Quillen metrics on modular curves

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Using Hodge theory, we can put the L^2-metric on $\lambda(E)$.

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The Poincaré metric on X and the metric on E inherited from the hermitian metric on $\mathbb{C}^r$ are then singular at the cusps, and the previous definition does not make sense.

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The Selberg zeta function associated to X and E is defined by

$$Z(s,\Gamma,\rho) = \prod_{\{\gamma\}_{\mathrm{hyp}}} \prod_{k=0}^{+\infty} \det\left(I - \rho(\gamma) N(\gamma)^{-s-k}\right) .$$

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This function exists on the half-plane $\operatorname{Re} s > 1$, and can be meromorphically continued.

First attempt at a Quillen metric

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Definition (Takhtajan-Zograf, 2007)

The regularized determinant is defined as

$$\det \Delta = \frac{\partial}{\partial s}_{|s=1} Z(s, \Gamma, Ad\rho)$$

where $Ad\rho$ is the adjoint representation, and the Quillen metric by

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Their aim was to get a curvature formula.

Conditions on the Quillen metric

As inspired by the compact case, this Quillen metric should satisfy :

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We will work to get a functorial Riemann-Roch theorem on modular curves, similar to the one proved by Deligne in 1987.

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Functorial isomorphism

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Theorem (Deligne, 1987)

We have an isomorphism of line bundles over S

$$\lambda(E)_{X/S}^{12} \simeq \langle \omega_{X/S}, \omega_{X/S} \rangle^{r} \langle \det E, \det E \otimes \omega_{X/S}^{-1} \rangle^{6} IC2_{X/S}(E)^{-12}$$

which is compatible with base change.

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The isometry part of this can be checked above each point of S.

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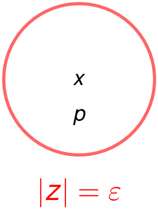

Poincaré metric : $\mathrm{d}s^2_{\mathrm{hyp}} = \dfrac{|\mathrm{d}z|^2}{(|z|\log|z|)^2}$

Metric ds^2_E on E : $\left\|e_j\right\|^2_z = |z|^{2\alpha_j}$

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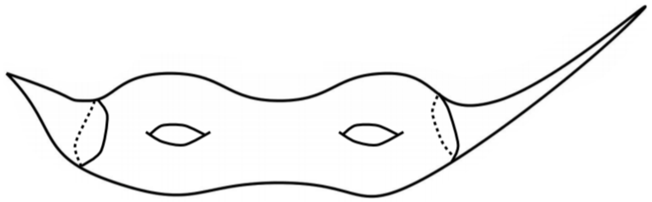

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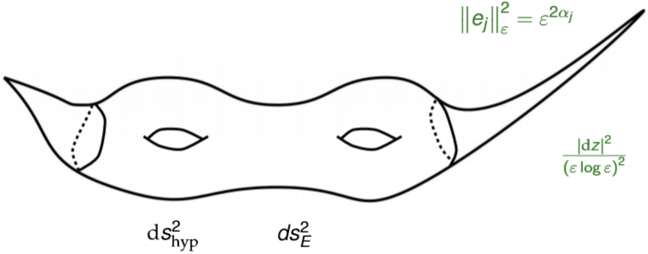

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$$\|\cdot\|_{Q,\varepsilon} = \left(\det \Delta_{\overline{\partial},E,\varepsilon}\right)^{-1/2} \|\cdot\|_{L^2} ,$$

where $\Delta_{\overline{\partial},E,\varepsilon}$ is the Dolbeault Laplacian acting on functions associated to the truncated metric.

ε-isometry

This ε-Quillen metric now fits into Deligne's result, which yields

$$\lambda(E)_{Q,\varepsilon} \simeq \left\langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon} \right\rangle^{r} \left\langle \det E_{\varepsilon}, \det E_{\varepsilon} \otimes \omega_{X,\varepsilon}^{-1} \right\rangle^{6} IC2\left(E_{\varepsilon}\right)^{-12},$$

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where every index ε means the metric has been truncated at radius ε at each cusp. The aim is now to let ε go to 0.

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Since none of the factors in the ε-isometry converges as ε goes to 0, we will need to regularize them, so as to extract the divergent part.

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The Deligne pairing $\langle \omega_{X,\varepsilon}(D), \omega_{X,\varepsilon}(D) \rangle$ then converges as ε goes to 0.

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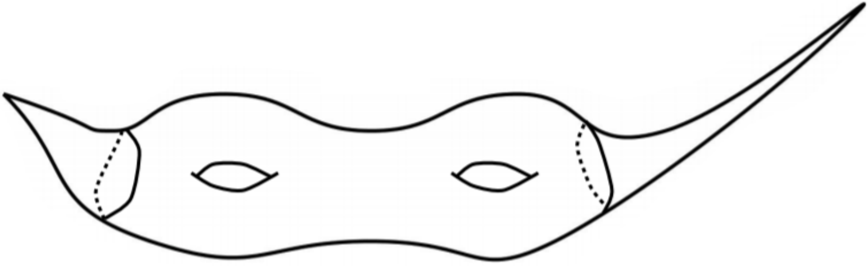

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This Quillen metric satisfies all three conditions required in the compact case (smoothness in family, spectral interpretation, Riemann-Roch theorem).