Regularity and asymptotic behavior of the Boltzmann equation with boundary conditions

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The Boltzmann equation

The Boltzmann equation is a mathematical model for collisional rarefied gas derived by James Clerk Maxwell and Ludwig Boltzmann. It is a PDE for probability density function $F(t,x,v)$,

$$\partial_t F + v \cdot \nabla_x F = Q(F,F).$$

$X(s;t,x,v)$: Position of a particle at time $s \leq t$ which was at (t,x,v).
$V(s;t,x,v)$: Velocity of a particle at time $s \leq t$ which was at (t,x,v).
With Hamiltonian

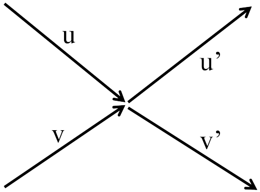
$$\frac{d}{ds}X(s;t,x,v) = V(s;t,x,v), \quad \frac{d}{ds}V(s;t,x,v) = 0$$

$$\frac{d}{ds}F(s,X(s),V(s))\bigg|_{s=t} = \partial_t F + v \cdot \nabla_x F,$$

"rate of change of probability density function" with velocity v at (t,x).

Collision operator and Maxwellian

Heuristic approach for Collision operator $Q(F,F)$: We assume elastic collision of hard sphere case.



with momentum conservation and energy conservation,

$$u + v = u' + v', \quad |u|^2 + |v|^2 = |u'|^2 + |v'|^2$$

Introducing two dimensional $\omega \in \mathbb{S}^2$,

$$u' = u + [(v-u)\cdot\omega]\omega, \quad v' = v - [(v-u)\cdot\omega]\omega.$$

Nonlinear Quadratic operator $Q(F, F)$ is written by

$$\begin{aligned} Q(F_1, F_2) &:= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \omega)\Big(F_1(u')F_2(v') - F_1(u)F_2(v)\Big) d\omega du \\ &:= Q_+(F_1, F_2) - Q_-(F_1, F_2), \quad \omega \in \mathbb{S}^2, \end{aligned}$$

where $B(v-u, \omega)$ is collision kernel for hard sphere collision,

$$B(v-u, \omega) = |(v-u) \cdot \omega|.$$

Maxwellian (Equilibrium state) $\mu(v)$ is a steady solution of the Boltzmann equation and has a form of (for example)

$$\mu(v) = e^{-\frac{|v|^2}{2}}.$$

In particular, $Q(\mu, \mu) = 0$.

Boundary conditions

In the case of boundary problem in domain Ω (let's say convex), we should impose relation between probability density function $F(t, x, v)$ on

$$\gamma_+ = \{(x, v) \ : \ v \cdot n(x) > 0\},$$
$$\gamma_- = \{(x, v) \ : \ v \cdot n(x) < 0\},$$

where $n(x)$ is outward unit normal vector on $x \in \partial\Omega$.

$$F(t, x, v)|_{\gamma_-} \sim \text{As a function of } \ F|_{\gamma_+},$$

For $x \in \partial\Omega$,
(1) **In flow** boundary condition

$$F(t, x, v) = g(t, x, v) \quad \text{for} \ \ v \cdot n < 0,$$

(2) **Bounce-back** boundary condition

$$F(t,x,v) = F(t,x,-v) \quad \text{for} \quad v \cdot n < 0,$$

(3) **Diffuse reflection** boundary condition

$$F(t,x,v) = c_\mu \mu \int_{v' \cdot n > 0} F(t,x,v')(n \cdot v') dv' \quad \text{for} \quad v \cdot n < 0,$$

(4) **Specular reflection** boundary condition (billiard model)

$$F(t,x,v) = F(t,x,R_x v) \quad \text{for} \quad x \in \partial\Omega,$$

where $R_x v = v - 2(n(x) \cdot v)n(x)$ is bouncing operator which changes the sign of normal direction.

Main idea of bootstrap and convergence to equilibrium

History : Asymptotic behaviors

Shizuta and Asano (1977) : Decay to Maxwellian, Full mathematical proof was not provided.

L.Devillettes and C.Villani (2005) : Almost exponential decay ($t^{-\infty}$) with large amplitute under assumption of apriori Sobolev bound .

Y.Guo (2010) : General boundary conditions, Small data, Analytic and uniformly convex boundary for specular BC.

M. Briant (2015) : Instant filling of the vacuum in convex domains with specular BC

R. Duan, F. Huang, Y. Wang, T. Yang (2017) : convergence to equilibrium with large amplitute data (periodic, torus, etc)

C. Kim and L (2018) : Specular BC in general smooth convex domain.

Formulation near Maxwellian

We explain the Boltzmann equation without external potential. With near Maxwellian expansion

$$F(t) = \mu + \sqrt{\mu} f(t) \geq 0,$$

yields

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f),$$

L is linear operator

$$Lf := -\frac{1}{\sqrt{\mu}} \Big[Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu) \Big]$$

and $\Gamma(\cdot, \cdot)$ is nonlinear quadratic operator

$$\Gamma(f, f) := \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} f)$$

Specular reflection BC gives

$$f(t, x, v) = f(t, x, R_x v) \quad \text{for} \quad x \in \partial\Omega.$$

Linear operator Lf

We split operator Lf as following

$$Lf := -\frac{1}{\sqrt{\mu}}\Big[Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\Big] = \nu f - Kf,$$

where the collision frequency $\nu(v)$

$$\nu(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-u)\cdot\omega|\sqrt{\mu}(u)\mathrm{d}\omega\mathrm{d}u.$$

For the hard sphere case, there are positive numbers C_0 and C_1 such that, ,

$$C_0\sqrt{1+|v|^2} \leq \nu(v) \leq C_1\sqrt{1+|v|^2}$$

Compact operator K on $L^2(\mathbb{R}^3_v)$, is defined as

$$Kf = \int_{\mathbb{R}^3} \mathbf{k}(v,u)f(u)\mathrm{d}u, \quad \mathbf{k}(v,u) \lesssim \frac{e^{-c|v-u|^2}}{|v-u|}.$$

*Projection of f onto $N(L)$. Hydrodynamic part of f.

$$\mathbf{P}f(t,x,v) \ := \ \Big\{a(t,x) + v\cdot b(t,x) + |v|^2 c(t,x)\Big\}\sqrt{\mu}.$$

We call a, b, and c as mass, momentum, and energy respectively.
* Semi-positiveness

$$\int_{\mathbb{R}^3} f \ Lf \mathrm{d}v \geq \delta_L \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})f\|^2_{L^2(\mathbb{R}^3)}.$$

*Missing $\|\mathbf{P}f\|_2$ estimate : Coercivity estimate

$$\|\mathbf{P}f\|_2 \lesssim \|(\mathbf{I}-\mathbf{P})f\|_2 \quad \text{in time interval} \quad t\in[0,1].$$

In L^2 energy estimate for f, coercivity estimate gives exponential decay for linearized Boltzmann.

$L^2 - L^\infty$ bootstrap argument

For the presentation, we consider a simplified linearized model

$$\partial_t f + v \cdot \nabla_x f + \nu(v) f = \int_{\mathbb{R}^3} \mathbf{k}(v,u) f(u) \mathrm{d}u$$

$$f(t,x,v) \lesssim \textit{initial datum's contributions} + O(\varepsilon)$$
$$+ \int_0^t e^{-\langle V(s)\rangle (t-s)} \int_u \mathbf{k}(V(s;t,x,v),u) f(s, X(s;t,x,v),u) duds$$

Using similar trajectory analysis for $f(s, X(s;t,x,v),u)$ again, we obtain estimate for double iteration

$$f(t,x,v) \lesssim \textit{initial datum's contributions} + O(\frac{1}{N})$$
$$+ \int_0^t e^{-(t-s)} \int_0^{s-\varepsilon} e^{-(s-s')} \iint_{|u|,|u'|\leq N} f(s', \underbrace{X(s';s,X(s;t,x,v),u)}, u')$$
$$\times \underbrace{\mathrm{d}u' \mathrm{d}u} \mathrm{d}s' \mathrm{d}s$$

Non-degeneracy condition

To use advantage of exponential decay of $\|f\|_2$, we recover $\|f\|_2$ through **non-degeneracy condition**,

$$\det\left(\frac{d}{du}X(s';s,X(s;t,x,v),u)\right) \geq \delta > 0,$$

which has closely related with uniform non-grazing $\frac{1}{|v\cdot\mathbf{n}|} \geq \delta > 0$. Then we get

$$f(t,x,v) = \textit{initial datum's contributions} + O(\varepsilon)$$
$$+ \int_0^t e^{-(t-s)} \int_0^{s-\varepsilon} e^{-(s-s')} \Big[\iint_{\Omega,|u'|\leq N} f(s',y,u')\mathrm{d}y\mathrm{d}u' \Big] \mathrm{d}s'\mathrm{d}s$$

where we have L^2 decay from

$$\iint_{\Omega,|u'|\leq N} f(s',y,u')\mathrm{d}y\mathrm{d}u' \leq C_{\delta,N}\|f\|_2.$$

and

$$(f \sim \text{initial data} + \int_0^t \|f\|_2 dt \sim \text{exponential decay}.$$

Main result 1 (2018, C. Kim and L)

Theorem

Let $w = (1+|v|)^\beta$ *for* $\beta > 5/2$*. Assume that the domain* $\Omega \subset \mathbb{R}^3$ *is* C^3 *and* ***strictly convex****. Assume that time-dependent external potential (not self-consistent)* $\Phi(t,x) \in C_x^{2,\gamma}$ *and decays to time independent potential sufficiently fast,i.e.,*

$$\sup_{t\geq 0} e^{\lambda t}\|\Phi(t,x) - \Phi^*(x)\|_{C^1} < C.$$

If initial data $\|wf_0\|_\infty \ll 1$*, relative entropy is sufficiently small, then the Boltzmann equation with specular reflection BC has a unique global solution. Moreover, we have mass conservation*

$$\iint_{\Omega\times\mathbb{R}^3} F(t) = \iint_{\Omega\times\mathbb{R}^3} F_0.$$

Since $\Phi(t,x)$ *is time-dependent, we cannot expect energy conservation.*

Main result 1 Cont'd

Theorem

We consider same small data assumptions as before, with time-independent external potential. Also we assume conservation of angular momentum if Ω is axisymmetric. For normalized initial data with normalization on initial data

$$\iint_{\Omega\times\mathbb{R}^3} \mu_E = \iint_{\Omega\times\mathbb{R}^3} F_0,$$

$$\iint_{\Omega\times\mathbb{R}^3} \Big(\frac{|v|^2}{2} + \Phi\Big)\mu_E = \iint_{\Omega\times\mathbb{R}^3} \Big(\frac{|v|^2}{2} + \Phi\Big)F_0,$$

We have a unique global solution $F = \mu_E + \sqrt{\mu_E} f \geq 0$ with asymptotic stability

$$\sup_{t\geq 0} e^{\lambda t}\|wf(t)\|_\infty \lesssim \|wf_0\|_\infty,$$

where $\mu_E = \mu e^{-\Phi(x)}$.

Theorem

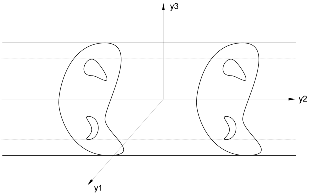
Furthermore, we have mass and energy conservations,

$$\iint_{\Omega\times\mathbb{R}^3} F(t) = \iint_{\Omega\times\mathbb{R}^3} F_0,$$

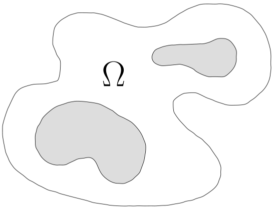
$$\iint_{\Omega\times\mathbb{R}^3} \Big(\frac{|v|^2}{2} + \Phi\Big) F(t) = \iint_{\Omega\times\mathbb{R}^3} \Big(\frac{|v|^2}{2} + \Phi\Big) F_0.$$

Main result 2 (2018, C. Kim and L)

Boltzmann eqaution (without external force) in a periodic cylindrical domain with non-convex analytic cross-section.



(a) periodic cylinder



(b) Cross-section

Grazing happens and we should classify traejctories to measure the size of "bad" sets in phase spaces.

Main result 2 (2018, C. Kim and L)

Theorem

Let $w = (1 + |v|)^{\beta}$ for $\beta > 5/2$. Assume periodic cylindrical domain U with general non-convex analytic cross-section Ω. If initial data is small $\|wf\|_{\infty} \ll 1$, then the Boltzmann equation with specular BC in U has a unique global solution which decays to Maxwellian exponentially. Moreover, the solution has mass and energy conservation under assumption of normalized initial data.

Main idea for regularity theory

History : Regularity

C. Kim (2011) : Formation and discontinuity with non-convex domain for diffuse reflection

Y.Guo, C. Kim, D.Tonon, A.Trescases (2016) : BV regularity for diffuse BC in **non-convex domains**

Y.Guo, C. Kim, D.Tonon, A.Trescases (2017)
: weighted C^1 regularity away from grazing set for specular BC,
: (weighted) $W^{1,p}$ regularity (away from grazing set) for diffuse BC both for **uniformly convex domains**
: Second order spatial derivative does not exist up to boundary (for diffuse BC, bounce-back BC)

Question : Regularity result in non-convex domains with specular reflection BC?

Exterior problem

We consider the Boltzmann equation in exterior region of unifomly convex Ω^c (uniformly non-convex) under specular reflection BC.
It is not easy to use high order Sobolev regularity setup for specular reflection BC. Hence we use mild solution using characteristics for $F = \sqrt{\mu} f$, i.e., for equation,

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \nu(f) f = \Gamma_{\text{gain}}(f,f) \quad x \in \Omega \\ f(t,x,v) = f(t,x,Rv) \quad x \in \partial\Omega, \end{aligned}$$

we have mild solution

$$\begin{aligned} f(t,x,v) &= e^{-\int_0^t \nu(f)(\tau, X(\tau;t,x,v), V(\tau;t,x,v)) \mathrm{d}\tau} f(0, X(0;t,x,v), V(0;t,x,v)) \\ &\quad + \int_0^t e^{-\int_s^t \nu(f)(\tau, X(\tau;t,x,v), V(\tau;t,x,v)) \mathrm{d}\tau} \\ &\qquad \times \Gamma_{\text{gain}}(f,f)(s, X(s;t,x,v), V(s;t,x,v)) \mathrm{d}s, \end{aligned}$$

with local in time solution $\sup_{0 \le t \le T} \| e^{\theta' |v|^2} f \|_\infty \lesssim P(\| e^{\theta |v|^2} f \|_\infty)$ for $0 < \theta' < \theta$.

Regularity of characteristics

In the case of convex domain,

$$|\nabla_x X(s;t,x,v)| \lesssim e^{c|v|(t-s)} \frac{|v|}{\sqrt{\alpha(x,v)}}, \quad |\nabla_v X(s;t,x,v)| \lesssim e^{c|v|(t-s)} \frac{1}{|v|},$$

$$|\nabla_x X(s;t,x,v)| \lesssim e^{c|v|(t-s)} \frac{|v|^3}{\alpha(x,v)}, \quad |\nabla_v X(s;t,x,v)| \lesssim e^{c|v|(t-s)} \frac{|v|}{\sqrt{\alpha(x,v)}}.$$

where kinetic distance $\alpha(x,v)$ is

$$\alpha(x,v) := |v \cdot \nabla \xi(x)|^2 - 2\{v \cdot \nabla^2 \xi(x) \cdot v\}\xi(x),$$

and ξ is parametrizaiton for boundary profile. Therefore, characteristics have C^1 regularity away from grazing phase.

In the case of **exterior domain case**, main observation is optimal regualrity of trajectory

$$X(s;t,x,v), V(s;t,x,v) \in C^{0,\frac{1}{2}}_{x,v}.$$

Fraction expansion

$$
\begin{aligned}
&\frac{|f(t,x,v+\zeta)-f(t,\bar{x},\bar{v}+\zeta)|}{|(x,v+\zeta)-(\bar{x},\bar{v}+\zeta)|^{\beta}} \\
\leq\quad & e^{-\int_0^t \nu(f)(\tau,X(\tau),V(\tau))\mathrm{d}\tau} \underbrace{\frac{|(X(0),V(0))-(\bar{X}(0),\bar{V}(0))|^{2\beta}}{|(x,v+\zeta)-(\bar{x},\bar{v}+\zeta)|^{\beta}}} \frac{|f(0,X(0),V(0))-f(0,\bar{X}(0),\bar{V}(0))|}{|(X(0),V(0))-(\bar{X}(0),\bar{V}(0))|^{2\beta}} \\
+\quad & \int_0^t e^{-\int_s^t \nu(f)(\tau,X(\tau),V(\tau))\mathrm{d}\tau} \underbrace{\frac{|(X(s),V(s))-(\bar{X}(s),\bar{V}(s))|^{2\beta}}{|(x,v+\zeta)-(\bar{x},\bar{v}+\zeta)|^{\beta}}} \\
& \qquad \times \underbrace{\frac{|\Gamma_{\text{gain}}(f,f)(s,X(s),V(s))-\Gamma_{\text{gain}}(f,f)(s,\bar{X}(s),\bar{V}(s))|}{|(X(s),V(s))-(\bar{X}(s),\bar{V}(s))|^{2\beta}}}_{\text{(Gamma fraction)}} ds \\
+\quad & \frac{\left|e^{-\int_0^t \nu(f)(\tau,X(\tau),V(\tau))\mathrm{d}\tau}-e^{-\int_0^t \nu(f)(\tau,\bar{X}(\tau),\bar{V}(\tau))\mathrm{d}\tau}\right|}{|(x,v+\zeta)-(\bar{x},\bar{v}+\zeta)|^{\beta}} |f(0,\bar{X}(0),\bar{V}(0))| \\
+\quad & \int_0^t \frac{\left|e^{-\int_s^t \nu(f)(\tau,X(\tau),V(\tau))\mathrm{d}\tau}-e^{-\int_s^t \nu(f)(\tau,\bar{X}(\tau),\bar{V}(\tau))\mathrm{d}\tau}\right|}{|(x,v+\zeta)-(\bar{x},\bar{v}+\zeta)|^{\beta}} |\Gamma_{\text{gain}}(f,f)(s,\bar{X}(s),\bar{V}(s))|\mathrm{d}s,
\end{aligned}
$$

Iteration form

From fraction of Γ_{gain} part, using Carlemann representation,

$$
\begin{aligned}
&\text{(Gamma fraction)} \\
&\lesssim \|wf\|_\infty \int_u \frac{e^{-\theta|u|^2}}{|u|} \frac{|f(X(s),u+V(s)) - f(X(s),u+\bar{V}(s))|}{|V(s)-\bar{V}(s)|^{2\beta}} du \\
&\quad + \|wf\|_\infty \int_u \frac{e^{-\theta|u|^2}}{|u|} \frac{|f(X(s),u+\bar{V}(s)) - f(\bar{X}(s),u+\bar{V}(s))|}{|X(s)-\bar{X}(s)|^{2\beta}} du \\
&\quad + \|wf\|_\infty^2 \big(e^{-\frac{\theta}{16}|v+\zeta|^2} + e^{-\frac{\theta}{16}|\bar{v}+\zeta|^2}\big)
\end{aligned}
$$

where

$$
\begin{aligned}
X(s) = X(s;t,x,v+\zeta), \quad V(s) = V(s;t,x,v+\zeta), \\
\bar{X}(s) = X(s;t,\bar{x},\bar{v}+\zeta), \quad \bar{V}(s) = V(s;t,\bar{x},\bar{v}+\zeta),
\end{aligned}
$$

Expansion for Iteration form

$$
\begin{aligned}
& e^{-w|v|t} \int_\zeta \frac{e^{-\theta|\zeta|^2}}{|\zeta|} \frac{|f(t,x,v+\zeta) - f(t,\bar{x},v+\zeta)|}{|x-\bar{x}|^{2\beta}} \\
\leq \quad & e^{-w|v|t} \int_\zeta \frac{e^{-\theta|\zeta|^2}}{|\zeta|} \frac{|V(0)-\bar{V}(0)|^{2\beta}}{|x-\bar{x}|^{2\beta}} \frac{|f(0,X(0),V(0)) - f(0,X(0),\bar{V}(0))|}{|V(0)-\bar{V}(0)|^{2\beta}} \\
+ \quad & e^{-w|v|t} \int_\zeta \frac{e^{-\theta|\zeta|^2}}{|\zeta|} \frac{|X(0)-\bar{X}(0)|^{2\beta}}{|x-\bar{x}|^{2\beta}} \frac{|f(0,X(0),\bar{V}(0)) - f(0,\bar{X}(0),\bar{V}(0))|}{|X(0)-\bar{X}(0)|^{2\beta}} \\
+ \quad & e^{-w|v|t} \int_0^t \int_\zeta \frac{e^{-\theta|\zeta|^2}}{|\zeta|} \frac{|(X(s),V(s)) - (\bar{X}(s),\bar{V}(s))|^{2\beta}}{|x-\bar{x}|^{2\beta}} \\
& \qquad \times \frac{|\Gamma_{\text{gain}}(f,f)(s,X(s),V(s)) - \Gamma_{\text{gain}}(f,f)(s,\bar{X}(s),\bar{V}(s))|}{|(X(s),V(s)) - (\bar{X}(s),\bar{V}(s))|^{2\beta}} \\
+ \quad & e^{-w|v|t} \int_\zeta \frac{e^{-\theta|\zeta|^2}}{|\zeta|} \frac{|e^{-\int_0^t \nu(f)(\tau,X(\tau),V(\tau))\mathrm{d}\tau} - e^{-\int_0^t \nu(f)(\tau,\bar{X}(\tau),\bar{V}(\tau))\mathrm{d}\tau}|}{|x-\bar{x}|^{2\beta}} |f(0,\bar{X}(0),\bar{V}(0))| \\
+ \quad & e^{-w|v|t} \int_0^t \int_\zeta \frac{e^{-\theta|\zeta|^2}}{|\zeta|} \frac{|e^{-\int_s^t \nu(f)(\tau,X(\tau),V(\tau))\mathrm{d}\tau} - e^{-\int_s^t \nu(f)(\tau,\bar{X}(\tau),\bar{V}(\tau))\mathrm{d}\tau}|}{|x-\bar{x}|^{2\beta}} \\
& \times |\Gamma_{\text{gain}}(f,f)(s,\bar{X}(s),\bar{V}(s))| \mathrm{d}s,
\end{aligned}
$$

Expansion for Iteration form with x difference

$$e^{-w|v|t}\int_\zeta \frac{e^{-\theta|\zeta|^2}}{|\zeta|}\frac{|f(t,x,v+\zeta)-f(t,\bar{x},v+\zeta)|}{|x-\bar{x}|^{2\beta}}$$

$\lesssim$ Initial data contribution

$$+\sup_t\|wf(t)\|_\infty\int_0^t e^{-w|v|(t-s)}\int_\zeta \underbrace{e^{-w|v|s}e^{w|V(s)|s}e^{-\frac{\theta}{2}|\zeta|^2}}_{(P)}\frac{e^{-\frac{\theta}{2}|\zeta|^2}}{|\zeta|}\frac{|V(s)-\bar{V}(s)|^{2\beta}}{|x-\bar{x}|^{2\beta}}\frac{1}{2|V(s)|^{2\beta}}$$

$$\times e^{-w|V(s)|s}\quad 2|V(s)|^{2\beta}\int_u \frac{e^{-\theta|u|^2}}{|u|}\frac{|f(X(s),V(s)+u)-f(X(s),\bar{V}(s)+u)|}{|V(s)-\bar{V}(s)|^{2\beta}}du$$

$$+\sup_t\|wf(t)\|_\infty\int_0^t e^{-w|v|(t-s)}\int_\zeta \underbrace{e^{-w|v|s}e^{w|V(s)|s}e^{-\frac{\theta}{2}|\zeta|^2}}_{(P)}\frac{e^{-\frac{\theta}{2}|\zeta|^2}}{|\zeta|}\frac{|X(s)-\bar{X}(s)|^{2\beta}}{|x-\bar{x}|^{2\beta}}$$

$$\times e^{-w|V(s)|s}\int_u \frac{e^{-\theta|u|^2}}{|u|}\frac{|f(X(s),\bar{V}(s)+u)-f(\bar{X}(s),\bar{V}(s)+u)|}{|X(s)-\bar{X}(s)|^{2\beta}}du$$

$+$ many other term with growth in v but without iteration form

Key estimate and scheme for fraction estimate

We need some smallness of

$$\int_0^t e^{-w|v|(t-s)} \int_\zeta \frac{e^{-\frac{\theta}{2}|\zeta|^2}}{|\zeta|} \frac{|X(s;t,x,v+\zeta) - X(s;t,\bar{x},v+\zeta)|^{2\beta}}{|x-\bar{x}|^{2\beta}}$$

(also other three-types)
We should perform fraction estimate. We parametrize $x - \bar{x}$ as line segment $x(\tau) = (1-\tau)\bar{x} + \tau x$.

$$\begin{aligned} &\frac{|X(s;t,x,v) - X(s;t,\bar{x},v)|}{|x-\bar{x}|} \\ &\leq \int_0^1 |\nabla_x X(s;t,x(\tau),v)| d\tau \\ &\lesssim \big(1 + |v|(t-s)\big) \int_0^1 \frac{|v|}{|v \cdot n(x_{\mathbf{b}}(x(\tau),v))|} d\tau \end{aligned}$$

Local geometric estimate

Proposition

We assume at least one trajectory from (x, v) or $(\tilde{x}, v)$ hit $\partial\Omega$ nongrazingly and $v \perp (x - \tilde{x})$ holds. Then, we can choose $\tau_-(x, \tilde{x}, v, \Omega) < \tau_* \leq \tau_{**} < \tau_+(x, \tilde{x}, v, \Omega)$ such that (for $|b - a| \leq 1$)

$$
\begin{aligned}
&\int_{\tau_-}^{\tau_+} \mathbf{1}_{[a,b]}(s) \frac{|v|}{|v \cdot n(x_{\mathbf{b}}(x(s), v))|} ds \\
&\lesssim_\Omega \mathbf{1}_{\tau_- \leq 1 \leq \tau_*} \frac{|v|}{|n(x_{\mathbf{b}}(x, v)) \cdot v|} + \mathbf{1}_{\tau_{**} \leq 0 \leq \tau_+} \frac{|v|}{|n(x_{\mathbf{b}}(\tilde{x}, v)) \cdot v|} \\
&\quad + \mathbf{1}_{\{\tau_- \leq 0 \leq \tau_+ \text{ or } \tau_- \leq 1 \leq \tau_+\}} \mathbf{1}_{\{\tau_* \leq 1 \text{ or } 0 \leq \tau_{**}\}} \\
&\quad \times \Big[\frac{|v|}{|n(x_{\mathbf{b}}(x(\tau_*), v)) \cdot v|} + \frac{|v|}{|n(x_{\mathbf{b}}(x(\tau_{**}), v)) \cdot v|} \Big].
\end{aligned}
$$

where $x(\tau_*)$ and $x(\tau_{**})$ are defined by

$$
x(\tau_*) = \tilde{x} + \tau_*(x - \tilde{x}), \quad x(\tau_{**}) = \tilde{x} + \tau_{**}(x - \tilde{x}),
$$

Continued

Proposition

(Continued) and τ_*, τ_{**} satisfy

$$1 \lesssim_\Omega |\hat{v} \cdot \hat{n}_{\|}(x_{\mathbf{b}}(x(\tau_*), v))| \leq 1,$$
$$1 \lesssim_\Omega |\hat{v} \cdot \hat{n}_{\|}(x_{\mathbf{b}}(x(\tau_{**}), v))| \leq 1.$$

where $n_{\|}$ is projection of $n(x_{\mathbf{b}})$ on the plane $x + \mathrm{span}(v, x - \bar{x})$.

(Key idea of proof) We estimate local profile of $|n(x_{\mathbf{b}}(x(\tau), v)) \cdot v|$ when the trajectory is nearly grazing.

$$|n(x_{\mathbf{b}}(x(\tau), v)) \cdot v| \simeq_\Omega \sqrt{\|\nabla n\|_\infty}|v|\sqrt{-(\dot{x} \cdot n(x_{\mathbf{b}}(x(\tau_-), v)))} \times |\tau - \tau_-|^{1/2},$$

Integrability

$$\int_0^t e^{-w|v|(t-s)} \int_\zeta \frac{e^{-\frac{\theta}{2}|\zeta|^2}}{|\zeta|} \frac{|v+\zeta|^{2\beta}}{|n(x_{\mathbf{b}}(x, v+\zeta)) \cdot (v+\zeta)|^{2\beta}} \big(1+|v+\zeta|(t-s)\big)^{2\beta} d\zeta ds$$

$$= \int_0^t e^{-w|v|(t-s)} \int_\zeta \frac{e^{-\frac{\theta}{2}|v-\zeta|^2}}{|v-\zeta|} \frac{|\zeta|^{2\beta}}{|n(x_{\mathbf{b}}(x, \zeta)) \cdot \zeta|^{2\beta}} \big(1+|\zeta|(t-s)\big)^{2\beta} d\zeta ds$$

(Roughly) We expect integrability $2\beta < 1$ and when $|\zeta| \simeq |v|$,

ζ-integration yields $|v|$ growth so

$$\lesssim \int_0^t e^{-w|v|(t-s)} \frac{1}{1-2\beta} |v| \big(1+|v|(t-s)\big)^{2\beta} d\zeta ds \lesssim \frac{1}{w} \ll 1.$$

Guess for Hölder regularity (Joint work with C. Kim)

We expect the following regularity result (rough version)

Guess If initial data $f_0 \in C^{0,2\beta}_{x,v}$ with $2\beta < 1$, (with some proper weight), local in time unique solution $f(t,x,v)$ enjoys weighted $C^{0,\beta}_{x,v}$ with $\beta < \frac{1}{2}$.

Further question : What about $\beta = \frac{1}{2}$ case?

Thank you!