Asymptotic stability of homogeneous equilibria for screened Vlasov-Poisson systems

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Introduction

We consider the following **Vlasov-Poisson** system with screening :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, & t \geq 0,\ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \\ E = -\nabla_x \phi, \\ \phi - \Delta_x \phi = \rho - 1, \quad \rho = \int_{\mathbb{R}^3} f \, dv \\ f|_{t=0} = f_0 \geq 0. \end{cases}$$

- Dynamics of **charged particles** (ions) in a **plasma**.
  $f(t,x,v) \geq 0$: distribution function in phase space $\mathbb{R}^3 \times \mathbb{R}^3$
  $\rho(t,x)$: density of ions
  $E(t,x)$: electric field
- Screening : Coulomb potential $\frac{1}{r} \hookrightarrow$ Yukawa potential $\frac{e^{-r}}{r}$.
- Equivalently, in Fourier space, low frequency regularization : $\frac{1}{|\xi|^2} \hookrightarrow \frac{1}{1+|\xi|^2}$.

The question of stability of homogeneous equilibria

Any $\mu(v) \geq 0$ (with the normalization $\int_{\mathbb{R}^3} \mu \, dv = 1$) is a trivial stationary solution of the system (with $E = 0$).

MAIN QUESTION :

Asymptotic Stability of such homogeneous equilibria $\mu(v)$?

Remarks :

- There exist (linearly) unstable equilibria [Penrose, 1960].
- As $\int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu \, dvdx = +\infty$: infinite mass solution.

Stability of the equilibrium $\mu = 0$

Theorem 1 (*[Bardos, Degond 1985]*)

Consider Vlasov-Poisson without the -1*. Assume* f_0 *is compactly supported and*

$$\|f_0\|_{L^\infty_{x,v}} + \|\nabla_{x,v} f_0\|_{L^\infty_{x,v}} \ll 1.$$

Then there exists a unique global solution to the Vlasov-Poisson system, satisfying

$$\|\rho(t)\|_{L^\infty_x} + \|\nabla_x \rho(t)\|_{L^\infty_x} + \|E(t)\|_{L^\infty_x} \searrow_{t\to+\infty} 0,$$

with algebraic decay.

Remarks :

- Originally written for Vlasov-Poisson **without** screening.
- Mainly based on **dispersion for the free transport operator** on $\mathbb{R}^3$.
- Higher derivatives : [Hwang, Rendall, Velázquez 2011], [Smulevici 2016] .
- In the screened case, works as well in 2D [Choi, Ha, Lee 2011].

Dispersion for the free transport equation

Lemma 1

Let f be the solution to

$$\partial_t f + v \cdot \nabla_x f = 0, \qquad f|_{t=0} = f_0 \geq 0.$$

Then $\rho = \int_{\mathbb{R}^3} f\, dv$ satisfies for all $t > 0$

$$\|\rho(t)\|_{L^\infty_x} \leq \frac{1}{t^3} \|f_0\|_{L^1_x L^\infty_v},$$

$$\|\rho(t)\|_{L^p_x} \leq \frac{1}{t^{\frac{3}{p'}}} \|f_0\|_{L^1_x L^p_v},$$

$$\|\nabla^k_x \rho(t)\|_{L^p_x} \lesssim \frac{1}{t^{k+\frac{3}{p'}}} \|\nabla^k_v f_0\|_{L^1_x L^p_v}.$$

Based on

$$f(t,x,v) = f_0(x - tv, v).$$

Lagrangian structure of the Vlasov-Poisson system

- Introduce the characteristics curves $(X_{s,t}, V_{s,t})$ solving

$$\begin{cases} \dfrac{d}{ds} X_{s,t}(x,v) = V_{s,t}(x,v), & X_{t,t}(x,v) = x, \\ \dfrac{d}{ds} V_{s,t}(x,v) = E(s, X_{s,t}(x,v)), & V_{t,t}(x,v) = v. \end{cases}$$

- The solution to the Vlasov-Poisson system satisfies

$$f(t,x,v) = f_0(X_{0,t}(x,v), V_{0,t}(x,v)).$$

  Ensuring that $(X_{0,t}, V_{0,t})$ is a **small perturbation of the free flow** yields the dispersive estimate

$$\|\rho(t)\|_{L^\infty_x} \lesssim 1/t^3,$$

  hence the theorem (prove this by **bootstrap**).

Stability for non-trivial μ

Theorem 2 (*[Bedrossian, Masmoudi, Mouhot 2018]*)

Assume μ satisfies the **Penrose stability condition**. *Let $n \gg 1$. Let $k > \frac{3}{2}$. Assume*

$$\|\langle v\rangle^k (f_0 - \mu)\|_{L^2_{x,v}} + \|\langle v\rangle^k \nabla^n_{x,v}(f_0 - \mu)\|_{L^2_{x,v}} \ll 1.$$

Then there exists a unique global solution to the Vlasov-Poisson system, satisfying

$$\|\rho(t) - 1\| + \|E(t)\| \searrow_{t\to+\infty} 0,$$

$$\exists g_\infty(x,v), \quad \|(f-\mu)(t, x+tv, v) - g_\infty(x,v)\| \searrow_{t\to+\infty} 0,$$

with algebraic decay.

Penrose stability condition : $\exists \kappa > 0$,

$$\boxed{\inf_{\gamma\geq 0} \inf_{\tau\in\mathbb{R}, \xi\in\mathbb{R}^3} \left| 1 - \int_0^{+\infty} e^{i\tau t - \gamma t} \frac{i\xi}{1+|\xi|^2} \cdot \widehat{\nabla_v \mu}(\xi t)\, dt \right| \geq \kappa.}$$

Any **radial and positive** equilibrium μ satisfies the Penrose stability condition.

Remarks on the paper of Bedrossian-Masmoudi-Mouhot

- The proof is mainly based on Fourier analysis and inspired by the study of **Landau damping** (same stability problem set on $\mathbb{T}^d \times \mathbb{R}^d$) [Mouhot, Villani 2011], [Bedrossian, Masmoudi, Mouhot 2016].
- On $\mathbb{T}^d$, dispersion is replaced by **phase mixing**.
- On $\mathbb{T}^d$, the main obstruction to asymptotic stability are the so-called **"plasma echoes"** that correspond to certain resonances appearing in the study of nonlinearities. Their effect is tamed using **high regularity solutions** (**Gevrey** or analytic spaces).
- On $\mathbb{R}^d$, **dispersion is used to show that these resonances are very rare**.
- The proof is not simpler in the case $\mu = 0$ or in higher dimensions.

Main result

Theorem 3 (*[HK, Nguyen, Rousset]*)

Assume μ *satisfies the* **Penrose stability condition**.
Let $k > 3$. *Assume*

$$\|\langle v\rangle^k (f_0-\mu)\|_{W^{1,\infty}} + \|f_0-\mu\|_{W^{1,1}} + \|f_0-\mu\|_{L^1_x L^\infty_v} + \|\nabla_{x,v}(f_0-\mu)\|_{L^1_x L^\infty_v} \ll 1.$$

Then there exists a unique global solution such that

$$\|\rho(t)-1\|_{L^1} + \langle t\rangle \|\nabla_x \rho(t)\|_{L^1} + \langle t\rangle^3 \|\rho(t)-1\|_{L^\infty} + \langle t\rangle^4 \|\nabla_x \rho(t)\|_{L^\infty} \ll \log(2+t),$$

and

$$\exists g_\infty(x,v), \quad \|(f-\mu)(t,x+tv,v) - g_\infty(x,v)\|_{L^\infty} \ll \frac{\log(2+t)}{t^2}.$$

Remarks on the result

- The strategy of the proof is based on the **lagrangian structure** of the Vlasov-Poisson system. It can be seen as a generalization of the Bardos-Degond result for $\mu = 0$.
- It does not rely on the strategy developed for Landau damping on the torus. We completely avoid the study of plasma echoes.
- We only ask for an initial control of one derivative as in the Bardos-Degond result.

Lagrangian structure of the Vlasov-Poisson system

- Write $f(t,x,v) = \mu(v) + g(t,x,v)$. The perturbation satisfies

$$\begin{cases} \partial_t g + v\cdot\nabla_x g + E\cdot\nabla_v g = -E\cdot\nabla_v\mu, \\ E = -\nabla_x\phi, \\ \phi - \Delta_x\phi = \rho, \quad \rho = \int_{\mathbb{R}^3} g\,dv \\ g|_{t=0} = g_0 \qquad (= f_0 - \mu). \end{cases}$$

- Duhamel formula :

$$g(t,x,v) = g_0(X_{0,t}(x,v), V_{0,t}(x,v)) - \int_0^t (E\cdot\nabla_v\mu)(s, X_{s,t}(x,v), V_{s,t}(x,v))\,ds.$$

- Compared to the case $\mu = 0$, there is a source term. Rewrite this as

$$\mathscr{L}g = g_0(X_{0,t}(x,v), V_{0,t}(x,v)) + \int_0^t E(s, x-(t-s)v)\cdot\nabla_v\mu(v)\,ds - \int_0^t (E\cdot\nabla_v\mu)(s, X_{s,t}(x,v), V_{s,t}(x,v))\,ds$$

where $\mathscr{L}$ is the linear operator defined as

$$\mathscr{L}g := g + \int_0^t E(s, x-(t-s)v)\cdot\nabla_v\mu(v)\,ds.$$

Comparison with Bedrossian-Masmoudi-Mouhot

$$\boxed{\mathscr{L}g := g - \int_0^t [\nabla_x (1-\Delta_x)^{-1}\rho](s, x-(t-s)v) \cdot \nabla_v \mu(v)\, ds.}$$

Whereas Bedrossian-Masmoudi-Mouhot rather see the Vlasov equation as

$$\partial_t g + v \cdot \nabla_x g + E \cdot \nabla_v \mu = -E \cdot \nabla_v g,$$

getting

$$\mathscr{L}g = g_0(x - tv, v) - \int_0^t (E \cdot \nabla_v g)\,(s, x-(t-s)v, v)\, ds$$

we rely on the lagrangian structure of the system, yielding

$$\begin{aligned}\mathscr{L}g &= g_0(X_{0,t}(x,v), V_{0,t}(x,v)) \\ &\quad + \int_0^t E(s, x-(t-s)v) \cdot \nabla_v \mu(v)\, ds - \int_0^t (E \cdot \nabla_v \mu)\,(s, X_{s,t}(x,v), V_{s,t}(x,v))\, ds.\end{aligned}$$

$\rightarrow$ **Same linearized operator, different way to write the source term**.

Plan of the proof

The strategy is as follows :

- There is a preferred quantity in order to propagate **global regularity** for the Vlasov-Poisson system, that is the **density**

$$\rho = \int_{\mathbb{R}^3} g \, dv.$$

- I) Obtain pointwise in time estimates on the L^1 or L^∞ norm of the **density** for the linear operator $\mathscr{L}$, **saturating** the dispersive estimates for the free flow.
- II) **Bootstrap** analysis
  $\to$ Prove that the characteristics are close to the free ones.
  $\to$ Use of elementary bilinear "dispersive" estimates to handle the nonlinear terms.

I) The linearized problem (1/3)

We study

$$\mathscr{L} g = \mathscr{S}(t,x,v), \quad t > 0.$$

Integrating in v and setting

$$S(t,x) := \int_{\mathbb{R}^3} \mathscr{S}(t,x,v)\, dv,$$

we obtain

$$\rho(t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x (1-\Delta_x)^{-1} \rho](s, x-(t-s)v) \cdot \nabla_v \mu(v)\, dv ds + S(t,x), \quad t \geq 0,$$

with ρ and S extended by zero for $t < 0$, so that we end up with

$$\rho(t,x) = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} \mathbb{1}_{(t-s)\geq 0} [\nabla_x (1-\Delta_x)^{-1} \rho](s, x-(t-s)v) \cdot \nabla_v \mu(v)\, dv ds + S(t,x),$$

for all $t \in \mathbb{R}$.

I) The linearized problem (2/3)

Theorem 4

Assume μ satisfies the Penrose condition. Then, there exists $M>0$ such that for all $S\in L^1(\mathbb{R},L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3))$, ρ satisfies the estimates

$$\|\rho(t)\|_{L^1}+t^3\|\rho(t)\|_{L^\infty}\leq M\log(1+t)\|S\|_{Y^0_t},$$
$$t\|\nabla\rho(t)\|_{L^1}+t^4\|\nabla\rho(t)\|_{L^\infty}\leq M\log(1+t)\|S\|_{Y^1_t},$$

for $t\geq 1$, where

$$\|S\|_{Y^0_t}=\sup_{[0,t]}\left(\|S(s)\|_{L^1}+(1+s)^3\|S(s)\|_{L^\infty}\right),$$
$$\|S\|_{Y^1_t}=\sup_{[0,t]}\left(\|S(s)\|_{L^1}+(1+s)\|\nabla S(s)\|_{L^1}+(1+s)^4\|\nabla S(s)\|_{L^\infty}\right).$$

By-product : solutions to the linearized Vlasov-Poisson system

$$\partial_t f+v\cdot\nabla_x f-\nabla_x(1-\Delta_x)^{-1}\rho\cdot\nabla_v\mu=0,\qquad f|_{t=0}=f_0.$$

decay as solutions to free transport, up to a log loss.

I) The linearized problem (3/3)

Idea of the proof :

By the **Penrose condition**, we can rewrite the equation for ρ as

$$\rho = S + G \star S,$$

where $\star$ denotes convolution in t and x, and

$$\widehat{G}(\tau,\xi) = \frac{\int_0^{+\infty} e^{i\tau t} \frac{i\xi}{1+|\xi|^2} \cdot \widehat{\nabla_v \mu}(\xi t)\, dt}{1 - \int_0^{+\infty} e^{i\tau t} \frac{i\xi}{1+|\xi|^2} \cdot \widehat{\nabla_v \mu}(\xi t)\, dt}.$$

Prove pointwise in time decay estimates for the L^1 or L^∞ norm of G, using the **fine properties of the symbols** and **Littlewood-Paley** decomposition in time and space.

This allows to go “beyond” Hörmander-Mikhlin and Calderón-Zygmund theories for this operator (that would only yield $L^p_t L^q_x$ type estimates, for $1 < p, q < +\infty$).

II) A glimpse of the Bootstrap analysis (1/2)

Set

$$\mathcal{N}(t) = \sup_{[0,t]} \frac{1}{\log(2+s)} \Big(\|\rho(s)\|_{L^1} + \langle s \rangle^3 \|\rho(s)\|_{L^\infty} + \langle s \rangle \|\nabla \rho(s)\|_{L^1} + \langle s \rangle^4 \|\nabla \rho(s)\|_{L^\infty} \Big).$$

- Local well-posedness theory in Sobolev spaces for Vlasov-Poisson allows to set up a bootstrap analysis.
  Let $T_0 > 0$ be the maximal existence time.
- For $\varepsilon > 0$ small enough introduce

$$T^\star := \sup \Big\{ t \in (0, T_0),\, \mathcal{N}(t) \leq \varepsilon \Big\}.$$

The goal is to show $T^\star = T_0 = +\infty$.

II) A glimpse of the Bootstrap analysis (2/2)

Recall

$$\mathscr{L}g = g_0(X_{0,t}(x,v), V_{0,t}(x,v)) \\ \underbrace{+\int_0^t E(s, x-(t-s)v)\cdot\nabla_v\mu(v)\,ds - \int_0^t (E\cdot\nabla_v\mu)\,(s, X_{s,t}(x,v), V_{s,t}(x,v))\,ds}_{=:\mathscr{S}(t,x,v)}$$

Thanks to the linear theorem

$$\begin{aligned}\mathscr{N}(t) &\lesssim \sup_{[0,t)} \|S\|_{Y_s^0} + \|S\|_{Y_s^1} \\ &\lesssim \varepsilon_0 + \text{"what comes from the nonlinear term"} \\ &\lesssim \varepsilon_0 + \varepsilon^2 \qquad \text{...hopefully...}\end{aligned}$$

To this end, prove that on $[0, T^\star)$, characteristics remain close to the free ones, and that they can be straightened up to a small error of order ε.

Thanks for your attention !