On the Cauchy problem for the Hall-MHD system without resistivity: wellposedness

Sung-Jin Oh (UC Berkeley), joint work with In-Jee Jeong (KIAS)

Nov. 26th, 2019

Conférence inaugurale France-Corée

Université de Bordeaux

Introduction

Hall- and electron-MHD

The *Hall-MHD system* (without resistivity) for the bulk plasma velocity field $\mathbf{u} : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ and the magnetic field $\mathbf{B} : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ associated to a plasma ($\nu \geq 0$) is:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \mathbf{p} - \nu \Delta \mathbf{u} = \mathbf{B} \cdot \nabla \mathbf{B}, \\ \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} + \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) = 0, \\ \nabla \cdot \mathbf{u} = 0, \qquad \nabla \cdot \mathbf{B} = 0. \end{cases} \tag{Hall-MHD}$$

The term in blue, called the Hall current term, represents the deviation from the ideal MHD equation.

The *electron-MHD system* is obtained by formally setting $\mathbf{u} = 0$:

$$\begin{cases} \partial_t \mathbf{B} + \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) = 0, \\ \nabla \cdot \mathbf{B} = 0. \end{cases} \tag{E-MHD}$$

Set-up

The Hall- and electron-MHD equations are *extended MHD models*, which take into account the relative motion of electrons compared to the positive ions through the Hall term $\nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B})$.

We neglect the effect of collision (resistivity), which is extremely small in many situations of physical interest.

In the case of (Hall-MHD), we assume incompressibility $\nabla \cdot \mathbf{u} = 0$ of the bulk plasma; in this setting, this equation was introduced by M. J. Lighthill.

Prelude

The Hall current term generates interesting new dynamics. The most striking example is the intriguing work of D. Chae and S. Weng, in which it was shown that (Hall-MHD) admits an *axisymmetric* finite time blow-up solution with regular initial data.

At the time of their discovery, however, the basic question of (local) wellposedness without symmetry assumptions had been open.

As we demonstrate in our work, the answer to this question turns out to be strikingly rich and markedly different from the known related models.

Prelude, cont.

A fundamental property of (Hall-MHD) (resp. (E-MHD)) is *conservation of energy*:

$$E[\mathbf{u}, \mathbf{B}](t) = \int |\mathbf{u}|^2(t,x) + |\mathbf{B}|^2(t,x)\,\mathrm{d}x,$$

$$\left(\text{resp. } E[\mathbf{B}](t) = \int |\mathbf{B}|^2(t,x)\,\mathrm{d}x\right).$$

On the other hand, the Hall current term incurs a *derivative loss* of order 1.

Previous mathematically rigorous investigations of (Hall-MHD) and (E-MHD) were mostly carried out either *with resistivity*, which gives rise to a dissipative term $\Delta\mathbf{B}$ compensating for this loss (cf. Chae–Degond–Liu, Chae–Lee, Chae–Wan–Wu, Chae–Wolf, Dai, Dai–Liu, Danchin–Tan etc.) or *in axisymmetry*, in which the second order terms vanish (cf. Chae–Weng, Jeong–Kim–Lee).

Main results, simplest versions

Our results demonstrate that:

- The derivative loss is unavoidable in certain cases. A striking example is the class of initial data near the trivial solution $(\mathbf{u}, \mathbf{B}) = 0$ (see *I.-J. Jeong*'s talk!).
- Nevertheless, the Cauchy problem is nevertheless (locally) well-posed for certain classes of initial data. An example is the class of initial data near a nonzero constant (uniform) magnetic field, which is the original set-up of Lighthill.

Main goal & key insight

In this talk, we will focus on the question of (local) wellposedness:

Goal

Find conditions on the initial data
which ensure wellposedness of the Cauchy problem.

(see *I.-J. Jeong*'s talk for illposedness!)

As we will see, both for proving wellposedness (this talk) and illposedness (I.-J. Jeong's talk), basic to our proof is the realization that the *Hall current term imparts the Hall- and electron-MHD with quasilinear dispersive characters*, in the absence of resistivity.

A case study: Perturbations of a uniform magnetic field

Linearized E-MHD around a nonzero uniform magnetic field

An instructive case to consider is that of perturbations of nonzero magnetic field (which is exactly the setting in M. J. Lighthill). We also only consider (E-MHD) for simplicity.

We take the ansatz

$$\mathbf{B} = \mathring{\mathbf{B}} + b = \bar{\mathbf{B}}\partial_1 + b, \quad \bar{\mathbf{B}} > 0.$$

Rewriting (E-MHD) for b, we obtain

$$\partial_t b + \bar{\mathbf{B}}\partial_1(\nabla \times b) = -\nabla \times ((\nabla \times b) \times b). \tag{1}$$

As we will see, (1) is a (nondegenerate) *quasilinear dispersive system* with many similarities to a quasilinear Schrödinger equation (cf. Kenig–Ponce–Vega, Kenig–Ponce–Rolvung–Vega, Marzuola–Metcalfe–Tataru).

Results around a nonzero uniform magnetic field

We start with an obvious result for the linearized system

$$\partial_t b + \bar{\mathbf{B}} \partial_x (\nabla \times b) = 0. \tag{2}$$

Theorem 1

The Cauchy problem for (2) *is well-posed for* $b(0) = b_0 \in H^\infty$.

Our next results concerns the full nonlinear system (1):

$$\partial_t b + \bar{\mathbf{B}} \partial_1 (\nabla \times b) = -\nabla \times ((\nabla \times b) \times b).$$

Theorem 2

The Cauchy problem for (1) *is well-posed for* $b(0) = b_0$ *satisfying*

$$\|b_0\|_{H^s} + \|\langle x^1 \rangle^s b_0\|_{L^2} \ll 1, \quad s > 7/2.$$

Ideas & Discussions

The linear result (Theorem 1) is a trivial consequence of constancy of $\bar{\mathbf{B}}$ and the energy identity.

We emphasize, however, that the linear result is not enough for nonlinear wellposedness. The culprit is the first order nonlinear terms (cf. I.-J. Jeong's talk).

To prove the nonlinear wellposedness result (Theorem 2), we need to exploit the *local smoothing effect* for (2) (and its perturbations):

$$\int_0^1 \int_{x^1 \in J} \frac{\lambda}{|J|} |P_\lambda b|^2 \,\mathrm{d}x\mathrm{d}t \lesssim \|b_0\|_{L^2}^2. \tag{3}$$

Here, P_λ is the Littlewood–Paley projection to frequencies $\xi \simeq \lambda$.

Wave packets and the local smoothing effect

The local smoothing estimate (3) may be heuristically understood using wave packets.

The linearized system (2), after diagonalization, can be split into two dispersive equations with dispersion relations

$$\omega(\xi) = \pm \bar{\mathbf{B}} \xi_1 |\xi|.$$

Wave packets follow, at least for short times, the group velocity

$$\mathbf{v}(\xi) = \nabla \omega(\xi) = \pm \bar{\mathbf{B}} |\xi| \begin{pmatrix} \frac{2\xi_1^2 + \xi_2^2 + \xi_3^2}{|\xi|^2} \\ \frac{\xi_1 \xi_2}{|\xi|^2} \\ \frac{\xi_1 \xi_3}{|\xi|^2} \end{pmatrix}.$$

The key properties is $\mathbf{v}^1 \simeq \bar{\mathbf{B}} |\xi|$.

Main results I: Linear setting

Linearization, principal symbol and bicharacteristics

For a general stationary vector field (but not necessarily a solution) $\mathbf{B} = \mathbf{B}(x)$, the linearization of (E-MHD) is

$$\partial_t b + \underbrace{\mathbf{B} \cdot \nabla(\nabla \times b)}_{\mathbf{P_B}(x,D)b+\cdots} - (\nabla \times b) \cdot \nabla \mathbf{B} + \nabla \times (\nabla \times \mathbf{B}) \times b) = 0. \quad (4)$$

The principal symbol $\mathbf{P_B}(x, D) = -\mathbf{B}(x) \cdot \xi(\xi \times)$ has two eigenvalues

$$\pm i p_{\mathbf{B}}(x, \xi) = \pm i \mathbf{B} \cdot \xi |\xi|.$$

The physical and frequency space centers $(X, \Xi)(t)$ of a wave packet solution to (4) follow the Hamiltonian ODE:

$$\begin{cases} \dot{X}(t) = \pm \nabla_\xi p_{\mathbf{B}}(x, \xi), \quad \dot{\Xi}(t) = \mp \nabla_x p_{\mathbf{B}}(x, \xi). \end{cases}$$

The solutions $(X, \Xi)(t)$ are also referred to as the bicharacteristic curves associated to $\mathbf{B}$.

Main result for linearized systems

Theorem 3 (Main linear result)

Let $\mathbf{B}$ *be a smooth stationary solution to* (E-MHD) *on* $M = \mathbb{R}^3/Z$, *and let* $S[\nabla \mathbf{B}] = \frac{1}{2}(\nabla \mathbf{B} + (\nabla \mathbf{B})^\top)$. *Assume that* $|\mathbf{B}|^{-1}|S[\nabla \mathbf{B}]|$ *is uniformly bounded and at each point*

$$S[\nabla \mathbf{B}]|_{\mathbf{B}^\perp} = 0 \quad (\textit{i.e.}, {}^{(\mathbf{B})}\pi^{jk} v_j w_k = 0 \textit{ if } \mathbf{B}^k v_k = \mathbf{B}^k w_k = 0). \quad (5)$$

Then the linearization of (E-MHD) *(resp.* (Hall-MHD)*) around* $\mathbf{B}$ *(resp.* $(0, \mathbf{B})$*) is well-posed in* H^∞ *(resp.* $H^\infty \times H^\infty$*).*

Although (5) seems complicated, it can be checked that $\mathbf{B} = f(y)\partial_x$ and $f(r)\partial_\theta$ satisfy it as long as f is uniformly smooth and uniformly bounded from below (cf. I.-J. Jeong's talk).

Main ideas

The quantity $S[\nabla \mathbf{B}]$ arises naturally from the computation of $|\Xi(t)|$ at the level of bicharacteristics:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Xi| = \mp S[\nabla \mathbf{B}]^{jk}(X)\Xi_j\Xi_k. \tag{6}$$

If $S[\nabla \mathbf{B}](X) = 0$, then $|\Xi|$ remains unchanged; this only happens if $\mathbf{B}$ is either a translation or a rotation vector field.

We use in addition the conservation of $p_{\mathbf{B}} = \mathbf{B}(X) \cdot \Xi|\Xi|$ along the flow, which is effective for Ξ in the direction parallel to $\mathbf{B}$. Thus we can prove that $|\Xi|$ remains bounded along any bicharacteristic only under (5).

The analysis at the level of bicharacteristics may be lifted to the linear PDE using pseudodifferential calculus and the energy identity.

Results in the nonlinear setting

Obstructions for local smoothing

We rely on the local smoothing effect to prove well-posedness of the nonlinear problem. Possible failures may be due to:

- (Incompleteness) A bicharacteristic fails to exist after finite time;
- (Trapping) A bicharacteristic fails to leave a compact region in finite time (e.g., periodic boundary condition);
- (Instability near degenerate stationary solutions) Certain stationary solutions lead to illposedness (c.f. I.-J. Jeong's talk)!

We seek for a general setting in which the above obstructions are avoided. Our key concept is that of a *compatible foliation* and *asymptotic uniformity*.

Compatible foliation

A *compatible foliation* plays the role played by the function x^1 in (3) in a more general setting:

Definition 4

A smooth function $\rho : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}$ is called a *compatible foliation* with respect to **B** if the following hold:

- (*nondegeneracy and completeness*) $\inf |\mathrm{d}\rho| > 0$ and ρ is onto;
- (*speed bounds*) $0 < \inf \mathbf{B}(\mathrm{d}\rho) \le \sup \mathbf{B}(\mathrm{d}\rho) < \infty$;
- (*transversality*) $\sup \left| \angle \left(\frac{\mathbf{B}}{|\mathbf{B}|}, \frac{\nabla\rho}{|\nabla\rho|} \right) \right| < \tan^{-1} \frac{1}{2\sqrt{2}}$.

These conditions ensure that $\frac{\mathrm{d}}{\mathrm{d}t}\rho(X) \simeq |\Xi|$ along any bicharacteristic $(X, \Xi)(t)$.

Weak asymptotic uniformity

We say that $\mathbf{B} = \mathbf{B}(x)$ is *(weakly) asymptotically uniform* if

$$\int_{-\infty}^{\infty} \sup_{\rho^{-1}(\alpha)} |S[\nabla \mathbf{B}]| \, \mathrm{d}\alpha < \infty,$$

where ρ is a compatible foliation. The following result holds.

Proposition

Consider the linearized system around $\mathbf{B}(x)$ (stationary but not necessarily a solution), which is weakly asymptotically uniform. Then every bicharacteristic is complete and nontrapped. Moreover, $\frac{\Xi(t_2)}{\Xi(t_1)}$ is uniformly bounded for any t_1, t_2.

For nonlinear applications, however, we need a stronger concept of asymptotic uniformity.

Strong asymptotic uniformity (continued)

We say that $\mathbf{B}$ is *strongly asymptotically uniform* if there exist a compatible foliation ρ and nonzero uniform magnetic fields $\mathbf{B}^{(+)}$ and $\mathbf{B}^{(-)}$ such that the followings holds:

$$\chi_{\pm}(\rho)(\mathbf{B} - \mathbf{B}^{(\pm)}) \in \ell^1_{\rho^*\mathcal{I}}\ell^\infty_{\mathcal{Q}} H^s,$$
$$\chi_{\pm}(\rho)\nabla(\mathbf{B} - \mathbf{B}^{(\pm)}),\ \chi_{\pm}(\rho)|\rho|(\mathbf{B} - \mathbf{B}^{(\pm)}) \in \ell^1_{\rho^*\mathcal{I}}\ell^\infty_{\mathcal{Q}} H^s,$$

where χ_+ is a smooth cutoff that equals 1 on $[\frac{1}{2}, \infty)$ and 0 on $(-\infty, -\frac{1}{2}]$, and $\chi_- = 1 - \chi_+$.

Here, the space $\ell^1_{\rho^*\mathcal{I}}\ell^\infty_{\mathcal{Q}} H^s$ is a refinement of H^s with stronger decay as $\rho \to \pm\infty$, and no decay in the directions tangential to ρ.

Strong asymptotic uniformity (continued)

More precisely, denote by $\mathcal{Q}_\lambda$ the partition of $\mathbb{R}^3$ into cubes of sidelength λ, and

$$\|u\|_{\ell^\infty_{\mathcal{Q}} L^2_\lambda} := \sup_{Q \in \mathcal{Q}_\lambda} \|\chi_Q u\|_{L^2}.$$

Let $\mathcal{I}_\lambda$ be the partition of $\mathbb{R}$ into intervals of sidelength λ, and

$$\|u\|_{\ell^1_{\rho^* \mathcal{I}} \ell^\infty_{\mathcal{Q}} L^2_\lambda} := \sum_{I \in \mathcal{I}} \sup_{Q \in \mathcal{Q}_\lambda} \|\chi_{\mathcal{I}}(\rho) \chi_Q u\|_{L^2}.$$

We introduce

$$\|u\|^2_{\ell^\infty_{\mathcal{Q}} H^s} = \sum_{k \in \mathbb{N}_0} \left(2^{sk} \|P_k u\|_{\ell^\infty_{\mathcal{Q}} L^2_k} \right)^2,$$

$$\|u\|^2_{\ell^1_{\rho^* \mathcal{I}} \ell^\infty_{\mathcal{Q}} H^s} = \sum_{k \in \mathbb{N}_0} \left(2^{sk} \|P_k u\|_{\ell^1_{\rho^* \mathcal{I}} \ell^\infty_{\mathcal{Q}} L^2_k} \right)^2.$$

Motivation for these definitions come from $|\dot{X}| \lesssim \lambda$ for $|\Xi| \simeq \lambda$.

Main nonlinear local wellposedness results

For (E-MHD), we have

Theorem 5 (Main nonlinear result for (E-MHD)**)**

The Cauchy problem for (E-MHD) *is locally well-posed for initial data* $\mathbf{B}_0$, *which is strongly asymptotically flat with* $s > \frac{7}{2}$

For (Hall-MHD), we have

Theorem 6 (Main nonlinear result for (Hall-MHD)**)**

The Cauchy problem for (Hall-MHD) *is locally well-posed for initial data* $(\mathbf{u}_0, \mathbf{B}_0)$, *where* $\mathbf{B}_0$ *is strongly asymptotically flat with* $s > \frac{7}{2}$ *and* $\mathbf{u}_0 \in \ell^\infty_{\mathcal{Q}} H^{s+2}$.

Main ideas

For (E-MHD), the main steps are as follows:

- Diagonalization and paralinearization: Focus on $P_\lambda b$.
- Two-point local smoothing estimate: Establish (3) with (essentially) $\|P_\lambda b(0)\|_{L^2} + \|P_\lambda b(T)\|_{L^2}$ on the RHS.
- Renormalization and energy boundedness: To bound $\|P_\lambda b(T)\|_{L^2}$, remove the dangerous first order term (coming from the commutator of P_λ and the principal term) by renormalization.

For (Hall-MHD), the case $\nu > 0$ is more straightforward. When $\nu = 0$, we use the good variables $(\mathbf{Z}, \mathbf{B}) = (\nabla \times \mathbf{u} + \mathbf{B}, \mathbf{B})$, which obey better equations.

Thank you for your attention!