Odd-odd continued fraction algorithm

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Joint work with Dong Han Kim and Lingmin Liao

Regular continued fractions (RCF)

A regular continued fraction is an expression of the form as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}} = [a_0; a_1, a_2, \cdots]$$

where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$.

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Remarks and Examples

1. $x \in \mathbb{Q} \Longleftrightarrow x$ has a finite RCF. e.g. $\frac{4}{5} = \frac{1}{1+\frac{1}{4}} = [0; 1, 4]$
2. **[Euler, Lagrange]** x quad. irr. $\Longleftrightarrow$ RCF is eventually periodic.
   e.g. Golden ratio $\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \cdots]$.

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- *A partial quotient* of x : $a_n(x) := a_n$
- *A (principal) convergent* of x : $\frac{p_n(x)}{q_n(x)} \left(= \frac{p_n}{q_n}\right) := [a_0; a_1, a_2, \cdots, a_n]$.

Gauss map

A continued fraction map (*Gauss map*) :

$$G([0; a_1, a_2, \cdots]) = [0; a_2, a_3, \cdots])$$

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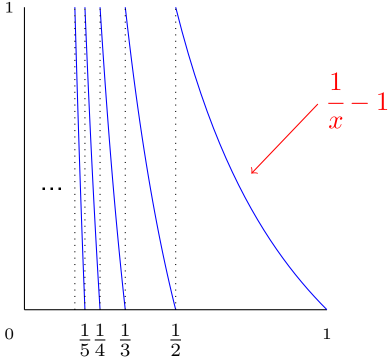
$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right] \quad \text{for } x \in (0, 1]$$



- $x \in [\frac{1}{n+1}, \frac{1}{n}]$:

$$G(x) = \frac{1}{x} - n$$

Farey map

$$F(x) = \begin{cases} \frac{x}{1-x}, & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1-x}{x}, & \text{if } \frac{1}{2} \le x < 1 \end{cases}$$

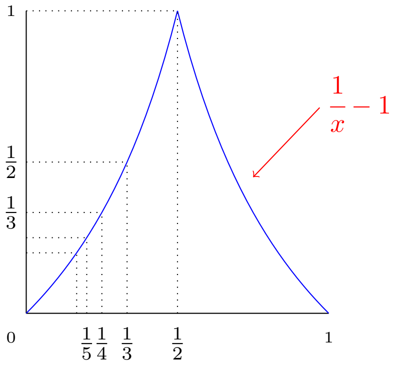


Figure: Graph of Farey map

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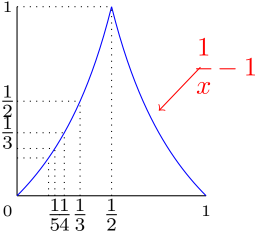

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \mapsto \cfrac{1}{a_1 - 1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

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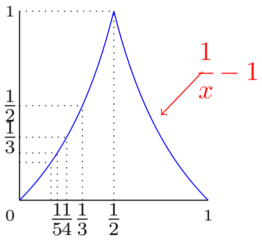

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$F^{a_1(x)}(x) = G(x)$, $\quad a_1(x) - 1$: the first return time of x to $[\frac{1}{2}, 1]$

Jump transformation

G is called *the jump transformation* of F w.r.t. $[\frac{1}{2}, 1]$.

T, T' : transformations, E : a subset of the domain

- *The first return time to E:*

$$n_E(x) = \min\{j \geq 0 : (T')^j(x) \in E\}$$

- *Jump transformation of T' w.r.t. E:*

$$T(x) = (T')^{n_E(x)+1}(x)$$

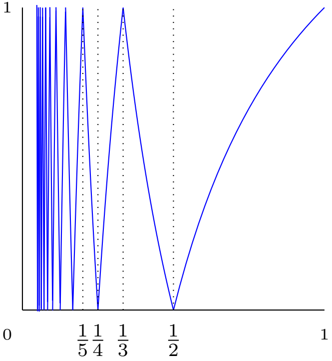
EICF: Even Integer Continued Fration

EICF: Continued fraction with Even entries

For $x \in (0,1]$,

$$x = \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{a_3 + \ddots}}}$$

where $a_n \in 2\mathbb{N}$, $\varepsilon_n \in \{1,-1\}$.

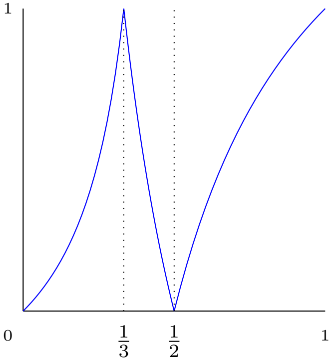


Continued fraction map:

$$T_{\text{EICF}}(x) = \begin{cases} \frac{1}{x} - 2k, \ x \in [\frac{1}{2k+1}, \frac{1}{2k}], & (a_1, \varepsilon_1) = (2k, +1), \\ 2k - \frac{1}{x}, \ x \in [\frac{1}{2k}, \frac{1}{2k-1}], & (a_1, \varepsilon_1) = (2k, -1). \end{cases}$$

Romik map

$$R(x) = \begin{cases} \frac{x}{1-2x}, & 0 \le x \le \frac{1}{3}, \\ \frac{1}{x} - 2, & \frac{1}{3} \le x \le \frac{1}{2}, \\ 2 - \frac{1}{x}, & \frac{1}{2} \le x \le 1. \end{cases}$$



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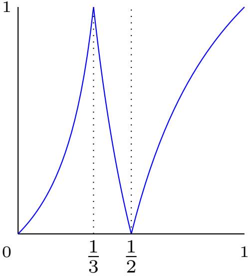

$$\cfrac{1}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{a_3 + \cdots}}} \mapsto \cfrac{1}{a_1 - 2 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{a_3 + \cdots}}}$$

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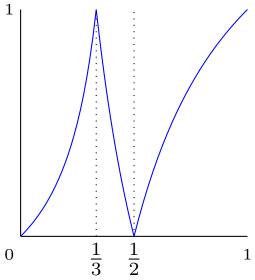

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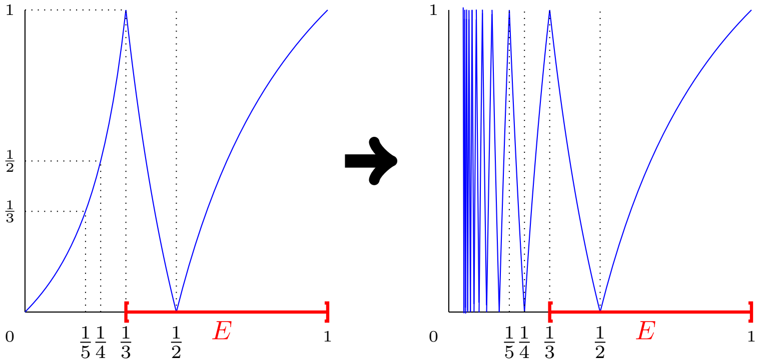
$$\cfrac{1}{2 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{a_3 + \cdots}}} \mapsto \cfrac{1}{a_2 + \cfrac{\varepsilon_2}{a_3 + \cdots}}$$

Romik → EICF

T_{EICF} is the jump transformation of R w.r.t. $E = [\frac{1}{3}, 1]$:

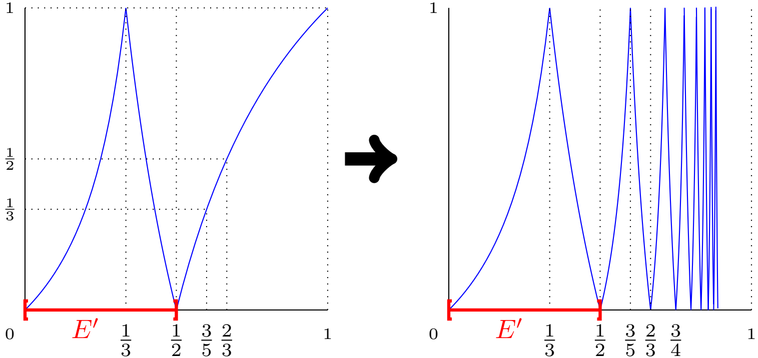


OOCF: Odd-Odd Continued Fraction

Let $E' = [0, \frac{1}{2}]$.

Jump transformation of R w.r.t. E':

$$T_{\mathrm{OOCF}}(x) = \begin{cases} \frac{kx-(k-1)}{k-(k+1)x}, & \frac{k-1}{k} \leq x \leq \frac{2k-1}{2k+1}, \\ \frac{k-(k+1)x}{kx-(k-1)}, & \frac{2k-1}{2k+1} \leq x \leq \frac{k}{k+1}. \end{cases}$$



$T := T_{\mathsf{OOCF}}$

$$\frac{1}{1-x} = \begin{cases} (k+1) - \frac{1}{2-(1-Tx)}, & x \in [\frac{k-1}{k}, \frac{2k-1}{2k+1}] \\ k + \frac{1}{2-(1-Tx)}, & x \in [\frac{2k-1}{2k+1}, \frac{k}{k+1}] \end{cases}$$

Odd-odd CF :

$$x = 1 - \cfrac{1}{\boldsymbol{a_1} + \cfrac{\boldsymbol{\varepsilon_1}}{2 - \cfrac{1}{\boldsymbol{a_2} + \cfrac{\boldsymbol{\varepsilon_2}}{2 - \ddots}}}},$$

where

$$(a_i, \varepsilon_i) \in \{(n,1) : n \geq 1\} \cup \{(n,-1) : n \geq 2\}.$$

A convergent of OOCF of x :

$$\frac{p_n}{q_n} := 1 - \cfrac{1}{\boldsymbol{a_1} + \cfrac{\boldsymbol{\varepsilon_1}}{2 - \cfrac{1}{\ddots \cfrac{1}{\boldsymbol{a_{n-1}} + \cfrac{\boldsymbol{\varepsilon_{n-1}}}{2 - \cfrac{1}{\boldsymbol{a_n} + \cfrac{\boldsymbol{\varepsilon_n}}{2}}}}}}}$$

A sub-convergent of OOCF of x :

$$\frac{p'_n}{q'_n} := 1 - \cfrac{1}{\boldsymbol{a_1} + \cfrac{\boldsymbol{\varepsilon_1}}{2 - \cfrac{1}{\ddots \cfrac{\boldsymbol{\varepsilon_{n-1}}}{2 - \cfrac{1}{\boldsymbol{a_n}}}}}}$$

Farey graph

On the hyperbolic plane $\mathbb{H}$ (as an upper half-plane model)

ℓ : the vertical line whose endpoints are 0 and ∞

Möbius transformations

$$\mathsf{SL}_2(\mathbb{R}) \circlearrowright \mathbb{H}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

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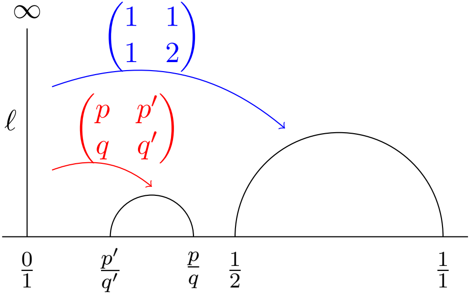


Farey graph

Farey graph:

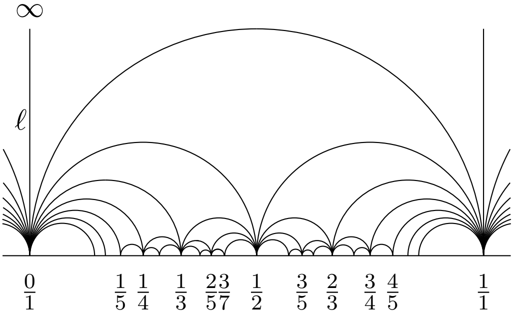
$$\mathcal{G} = \bigcup_{\gamma \in \mathsf{SL}_2(\mathbb{Z})} \gamma(\ell)$$

Vertices of $\mathcal{G}$:

$$\mathbb{Q} \cup \{\infty\}$$

p/q is adjacent to p'/q'

$$\Leftrightarrow \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \pm 1$$



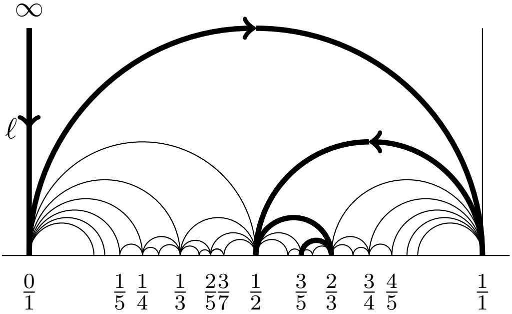
Farey graph (RCF)

$\varphi = \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$

Convergents of RCF:

$$p_1/q_1 = 1, p_2/q_2 = 1/2, p_3/q_3 = 2/3, p_4/q_4 = 3/5, \cdots$$

The corresponding path on Farey graph:



Farey tree (EICF) [Short-Walker, 2014]

$$\Theta = \left\{ \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} \text{ or } \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}$$

Farey tree $\mathcal{F} = \bigcup_{\gamma \in \Theta} \gamma(\ell)$

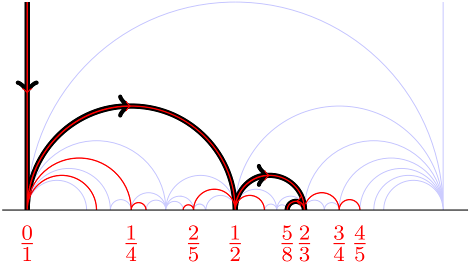
∞-rationals: Vertices of $\mathcal{F}$: $\Theta(\infty) = \{\text{even/odd or odd/even}\}$

Convergents of EICF are ∞-rationals.

e.g.

$$\varphi = \frac{\sqrt{5}-1}{2} = \cfrac{1}{2 + \cfrac{-1}{2 + \cfrac{1}{2 + \cfrac{-1}{\cdots}}}}$$

Convergents: $\frac{1}{2}, \frac{2}{3}, \frac{5}{8}, \frac{8}{13}, \cdots$



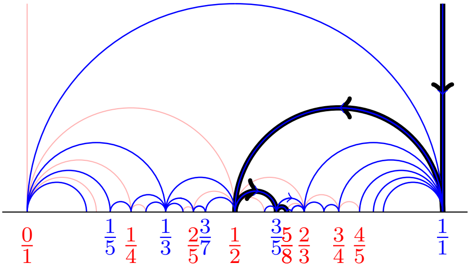
Farey graph (OOCF)

OOCF corresponds a path on Farey graph - Farey tree

Convergents of OOCF are 1-rationals

1-rationals : $\Theta(1) = \left\{\dfrac{\text{odd}}{\text{odd}}\right\}$, $\Theta = \left\{ \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} \text{ or } \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}$

$$\varphi = \frac{\sqrt{5}-1}{2} = 1 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{1}}{2 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{1}}{2 - \cfrac{1}{\mathbf{2} + \cdots}}}}}$$



Sub-convergents: $1/2, 5/8, 21/34, \cdots$

Convergents: $3/5, 13/21, 55/89, \cdots$

Finite OOCFs

Theorem 1 [Dong Han Kim - L. - Lingmin Liao]

1. finite OOCF $\iff$ 1-rational
2. periodic OOCF $\iff$ quad. irr. or ∞-rational

$$1-\cfrac{1}{\boldsymbol{a_1}+\cfrac{\boldsymbol{\varepsilon_1}}{2-\cfrac{1}{\ddots\cfrac{1}{\boldsymbol{a_{n-1}}+\cfrac{\boldsymbol{\varepsilon_{n-1}}}{2-\cfrac{1}{\boldsymbol{a_n}+\cfrac{\boldsymbol{\varepsilon_n}}{2}}}}}}} : \text{odd}/\text{odd}$$

T_{OOCF}(odd/odd): 1-rational $\Longrightarrow \exists N$ s.t. T_{OOCF}^{N}(odd/odd) $= 1$

Analogue of Euler-Lagrange Theorem

Theorem 1 [Dong Han Kim - L. - Lingmin Liao]

1. finite OOCF $\Longleftrightarrow$ 1-rational
2. periodic OOCF $\Longleftrightarrow$ quad. irr. or ∞-rational

$$\zeta_i = 1 - \cfrac{1}{\boldsymbol{a_i} + \cfrac{\boldsymbol{\varepsilon_i}}{2 - \cfrac{1}{\ddots \cfrac{1}{\boldsymbol{a_{i+n}} + \cfrac{\boldsymbol{\varepsilon_{i+n}}}{2 + \ddots}}}}} \Longrightarrow \quad x = \frac{(p_i - p_i') + p_i'\zeta_{i+1}}{(q_i - q_i') + q_i'\zeta_{i+1}}$$

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Perodic OOCF :

$$\Longrightarrow x = \frac{(p_i - p_i') + p_i' x}{(q_i - q_i') + q_i' x}$$

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quad. irr x **:** $a_1 x^2 + b_1 x + c_1 = 0 \implies a_i \zeta_i^2 + b_i \zeta_i + c_i = 0$

Analogue of Euler-Lagrange Theorem

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∞**-rational** $\frac{m}{n}$ **:** $T_{\mathsf{OOCF}}\left(\frac{m}{n}\right)$: ∞-rational $\Longrightarrow \exists N$ s.t. $T^N_{\mathsf{OOCF}}\left(\frac{m}{n}\right) = 0$

Analogue of Euler-Lagrange Theorem

Theorem 1 [Dong Han Kim - L. - Lingmin Liao]

1. finite OOCF $\Longleftrightarrow$ 1-rational
2. periodic OOCF $\Longleftrightarrow$ quad. irr. or ∞-rational

CF	RCF	EICF	OOCF
finite	rationals	∞-rationals [1] $\left(\frac{\text{even}}{\text{odd}} \text{ or } \frac{\text{odd}}{\text{even}}\right)$	1-rationals e.g. $\frac{7}{11} = 1 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{1}}{2 - \cfrac{1}{\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}}}}$
periodic	quad. irr.	quad. irr [2] & 1-rationals [1] $\left(\frac{\text{odd}}{\text{odd}}\right)$	quad. irr. & ∞-rationals e.g. in the next slides

[1] [Short - Walker, 2014]

[2] [Boca - Merriman, 2018]

Examples

$$0 = 1 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{-1}}{2 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{-1}}{2 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{-1}}{2 - \ddots}}}}}}$$

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$$\frac{1}{2} = 1 - \cfrac{1}{\mathbf{1} + \cfrac{\mathbf{1}}{2 - (1 - {\color{red}0})}} = 1 - \cfrac{1}{\mathbf{1} + \cfrac{\mathbf{1}}{2 - {\color{red}\frac{1}{\mathbf{2} + \frac{\mathbf{-1}}{2 - \frac{1}{\mathbf{2} + \cdots}}}}}} = 1 - \cfrac{1}{\mathbf{3} + \cfrac{\mathbf{-1}}{2 - (1 - {\color{red}0})}}$$

Examples

$$\frac{\sqrt{5}-1}{2} = 1 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{1}}{2 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{1}}{2 - \cfrac{1}{\mathbf{2} + \ddots}}}}}$$

Diophantine approximations

A Diophantine question:

For given $x \notin \mathbb{Q}$ and $N \in \mathbb{Z}$,
which rational p/q s.t. $0 < q \leq N$ minimize $|qx - p|$?

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Definition p/q is *a best (rational) approximation of x* if

$$|qx - p| < |bx - a| \quad \text{for any } a/b \neq p/q \text{ s.t. } 0 < b \leq q.$$

Theorem [Lagrange]

Every best approximation of x is a convergent of RCF of x, and vice versa.

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Every best approximation of x is a convergent of RCF of x, and vice versa.

Best ∞-rational approximations

Definition $p/q \in \Theta(\infty)$ is *a best ∞-**rational** approximation of x* if

$$|qx - p| < |bx - a| \quad \text{for any } \infty\text{-}\textbf{rational } a/b \neq p/q \text{ s.t. } 0 < b \leq q.$$

Theorem [Short and Walker, 2014]

Every best ∞-**rational** approximation of x is a convergent of **EICF** of x, and vice versa.

Best 1-rational approximations

Definition $p/q \in \Theta(1)$ is *a best* 1-**rational** *approximation of* x if

$$|qx - p| < |bx - a| \quad \text{for any } 1\text{-}\textbf{rational } a/b \neq p/q \text{ s.t. } 0 < b \leq q.$$

Theorem 2. [Dong Han Kim - L. - Lingmin Liao]

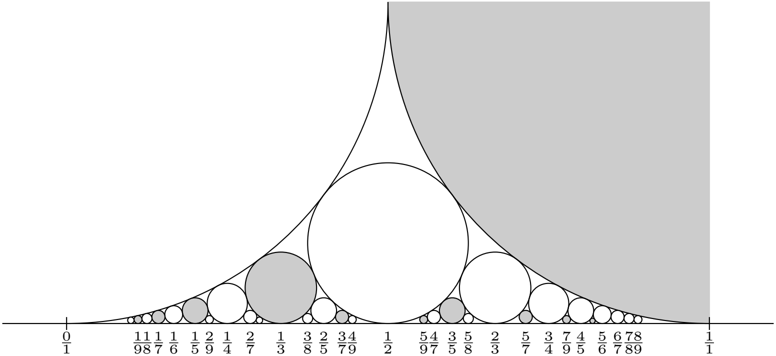
Every best 1-**rational** approximation of x is a convergent of **OOCF** of x, and vice versa.

Ford circles

A Ford circle $C_{\frac{a}{b}}$:

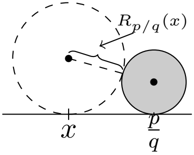
A horocycle based at $\frac{a}{b} \in \mathbb{Q} \cup \{\infty\}$ whose radius is $\frac{1}{2b^2}$.

- $C_{p/q}$ tangent to $C_{p'/q'} \Leftrightarrow \left| \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \right| = 1$



- $R_{p/q}(x)$: the Euclidean radius of the horocycle based at x which is tangent to $C_{p/q}$.

$$R_{p/q}(x) = \frac{1}{2}|qx - p|^2$$



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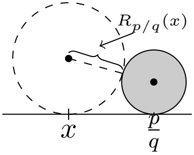

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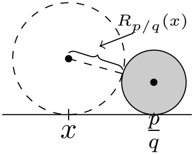

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Theorem 2. [Dong Han Kim - L. - Lingmin Liao]

A convergent of **OOCF** of x is a best 1-**rational** approximation of x, and vice versa.

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- $x \notin \mathbb{Q}$
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- a/b : a 1-rational s.t. $b \leq q$.
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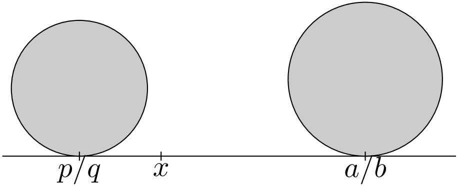

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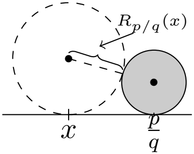

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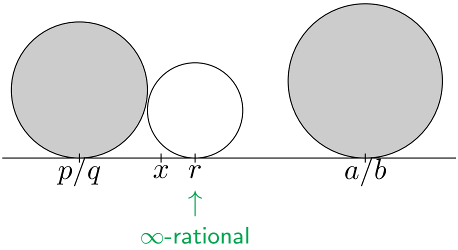

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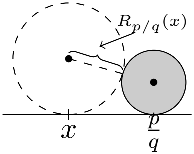

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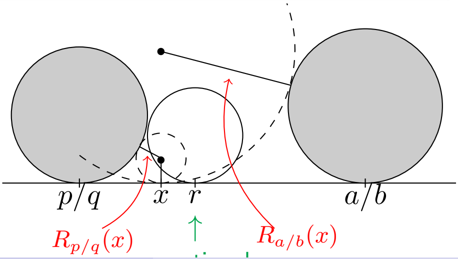

1-OOCF VS. EICF

p_n/q_n: Convergents of OOCF

$1 - p_n/q_n$: the form of even/odd.

$$\{even/odd \text{ among the convergents of EICF}\} \subset \{1 - p_n/q_n\}$$

Example ($x = \pi^2 - 9$)

EICF: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{20}{23}, \frac{967}{1112}, \frac{9650}{11097}, \cdots$

1−OOCF:
$\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{20}{23}, \frac{1914}{2201}, \frac{3848}{4425}, \frac{5782}{6649}, \frac{7716}{8873}, \frac{9650}{11097}, \cdots$

$$|1112x - 967| = 0.0000940113\cdots < |2201x - 1914| = 0.00071320\cdots$$
$$< |23x - 20| = 0.00090122\cdots$$

Romik Dynamical system

Theorem T_{OOCF} and T_{EICF} are conjugate.

More precisely, $f \circ T_{\mathsf{OOCF}} = T_{\mathsf{EICF}} \circ f$ where $f(t) = \dfrac{1-t}{1+t}$.

On $\mathcal{Q} = \{(x,y) : x^2 + y^2 = 1, x \geq 0, y \geq 0\}$,

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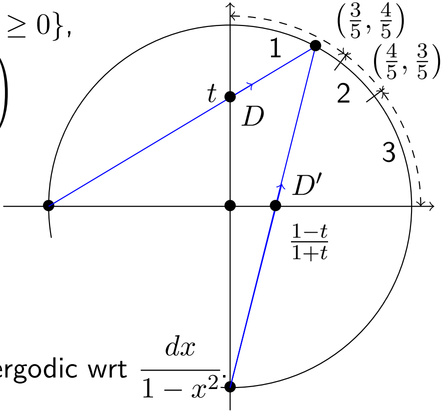

$$D(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$$

Theorem [Schweiger, '82] T_{EICF} is ergodic wrt $\dfrac{dx}{1-x^2}$.

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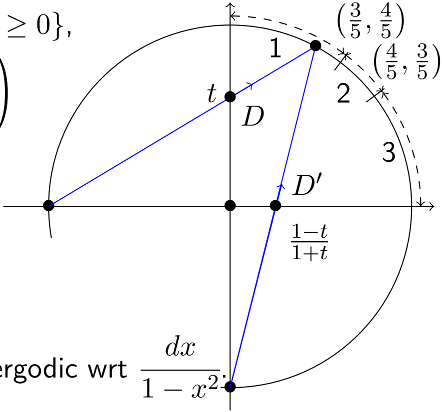

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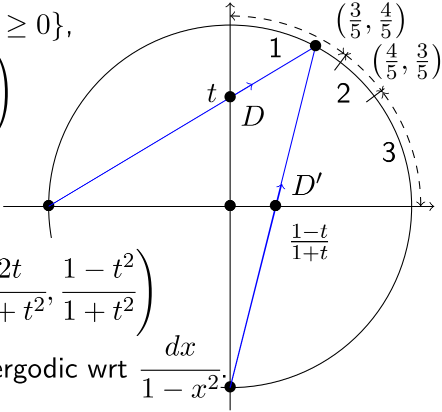

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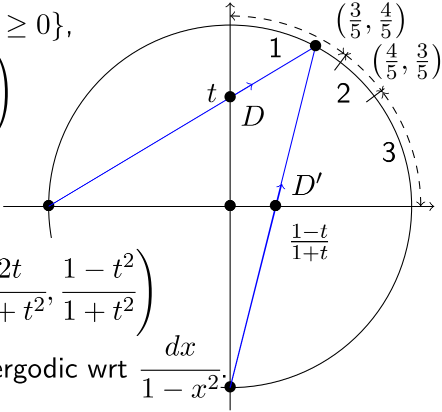

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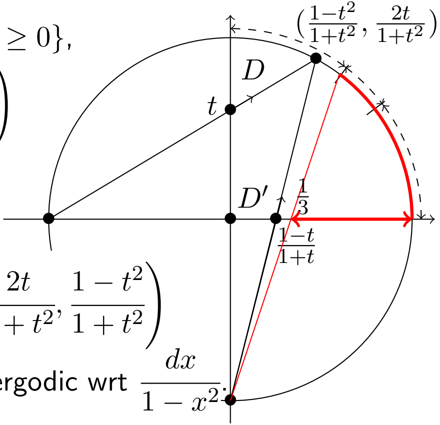

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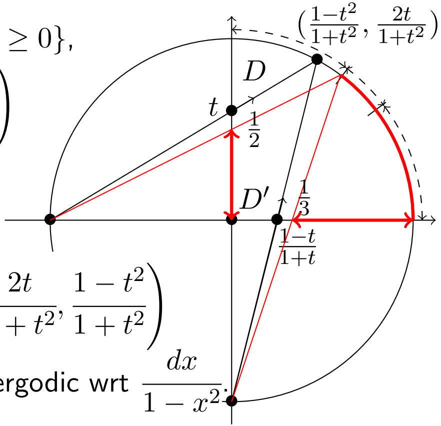

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