R}^d} (\nabla DE(U)) : A(U^\varepsilon|U)\,dxds$$

$$+ \int_0^t\int_{\mathbb{R}^d} \left(\frac{n^\varepsilon}{n}\rho - \rho^\varepsilon\right)(v-v^\varepsilon)(u-v)\,dxds$$

$$+ 2\int_0^t\int_{\mathbb{R}^d} \left(\frac{n^\varepsilon}{n} - 1\right)(\nabla\cdot(\nu(n)\mathbb{D}v))\cdot(v-v^\varepsilon)\,dxds$$

$$+ 2\int_0^t\int_{\mathbb{R}^d} (\nabla\cdot((\nu(n)-\nu(n^\varepsilon))\,\mathbb{D}v))\cdot(v-v^\varepsilon)\,dxds$$

$$=: \sum_{k=1}^{7}\mathcal{I}_k.$$

- $\mathcal{I}_1 = \mathcal{O}(\sqrt{\varepsilon})$ by **(H2)**.
- For $\mathcal{I}_2$, we use $\int_{\mathbb{R}^d} E(U^\varepsilon)dx \le \mathcal{F}(f^\varepsilon, n^\varepsilon, v^\varepsilon)$, **(H1)**, and $(\star)$ to get

$$\mathcal{I}_2 \le C(T)\varepsilon + \mathcal{F}(f^\varepsilon, n^\varepsilon, v^\varepsilon) - \int_{\mathbb{R}^d} E(U_0)dx \le \mathcal{O}(\sqrt{\varepsilon})$$

- From the uniform-in-ε estimates,

$$\mathcal{I}_3 \le \|\nabla u\|_{L^\infty} \left(\int_{\mathbb{R}^{2d}} |\xi|^2 f^\varepsilon dxd\xi\right)^{1/2} D_1(f^\varepsilon)^{1/2} \le C(T)\sqrt{\varepsilon}.$$

- Easily, one gets

$$\mathcal{I}_4 \le C\int_0^t \int_{\mathbb{R}^d} \mathcal{H}(U^\varepsilon|U)dxds.$$

- Careful usage of Hölder inequality and G-N-S inequality gives

$$\mathcal{I}_5 \le C\int_0^t \int_{\mathbb{R}^d} \mathcal{H}(U^\varepsilon|U)dxds + \frac{1}{4}\int_0^t \int_{\mathbb{R}^d} \nu(n^\varepsilon)|\mathbb{D}(v - v^\varepsilon)|^2 dxds$$
$$+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |(u^\varepsilon - v^\varepsilon) - (u - v)|^2 dxds.$$

- Similarly to $\mathcal{I}_5$,

$$\mathcal{I}_6 \leq C \int_0^t \int_{\mathbb{R}^d} \mathcal{H}(U^\varepsilon | U) dxds + \frac{1}{8} \int_0^t \int_{\mathbb{R}^d} \nu(n^\varepsilon) |\mathbb{D}(v - v^\varepsilon)|^2 dxds.$$

- We use the condition on ν to obtain

$$\mathcal{I}_7 \leq C \int_0^t \int_{\mathbb{R}^d} \mathcal{H}(U^\varepsilon | U) dxds + \frac{1}{8} \int_0^t \int_{\mathbb{R}^d} \nu(n^\varepsilon) |\mathbb{D}(v - v^\varepsilon)|^2 dxds.$$

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Combine all the estimates for $\mathcal{I}_k$'s to get

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathcal{H}(U^\varepsilon | U)\, dx + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \nu(n^\varepsilon) |\mathbb{D}(v - v^\varepsilon)|^2\, dxds \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |(u^\varepsilon - v^\varepsilon) - (u - v)|^2\, dxds \\ &\leq C \left(\int_0^t \int_{\mathbb{R}^d} \mathcal{H}(U^\varepsilon | U)\, dxds + \sqrt{\varepsilon} \right), \end{aligned}$$

where C is a positive constant depending on ν_{Lip}, c_0, γ, $n_* := \inf_{x \in \mathbb{R}^d} n(x)$, ν_*, $\|\rho\|_{L^\infty}$, $\|u - v\|_{L^d \cap L^\infty}$, $\|n\|_{L^\infty}$, $\|\mathbb{D}v\|_{L^\infty}$, $\|\nabla \cdot (\nu(n)\mathbb{D}v)\|_{L^d \cap L^\infty}$ and $\|\nabla u\|_{L^\infty}$.

Convergence toward limits

- The convergence of ρ^ε, $\rho^\varepsilon u^\varepsilon$, and $\rho^\varepsilon |u^\varepsilon|^2$: follows from the same argument as in [Karper-Mellet-Trivisa, '15].
- For the convergence of n^ε, $n^\varepsilon v^\varepsilon$ and $n^\varepsilon |v^\varepsilon|^2$, the following inequality is useful: if $x, y > 0$ and $0 < y_{min} \le y \le y_{max} < \infty$, then

$$\begin{aligned}\tilde{P}(x|y) &= K(x) - K(y) - K'(y)(x-y)\\ &\ge \begin{cases} \gamma (2y_{max})^{\gamma-2}|x-y|^2 & \text{if } y/2 \le x \le 2y,\\ \dfrac{\gamma y_{min}^\gamma}{4(1+y_{min}^\gamma)}(1+x^\gamma) & \text{otherwise.}\end{cases}\end{aligned}$$

Convergence $n^\varepsilon \to n$

For $\Omega \subset \mathbb{R}^d$ with $|\Omega| < \infty$, we estimate

$$\begin{aligned}\int_\Omega |n^\varepsilon - n|^\gamma \, dx &= \int_{\Omega \cap \{n/2 \le n^\varepsilon \le 2n\}} |n^\varepsilon - n|^\gamma \, dx + \int_{\Omega \cap \{n/2 \le n^\varepsilon \le 2n\}^c} |n^\varepsilon - n|^\gamma \, dx \\ &=: L_1^\varepsilon + L_2^\varepsilon.\end{aligned}$$

For L_1^ε,

$$L_1^\varepsilon \le C \left(\int_{\Omega \cap \{n/2 \le n^\varepsilon \le 2n\}} \mathcal{H}(U^\varepsilon | U) \, dx \right)^{\frac{\gamma}{2}} \left(\left(2\|n\|_{L^\infty} \right)^\gamma |\Omega| \right)^{\frac{2-\gamma}{2}} \longrightarrow 0,$$

as $\varepsilon \to 0$, where $C = C(\gamma)$ is independent of ε. For L_2^ε,

$$\begin{aligned} L_2^\varepsilon &\le \int_{\Omega \cap \{n/2 \le n^\varepsilon \le 2n\}^c} \|n\|_{L^\infty}^\gamma \left| \frac{n^\varepsilon}{n} + 1 \right|^\gamma dx \\ &\le C \int_{\Omega \cap \{n/2 \le n^\varepsilon \le 2n\}^c} (1 + (n^\varepsilon)^\gamma) \, dx \\ &\le C \int_{\Omega \cap \{n/2 \le n^\varepsilon \le 2n\}^c} \mathcal{H}(U^\varepsilon | U) \, dx \longrightarrow 0, \end{aligned}$$

as $\varepsilon \to 0$, where $C = C(\|n\|_{L^\infty}, n_*, \gamma)$ is independent of ε.

Convergence $n^\varepsilon v^\varepsilon \to nv$

We estimate

$$\int_\Omega |n^\varepsilon v^\varepsilon - nv|\,dx \le \int_\Omega (n^\varepsilon |v^\varepsilon - v| + |n^\varepsilon - n||v|)\,dx =: L_3^\varepsilon + L_4^\varepsilon.$$

For L_3^ε,

$$\begin{aligned} L_3^\varepsilon &\le \left(\int_\Omega n^\varepsilon |v^\varepsilon - v|^2\,dx\right)^{1/2} \left(\int_\Omega n^\varepsilon\,dx\right)^{1/2} \\ &\le \left(\int_\Omega \mathcal{H}(U^\varepsilon | U)\,dx\right)^{1/2} \left(\int_\Omega n^\varepsilon\,dx\right)^{1/2} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

since n^ε is locally integrable in $\mathbb{R}^d$. For the estimate of L_4^ε, we obtain

$$L_4^\varepsilon \le \|v\|_{L^\infty} |\Omega|^{\frac{\gamma-1}{\gamma}} \left(\int_\Omega |n^\varepsilon - n|^\gamma\,dx\right)^{1/\gamma} \longrightarrow 0,$$

as $\varepsilon \to 0$.

Convergence $n^\varepsilon|v^\varepsilon|^2 \to n|v|^2$

Note that the following identity holds:

$$n^\varepsilon|v^\varepsilon|^2 - n|v|^2 = n^\varepsilon|v^\varepsilon - v|^2 + 2v\cdot(n^\varepsilon v^\varepsilon - nv) + |v|^2(n - n^\varepsilon).$$

This relation together with the previous convergence results yields the desired strong convergence of $n^\varepsilon|v^\varepsilon|^2$.

Outline

Conclusion

- We show that the two-phase fluid systems can be derived from the kinetic-fluid systems provided that weak solution exists.

- Thus, once the existence of weak solutions to the kinetic-fluid systems on the whole domain is proved, our derivation becomes fully rigorous.

Future works

- Global existence of weak solutions in the whole space
- Hydrodynamic limits of VFP/compressible NS on bounded domains
- Hydrodynamic limits of Vlasov-Navier-Stokes systems in a strong local alignment regime
- ...

The end

Thank you for your attention.