Stationary flows in a slab for the ES-BGK model with correct Prandtl number

Seok-Bae Yun

Department of Mathematics
Sungkyunkwan University (SKKU), Korea

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Institut de Mathmatiques de Bordeaux,
Bordeaux, France

Joint work with Stephane Brull (IMB, U of Bordeaux)

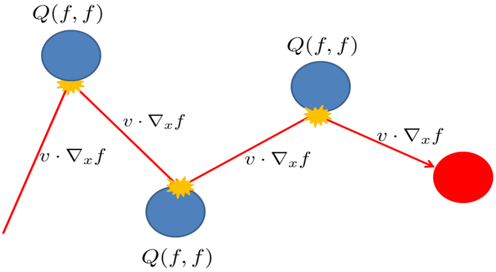
Boltzmann equation

The Boltzmann equation

- For non-ionized monatomic rarefied gas (1872):

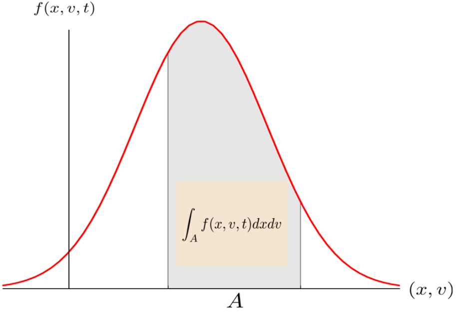
$$\partial_t f + v \cdot \nabla_x f = Q(f,f),$$
$$f(x,v,0) = f_0(x,v).$$

- Transport+collision



Velocity distribution function

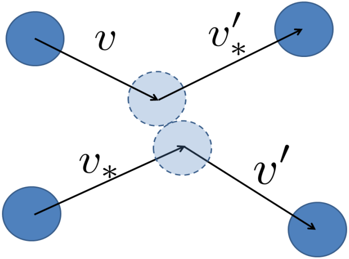
- Maxwell(1860), Boltzmann(1872)
- How particles are distributed in the phase space?
- $\int_A f(x,v,t)dxdv = \#$ of particles such that $(x,v) \in A$ at time t



Collision Operator

$$Q(f,f)(v) \equiv \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} B(v - v_*, \omega)(f(v')f(v'_*) - f(v)f(v_*))d\omega dv_*.$$

$$v' = v - [(v - v_*) \cdot \omega]\omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega.$$



Q satisfies

- Q satisfies

$$\int_{\mathbb{R}^3} Q(f,f)(1,v,|v|^2)dv = 0$$

and

$$\int_{\mathbb{R}^3} Q(f,f)\ln f dv \leq 0$$

Local Maxwellian

- Equilibrium

$$Q(\mathcal{M},\mathcal{M}) = 0$$

- Due to the conservation laws, we get

$$\mathcal{M}(f)(x,v,t) = \frac{\rho(x,t)}{\sqrt{(2\pi T(x,t))^3}} \exp\Big(-\frac{|v-U(x,t)|^2}{2T(x,t)}\Big).$$

where

$$\begin{aligned}
\rho(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)dv \\
\rho(x,t)U(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)vdv \\
\rho(x,t)T(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)|v-U(x,t)|^2 dv.
\end{aligned}$$

BGK model

BE: fundamental but not practical

- hard to develop fast & efficient numerical methods.
- Most difficulties and costs arise in the computation of Q.

The Boltzmann-BGK model

- BGK Model (Bhatnagar-Gross-Krook [1954]):

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= \frac{1}{\kappa}(\mathcal{M}(f) - f), \\ f(x, v, 0) &= f_0(x, v), \end{aligned}$$

- $1/\kappa$: collision frequency

- Local Maxwellian where

$$\mathcal{M}(f)(x,v,t)=\frac{\rho(x,t)}{\sqrt{(2\pi T(x,t))^3}}\exp\Big(-\frac{|v-U(x,t)|^2}{2T(x,t)}\Big).$$

where

$$\begin{aligned}\rho(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)dv\\ \rho(x,t)U(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)vdv\\ \rho(x,t)T(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)|v-U(x,t)|^2dv.\end{aligned}$$

Recall

- Q satisfies

$$\int_{\mathbb{R}^3} Q(f,f)(1,v,|v|^2)dv = 0$$

and

$$\int_{\mathbb{R}^3} Q(f,f)\ln f dv \leq 0$$

BGK also satisfies

- BGK operator also satisfies

$$\int_{\mathbb{R}^3} \{\mathcal{M}(f) - f\} (1, v, |v|^2) dv = 0$$

and

$$\int_{\mathbb{R}^3} \{\mathcal{M}(f) - f\} \ln f dv \leq 0$$

- Collision process ⇒ Relaxation process

- Much lower computational cost

- Still shares important features with the BE:
  - Conservation laws
  - H-theorem
  - Relaxation to equilibrium.
  - Correct Euler Limit

- Very popular model for numerical experiments in kinetic theory (Google cite 7611)

Navier-Stokes limit

- Macroscopic limit at the Euler equation level is O.K.
- How about the Navier-Stokes limit?

Prandtl number

- Prandtl number: ratio between diffusivity and viscosity.
- Boltzmann equation : 2/3
- BGK model : 1.
- Therefore, compressible NS limit of the BGK model is not correct.

Ellipsoidal BGK model

The Ellipsoidal-BGK model

- ES-BGK Model $(-1/2 \leq \nu < 1)$ [Halway, 1964] :

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= \frac{1}{\kappa} A_\nu (\mathcal{M}_\nu(f) - f), \\ f(x, v, 0) &= f_0(x, v), \end{aligned}$$

- κ: Knudsen number: How rarefied the gas is.
- A_ν: collision frequency.

$$A_\nu = \frac{\rho}{1-\nu}$$

- $\mathcal{M}_\nu(f)$?

- The local Maxwellian is generalized to the ellipsoidal Gaussian:

$$\mathcal{M}_{\nu}(f) = \frac{\rho}{\sqrt{\det(2\pi\mathcal{T}_{\nu})}} \exp\left(-\frac{1}{2}(v-U)^{\top}(\mathcal{T}_{\nu})^{-1}(v-U)\right)$$

- The local Maxwellian is generalized to the ellipsoidal Gaussian:

$$\mathcal{M}_{\nu}(f) = \frac{\rho}{\sqrt{\det(2\pi\mathcal{T}_{\nu})}} \exp\left(-\frac{1}{2}(v-U)^{\top}(\mathcal{T}_{\nu})^{-1}(v-U)\right)$$

- $\mathcal{T}_{\nu}$?

- The local Maxwellian is generalized to the ellipsoidal Gaussian:

$$\mathcal{M}_{\nu}(f) = \frac{\rho}{\sqrt{\det(2\pi\mathcal{T}_{\nu})}} \exp\left(-\frac{1}{2}(v-U)^{\top}(\mathcal{T}_{\nu})^{-1}(v-U)\right)$$

- $\mathcal{T}_{\nu}$:Temperature Tensor parametrized by ν

- Temperature Tensor: $(-1/2 \leq \nu < 1)$

$$\begin{aligned}\mathcal{T}_\nu(x,t) &= (1-\nu)T(x,t)Id + \nu\Theta(x,t)\\ &= \begin{pmatrix} (1-\nu)T + \nu\theta_{11} & \nu\theta_{12} & \nu\theta_{13} \\ \nu\theta_{21} & (1-\nu)T + \nu\theta_{22} & \nu\theta_{23} \\ \nu\theta_{31} & \nu\theta_{32} & (1-\nu)T + \nu\theta_{33} \end{pmatrix}\end{aligned}$$

where Θ denotes the stress Tensor:

$$\Theta(x,t) = \frac{1}{\rho}\int_{R^3} f(x,v,t)(v-U)\otimes(v-U)dv.$$

- Prandtl number: $\frac{1}{1-\nu}$.

- 2 important cases:
  - $\nu = 0$: Classical BGK model
  - $\nu = -1/2$: ES-BGK with correct Prandtl number.

- H-theorem: Andries-Le Tallec-Perlat-Perthame (2001)
- Rigourous derivation: Brull-Schnieder (2008)
- Entropy production estimate: Yun (2016)

Stationary problem in a slab

Stationary BGK model in a slab

- Stationary solution in a slab: $f = f(x,v)$, $x, v \in [0,1] \times \mathbb{R}^3$
- Boundary value problem:

$$v_1 \frac{\partial f}{\partial x} = \frac{\rho}{\tau}(\mathcal{M}_\nu(f) - f),$$

  on a finite interval $[0,1]$.
- Mixed boundary conditions ($\delta_1 + \delta_2 = 1$):

$$f(0,v) = \delta_1 f_L(v) + \delta_2 \left(\int_{|v_1|<0} f(0,v)|v_1|dv \right) M_w, \quad (v_1 > 0)$$

$$f(1,v) = \delta_1 f_R(v) + \delta_2 \left(\int_{|v_1|>0} f(1,v)|v_1|dv \right) M_w. \quad (v_1 < 0)$$

- δ_1: Inflow and δ_2: Diffusive.

Literatures

- BGK
  - Ukai (91): Weak solution with inflow boundary data
  - Nouri (08): QBGK: Weak solution with diffusive boundary data
  - Y. et al (16.18): ES-BGK, QBGK, RBGK.
- Boltzmann
  - Arkeryd-Cercignani-Illner (91): Measure-Valued Solutions.
  - Maslova: Mild Solutions (93)
  - Arkeryd-Nouri (98,99,00...): Weak solutions
  - Brull (08): Gas mixture
  - Guo-Kim-Esposito-Marra (13,18): Near Maxwellian

Notations

We first set up notational conventions and define norms:

- abbreviate notation:
$$f_{LR}(v) = f_L(v)1_{v_1>0} + f_R(v)1_{v_1<0}.$$
We also define the following quanities:

Notations

- Norm:

$$\sup_x \|f\|_{L^1_{2,}} = \sup_x \left\{ \int_{\mathcal{R}^3} |f(x,v)|(1+|v|^2)dv \right\},$$

- Trace norms ($n(i)$: outward normal at $x = i$ ($i = 0, 1$)):

$$\|f\|_{L^1_{\gamma,|v_1|}} = \sum_{i=1,2} \int_{v\cdot n(i)<0} |f(i,v)||v_1|dv + \int_{v\cdot n(i)>0} |f(i,v)||v_1|dv,$$

$$\|f\|_{L^1_{\gamma,\langle v\rangle}} = \sum_{i=1,2} \int_{v\cdot n(i)<0} |f(i,v)|\langle v\rangle dv + \int_{v\cdot n(i)>0} |f(i,v)|\langle v\rangle dv,$$

where $\langle v\rangle = (1+|v|^2)$.

Conditions on f_{LR}

(P_1) Finite flux, No concentration around $v_1 = 0$:

$$\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \left\|\frac{f_{LR}}{|v_1|}\right\|_{L^1_{\gamma,\langle v\rangle}} < \infty$$

(P_2) No vertical inflow at the boundary:

$$\int_{\mathbb{R}^2} f_L v_i dv = \int_{\mathbb{R}^2} f_R v_i dv = 0 \quad (i = 2, 3)$$

$(\mathcal{P}_3)$ Lower bound $(-1/2 < \nu < 1)$ $(i = 1, 2)$

$$\left(\int_{v_1>0} e^{-\frac{a_{u,i}}{|v_1|}} f_L(v)|v_1|dv\right)\left(\int_{v_1<0} e^{-\frac{a_{u,i}}{|v_1|}} f_R(v)|v_1|dv\right) > \gamma_{\ell,i} > 0.$$

(P_4) Lower bound $(\nu = -1/2)$ $(i = 1, 2)$

$$\inf_{|\kappa|=1}\int_{\mathbb{R}^3} e^{-\frac{2C_{LM,i}}{|v_1|}} f_{LR}\left\{|v|^2 - (v\cdot\kappa)^2\right\} dv \equiv a_{-1/2,i} > 0,$$

Mild Solution

Definition

$f \in L^1_2([0,1]_x \times \mathbb{R}^3_v)$ is a mild solution if

$$f(x,v) = e^{-\frac{1}{\tau|v_1|}\int_0^x \rho_f(y)dy} f(0,v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \rho_f(z)dz}\rho_f(y)\mathcal{M}(f)dy \quad \text{if } v_1 > 0$$

where

$$f(0,v) = \delta_1 f_L(v) + \delta_2\left(\int_{|v_1|<0} f(0,v)|v_1|dv\right)M_w, \quad (v_1 > 0)$$

$$f(1,v) = \cdots$$

Mild solution

For $v_1 > 0$

$$|v_1|\partial_x f = \frac{\rho}{\tau}\big(\mathcal{M}_\nu(f) - f\big)$$

$$\partial_x f + \frac{\rho}{\tau|v_1|} f = \frac{\rho}{\tau|v_1|}\mathcal{M}_\nu(f)$$

$$\frac{d}{dx}\left(e^{\frac{\int_0^x \rho(y)dy}{|v_1|\tau}} f(x,v)\right) = \frac{1}{\tau|v_1|} e^{\frac{\int_0^x \rho(y)dy}{|v_1|\tau}} \rho(x)\mathcal{M}_\nu(f).$$

The case for $v_1 < 0$ is the same.

Main result: Inflow dominant case $\delta_2 \ll 1$

- : Non-critical case: $-1/2 < \nu < 1$:

Theorem (Brull-Y. 19)

Let $-1/2 < \nu < 1$. Suppose f_{LR} satisfies (P_1), (P_2) and (P_3). Then, for sufficiantly small δ_2 and τ^{-1}, there exists a unique mild solution $f \geq 0$ for BVP.

- : Critical case: $\nu = -1/2$:

Theorem (Brull-Y. 19)

Let $\nu = -1/2$: Suppose f_{LR} satisfies (P_1), (P_2) and (P_4). Assume further that

$$\left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right| \ll 1,$$

Then, for sufficiantly small δ_2 and τ^{-1}, there exists a unique mild solution $f \geq 0$ for BVP.

Main result: Diffusive dominant case: $\delta_1 \ll 1$

- : Non-critical case: $-1/2 < \nu < 1$:

Theorem (Brull-Y. 19)

Let $-1/2 < \nu < 1$. Suppose f_{LR} satisfies (P_1), (P_2) and (P_3). Assume furthe that f satisfies

$$\int_{v_1<0} f(0,v)|v_1|dv + \int_{v_1>0} f(1,v)|v_1|dv = 1. \tag{2.1}$$

Then, for sufficiantly small δ_2 and τ^{-1}, then there exists a unique mild solution $f \geq 0$ for BVP.

- : Critical case: $\nu = -1/2$:

Theorem (Brull-Y. 19)

Let $\nu = -1/2$: Suppose f_{LR} satisfies (P_1), (P_2) and (P_4). Assume the flux satisfies

$$\int_{v_1<0} f(0,v)|v_1|dv + \int_{v_1>0} f(1,v)|v_1|dv = 1. \tag{2.2}$$

Then, for sufficiantly small δ_2 and τ^{-1}, then there exists a unique mild solution $f \geq 0$ for BVP.

Approximate Scheme

We define our approximate scheme by

$$f^n = f^n_{v_1>0} + f^n_{v_1<0},$$

where

$$f^{n+1}(x,v) = e^{-\frac{1}{\tau|v_1|}\int_0^x \rho_n(y)dy} f^{n+1}(0,v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \rho^n(z)dz} \rho_n(y)\mathcal{M}_\nu(f^n)dy \quad \text{if } v_1 > 0$$

and

$$f^{n+1}(0,v) = \delta_1 f_L(v) + \delta_2 \left(\int_{|v_1|<0} f^n(0,v)|v_1|dv \right) M_w, \quad (v_1 > 0)$$

Solution Space

$$\Omega = \left\{ f \in L^1_2 \;\middle|\; f \text{ satisfies } (\mathcal{A}), (\mathcal{B}), (\mathcal{C}) \right\}$$

where

- $(\mathcal{A})$ f is non-negative:
$$f(x,v) \geq 0 \text{ a.e}$$
- $(\mathcal{B})$ Lower bounds $(|\kappa| = 1)$:
$$\rho \geq C_1. \qquad \kappa^{\top}\left\{\mathcal{T}_{\nu}\right\}\kappa \geq C_2$$
- $(\mathcal{C})$ Norm bounds
$$\|f\|_{L^1_2}, \quad \|f\|_{L^1_{\gamma,|v_1|}}, \; \|f\|_{L^1_{\gamma,\langle v\rangle}} \leq C_3$$

For large τ

- Banach fixed point theorem $\rightarrow$ Something has to be small.
- Usually, $\|f\|$ small in a suitable norm.
- We take τ large instead. (Maslova)

We want

- Uniform estimate:

$$f^n \in \Omega \quad \text{for all } n.$$

- Contractivity:

$$\|f^{n+1} - f^n\| \leq \delta \|f^{n+1} - f^n\|$$

for appropriate norm and $\delta < 1$.

Difficulties

- Singularities may arise near $v_1 = 0$:

$$\partial_x f = \frac{\rho}{\tau v_1}(\mathcal{M}_\nu - f).$$

- Singularities may arise near $\mathcal{T}_\nu = 0$ since $\mathcal{M}_\nu$ contains $\mathcal{T}_\nu^{-1}$ and $(\det \mathcal{T}_\nu)^{-1}$

- Dichotomy: $(-1/2 < \nu < 1$: $T \sim \mathcal{T}_\nu$ Id$)$ VS $(\nu = -1/2$: $T \nsim \mathcal{T}_{-1/2}$ Id $)$

First Problem: $\dfrac{1}{|v_1|}$

The main decay estimate:

Lemma

Let $f \in \Omega_i$ $(i = 1, 2)$. Then we have

$$\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|} e^{-\frac{\int_y^x \rho_f(z)dz}{\tau|v_1|}} \rho_f(y)\mathcal{M}_\nu(f)(1+|v|^2)dydv \leq C\left(\frac{\ln\tau + 1}{\tau}\right)$$

Proof

For $f \in \Omega$, we can reduce

$$\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|} e^{-\frac{\int_y^x \rho_f(z)dz}{\tau|v_1|}} \rho_f(y)\mathcal{M}_\nu(f)|v|^2 dydv \leq \int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|} e^{-\frac{a_{\ell,1}(x-y)}{\tau|v_1|}} e^{-Cv_1^2} dydv$$

Divide the domain of integration:

$$\left\{\int_0^x \int_{|v_1|<\frac{1}{\tau}} + \int_0^x \int_{\frac{1}{\tau}\leq|v_1|<\tau} + \int_0^x \int_{|v_1|\geq\tau}\right\} \frac{1}{\tau|v_1|} e^{-\frac{a_{\ell,1}(x-y)}{\tau|v_1|}} e^{-Cv_1^2} dv_1 dy \equiv I_1 + I_2 + I_3.$$

(a) Estimate of I_1: Integrate on x:

$$\begin{aligned}
&\int_{|v_1|<\frac{1}{\tau}} \left\{ \int_0^x \frac{1}{\tau|v_1|} e^{-\frac{a_{\ell,1}(x-y)}{\tau|v_1|}} dy \right\} e^{-Cv_1^2} dv_1 \\
&= \frac{1}{a_{\ell,1}} \int_{|v_1|<\frac{1}{\tau}} \left\{ 1 - e^{-\frac{a_{\ell,1}x}{\tau|v_1|}} \right\} e^{-Cv_1^2} dv_1 \\
&\leq \frac{1}{a_{\ell,1}} \int_{|v_1|<\frac{1}{\tau}} dv_1 \\
&\leq \frac{1}{a_{\ell,1}\tau}
\end{aligned}$$

(b) Estimate of I_2: We first integrate on x:

$$I_2 \leq \frac{1}{a_{\ell,1}} \int_{\frac{1}{\tau} \leq |v_1| \leq \tau} 1 - e^{-\frac{a_{\ell,1} x}{\tau |v_1|}}\, dv_1$$

Apply the Tyalor expasion to $1 - e^{-\frac{a_{\ell,1}}{\tau|v_1|}}$:

$$\begin{aligned}
I_2 &= \frac{1}{a_{\ell,1}} \int_{\frac{1}{\tau} < |v_1| < \tau} \left\{ \left(\frac{a_{\ell,1}}{\tau|v_1|} \right) - \frac{1}{2!} \left(\frac{a_{\ell,1}}{\tau|v_1|} \right)^2 + \frac{1}{3!} \left(\frac{a_{\ell,1}}{\tau|v_1|} \right)^3 + \cdots \right\} dv_1 \\
&= \frac{1}{\tau} \ln \tau^2 + \frac{1}{2!} \frac{a_{\ell,1}}{\tau^2} \frac{\tau^2 - 1}{\tau} + \frac{1}{2 \cdot 3!} \frac{a_\ell^2}{\tau^3} \frac{\tau^4 - 1}{\tau^2} + \frac{1}{3 \cdot 4!} \frac{a_\ell^3}{\tau^4} \frac{\tau^6 - 1}{\tau^3} \cdots \\
&\leq \frac{1}{\tau} \ln \tau^2 + \frac{e^{a_{\ell,1}}}{a_{\ell,1}} \frac{1}{\tau}.
\end{aligned}$$

I_3

(c) Estimate of I_3:

$$\begin{aligned} I_3 &\leq \int_0^1 \int_{|v_1|>\tau} \frac{1}{\tau|v_1|} e^{-Cv_1^2} dv_1 dy \\ &\leq \frac{1}{\tau^2} \int_{\mathbb{R}^3} e^{-Cv_1^2} dv_1 \\ &\leq C_{\ell,u} \frac{1}{\tau^2}. \end{aligned}$$

We combine the estimates (a), (b), (c) to obtain the desired result.

Second problem: $\mathcal{T}_\nu^{-1}$, $(\det \mathcal{T}_\nu)^{-1}$

- We will derive lower bounds for $\kappa^\top \mathcal{T}_\nu \kappa$.
- We first need some control on bulk velocity:

Lemma

Let $f^n \in \Omega$.
(1) *For $i = 1$, we have*

$$\Big|\int_{\mathbb{R}^3} f^{n+1} v_1 dv\Big| \leq \delta_1 \Big|\int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv\Big| + O(\delta_2, 1/\tau).$$

(2) *For $i = 2, 3$, we have*

$$\Big|\int_{\mathbb{R}^3} f^{n+1} v_i dv\Big| \leq C_{\ell,u}\left(\frac{\ln \tau + 1}{\tau}\right).$$

- size of U_1 $\sim$: Depends on the discrepance of the boundary flux
- size of U_2, U_3: Small

Uniform Lower bounds for $\mathcal{T}_\nu$

Lemma

(1) *Let* $-1/2 \leq \nu < 1$. *Assume* $f^n \in \Omega$. *Then, for sufficiently large* τ, *we have*

$$\kappa^\top \left\{\mathcal{T}_\nu^{n+1}\right\} \kappa \geq C.$$

for some $C > 0$ *indepdent of* n.

- We divide the proof into $-1/2 < \nu < 1$ and $\nu = -1/2$.

The Proof for $-1/2 < \nu < 1$: T Id $\approx \mathcal{T}_\nu$

Lemma

Let $-1/2 \lneq \nu < 1$. *Then we have*

$$\min\{1-\nu, 1+2\nu\} T Id \leq \mathcal{T}_\nu \leq \min\{1-\nu, 1+2\nu\} T Id,$$

- Therefore, it is enough to estimate T.

Proof: $\mathcal{T}_\nu \approx T$ Id

- By definition

$$\mathcal{T}_\nu = \frac{1}{\rho}\int_{\mathbb{R}^3} F\Big\{\frac{1-\nu}{3}|v-U|^2 Id + \nu(v-U)\otimes(v-U)\Big\}dv.$$

- For any $|k|=1$ in $\mathbb{R}^3$

$$k^T\{\rho\mathcal{T}_\nu\}k = \frac{1}{\rho}\int_{\mathbb{R}^3} F\Big\{\frac{(1-\nu)}{3}|v-U|^2 + \nu\{(v-U)\cdot k\}^2\Big\}dv.$$

- Split into 2 cases, $0 \le \nu < 1$, $-\frac{1}{2} < \nu < 0$.

Estimate of T

By Cauchy-Schwarz

$$
\begin{aligned}
3\{\rho^{n+1}\}^2 T^{n+1} &= \left(\int_{\mathbb{R}^3} f^{n+1} dv\right)\left(\int_{\mathbb{R}^3} f^{n+1}|v|^2 dv\right) - \left|\int_{\mathbb{R}^3} f^{n+1} v dv\right|^2 \\
&\geq \left(\int_{\mathbb{R}^3} f^{n+1}|v_1| dv\right)^2 - \left|\int_{\mathbb{R}^3} f^{n+1} v dv\right|^2 \\
&= \left(\int_{\mathbb{R}^3} f^{n+1}|v_1| dv\right)^2 - \left(\int_{\mathbb{R}^3} f^{n+1} v_1 dv\right)^2 - R.
\end{aligned}
$$

where

$$
R = \sum_{(i,j)\neq(1,1)} \left|\int_{\mathbb{R}^3} f^{n+1} v_i dv\right| \left|\int_{\mathbb{R}^3} f^{n+1} v_j dv\right|,
$$

R small

By the smallness of vertical flow, R is small:

$$R \leq C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right).$$

I bounded from below

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^3} f^{n+1}|v_1|dv\right)^2-\left(\int_{\mathbb{R}^3} f^{n+1}v_1\,dv\right)^2\\
&\quad=\left\{\int_{\mathbb{R}^3} f^{n+1}(|v_1|+v_1)dv\right\}\left\{\int_{\mathbb{R}^3} f^{n+1}(|v_1|-v_1)dv\right\}\\
&\quad=4\left\{\int_{v_1>0} f^{n+1}|v_1|dv\right\}\left\{\int_{v_1<0} f^{n+1}|v_1|dv\right\}\\
&\quad\geq 4\delta_1^2\left(\int_{v_1>0} e^{-\frac{a_{u,1}}{\tau|v_1|}}f_L|v_1|dv\right)\left(\int_{v_1<0} e^{-\frac{a_{u,1}}{\tau|v_1|}}f_R|v_1|dv\right)\\
&\quad\geq 4\delta_1^2\gamma_{\ell,1}.
\end{aligned}
$$

where we used

$$
f^{n+1}\geq \delta_1 e^{-\frac{a_{u,1}}{\tau|v_1|}}f_L 1_{v_1>0}+\delta_1 e^{-\frac{a_{u,1}}{\tau|v_1|}}f_R 1_{v_1<0}.
$$

Estimate of I and R

Therefore, for sufficiently large τ, we can get

$$T^{n+1} \geq \frac{1}{3\{\rho^{n+1}\}^2}\left\{4\delta_1^2\gamma_{\ell,1} - C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right)\right\} \geq C_1 \tag{2.3}$$

Critical Case: $\nu = -1/2$

- In the critical case, we don't enjoy the equivalence type estimate.

$$\mathcal{T}_{-1/2} \not\approx T.$$

- By definition and explicit computation:

$$\begin{aligned}
&\rho^{n+1}\kappa^{\top}\left\{\mathcal{T}^{n+1}_{-1/2}\right\}\kappa \\
&\quad = \int_{\mathbb{R}^3} f^{n+1}\left\{|v|^2 - (v\cdot\kappa)^2\right\} dv - \left\{\rho^{n+1}|U^{n+1}|^2 - \rho^{n+1}(U^{n+1}\cdot\kappa)^2\right\} \\
&\quad \equiv I + II,
\end{aligned}$$

for $|\kappa| = 1$.

Lower bound of I

From the assumption on f_{LR}:

$$\begin{aligned} I &= \int_{\mathbb{R}^3} f^{n+1}\left\{|v|^2-(v\cdot\kappa)^2 dv\right\} \\ &\geq \delta_1\int_{\mathbb{R}^3}\left\{e^{-\frac{\int_0^x \rho^n dy}{|v_1|}} f_L 1_{v_1>0} + e^{-\frac{\int_x^1 \rho^n dy}{|v_1|}} f_R 1_{v_1<0}\right\}\left\{|v|^2-(v\cdot\kappa)^2\right\} dv \\ &\geq \delta_1 \inf_{|\kappa|=1}\int_{\mathbb{R}^3} e^{-\frac{2}{|v_1|}\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}}} f_{LR}\left\{|v|^2-(v\cdot\kappa)^2\right\} dv \\ &\geq \delta_1 a_{-1/2,1}. \end{aligned}$$

Control of *II*

We can control *II* by fact that the macro flow in x direction is controlled by the discrepance of the boundary flux, and the vertical flows are small:

$$
\begin{aligned}
II &\leq \frac{|\rho^{n+1}U^{n+1}|^2}{\rho^{n+1}} \\
&\leq \frac{1}{a_{\ell,1}}\sum_{i=1}^{3}\left|\int_{\mathbb{R}^3} f^{n+1}v_i\,dv\right|^2 \\
&\leq 2\delta_1^2\left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right|^2 + O(\delta_2,\tau^{-1}).
\end{aligned}
$$

Therefore,

$$\kappa^\top \left\{ \mathcal{T}^{n+1}_{-1/2} \right\} \kappa \geq \delta_1 \inf_{|\kappa|=1} \int_{\mathbb{R}^3} e^{-\frac{2C_{LM}}{|v_1|}} f_{LR} \left\{ |v|^2 - (v\cdot\kappa)^2 \right\} dv$$
$$- \left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right|^2$$

up to small error.

Lip Continuity of $\mathcal{M}_\nu$

Lemma

Let f, g be elements of Ω_i. Then $\mathcal{M}_\nu$ satisfies

$$|\mathcal{M}_\nu(f)-\mathcal{M}_\nu(g)| \leq C_{\ell,u} \sup_x \|f-g\|_{L^1_2} e^{-C_{\ell,u}|v|^2}.$$

We expand $\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)$ as

$$\begin{aligned}\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g) &= (\rho_f - \rho_g)\int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \rho}d\theta \\ &+ (U_f - U_g)\int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial U}d\theta \\ &+ (\mathcal{T}_f - \mathcal{T}_g)\int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \mathcal{T}_\nu}d\theta.\end{aligned} \tag{2.4}$$

Roughly,

$$|\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)| \leq C\left(\frac{1}{\rho} + \frac{1}{T^{5/2}}\right)\|f - g\|$$

Contraction

Lemma

Suppose $f^{n+1}, f^n \in \Omega$. Then, under the assumption of Theorem 2.2, we have

$$\sup_x \|f^{n+1} - f^n\|_{L^1_2} + \|f^{n+1} - f^n\|_{L^1_{\gamma,|v_1|}} + \|f^{n+1} - f^n\|_{L^1_{\gamma,\langle v\rangle}}$$
$$\leq K(\delta_1, \tau, f_{LR}) \sup_x \|f_n - f_{n-1}\|_{L^1_2} + \delta_2 C\|f^n - f^{n-1}\|_{L^1_{\gamma,|v_1|}} + \delta_3 C\|f^n - f^{n-1}\|_{L^1_{\gamma,\langle v\rangle}}$$

where $K(\delta_1, \tau, f_{LR})$ denotes

$$K(\delta_1, \tau, f_{LR}) = \frac{\delta_1}{\tau}\left(\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \left\|f_{LR}|v_1|^{-1}\right\|_{L^1_{\gamma,\langle v\rangle}}\right) + \frac{\ln t + 1}{\tau\delta_1^3}.$$

Future

- Boltzmann equation
- Various BGK models: Q,R,Reactive
- Weak solutions with no smallness.

Thank You Very Much!

Thank you for your attention!