Boundary Regularity for the Steady Shear Thickening Flow

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Coworker: Jörg Wolf

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Outline

1. Introduction
   - Power-law non-Newtonian system
     - Review of Incompressible Fluids
     - Known Facts

2. Main Result
   - Definitions of Weak and Strong Solutions
   - Main Theorem

3. Weighted Sobolev Embedding

4. Main Proof: Gradient estimates of stress near boundary
   - Estimation of tangential derivatives
   - Estimation of derivative in normal direction $y_3$
   - Higher $L^p$ estimate of $\nabla u$ via weighted Sobolev embedding
   - $\varepsilon \to 0$
   - $N \to \infty$
   - Proof of Main Theorem
\[ -\nabla \cdot \sigma + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad (1.1) \]
\[ \nabla \cdot \mathbf{u} = 0. \quad (1.2) \]

**BC:** \( \mathbf{u} = 0 \) on \( \partial \Omega \times (0, T) \) \( (1.3) \)

- \( \mathbf{u} = (u^1, u^2, u^3)^T \): unknown velocity
- \( p \): unknown pressure
- \( \mathbf{f} \) given external force
- \( \sigma = (\sigma_{ij}) \) shear stress;

\[ \sigma_{ij} = |\mathbf{D}(\mathbf{u})|^{q-2}D_{ij}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}(\mathbf{D}\mathbf{u} + (\mathbf{D}\mathbf{u})^T) \]
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H.-O. Bae (Ajou University)
Motivation

(a) Shear thickening fluid (oobleck)  (b) Shear thinning fluid (whipped cream)

**Figure:** Movie for non-Newtonian Fluids
Nonsteady Incompressible Fluids

\[ \begin{align*}
  u_t + (u \cdot \nabla) u - \text{div} \sigma + \nabla p &= f, \\
  \text{div} u &= 0, \\
  \text{IC} : \quad u|_{t=0} &= a, \\
  \text{BC} : \quad u|_{\partial \Omega} &= 0.
\end{align*} \]

\( a \) : given initial data
Two type of power-law models

\[ \sigma = 2|D|^{q-2}D, \]  \hspace{1cm} (1.4)

\[ \sigma = (\mu_0 + \mu_1|D|^{q-2})D, \]  \hspace{1cm} (1.5)

- For \( \mu_1 = 0 \) or \( q = 2 \), Navier-Stokes equations.
- The non-Newtonian fluids: nonlinear relation between the shear stress and the strain rate — some fluids such as ketchup, starch, blood.
- The case \( q > 2 \) describes dilatant (shear thickening) fluids whose viscosity increases with the rate of shear. (movie (a))
- Pseudo-plastic (shear thinning) fluids correspond to the case \( 1 < q < 2 \), where viscosity decreases with increasing rate of shear. (movie (b))
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\begin{align*}
\text{Newtonian} & \quad \text{for } \mu_0 > 0, \mu_1 = 0, \\
\text{Rabinowitsch} & \quad \text{for } \mu_0, \mu_1 > 0, \text{ and } q = 4, \\
\text{Smagorinsky} & \quad \text{for } \mu_0, \mu_1 > 0, \text{ and } q = 3, n = 3 \\
\text{Ellis} & \quad \text{for } \mu_0, \mu_1 > 0, \text{ and } q > 2, \\
\text{Ostwald-de Waele} & \quad \text{for } \mu_0 = 0, \mu_1 > 0, \text{ and } q > 1, \\
\text{Bingham} & \quad \text{for } \mu_0, \mu_1 > 0, \text{ and } q = 1.
\end{align*}

- paper pulp $\mu_1 = 0.418, q = 1.575$
- carboxymethyl cellulose (CMC) in water $\mu_1 = 0.194, q = 1.566$. 
Figure: non-Newtonian fluids (From Wikipedia...)
**Non-Newtonian, viscoelastic properties**

<table>
<thead>
<tr>
<th>Viscoelastic</th>
<th>Kelvin Material, Maxwell material</th>
<th>&quot;Parallel&quot; linear elastic combination of elastic and viscous effects</th>
<th>Some lubricants, whipped cream, Silly Putty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time dependent viscosity</td>
<td>Rheopecty</td>
<td>Apparent viscosity increases with duration of stress</td>
<td>printer ink, gypsum paste</td>
</tr>
<tr>
<td></td>
<td>Trixotropic</td>
<td>Apparent viscosity decreases with duration of stress</td>
<td>Yogurt, xanthan gum solutions, aqueous iron oxide gels, gelatin gels, pectin gels, synovial fluid, hydrogenated castor oil, some clays (bentonite, montmorillonite), carbon black suspension in molten tire rubber, some drilling muds, many paints, many floc suspensions, many colloidal suspensions</td>
</tr>
<tr>
<td>Time independent viscosity</td>
<td>shear thickening (dilatant)</td>
<td>Apparent viscosity increases with increased stress</td>
<td>Suspensions of corn starch in water(oobleck), sand in water</td>
</tr>
<tr>
<td></td>
<td>shear thinning (pseudoplastic)</td>
<td>Apparent viscosity decreases with increased stress</td>
<td>nail polish, whipped cream, ketchup, molasses, syrups, paper pulp in water, latex paint, ice, blood, some silicone oils, some silicone coatings</td>
</tr>
<tr>
<td></td>
<td>generalized Newtonian fluids</td>
<td>Viscosity is constant. Stress depends on normal and shear strain rates and also pressure applied on it</td>
<td>blood plasma, custard, water</td>
</tr>
</tbody>
</table>

(from Wikipedia)
Similar models:

\[ T_{ij} = (\mu_0 + \mu_1 |D|^2)^{(q-2)/2} D_{ij} \]  

\[ T_{ij} = (\mu_0 + \mu_1 |D|)^{q-2} D_{ij} \]  

- \( p \)-Laplacian equations
- Korns’ inequality
- asymptotic behaviors for \( q > 2 \), \( q = 2 \) and \( q < 2 \).
○ Similar models:

\[ T_{ij} = (\mu_0 + \mu_1 |D|^2)^{(q-2)/2} D_{ij} \] (1.6)

\[ T_{ij} = (\mu_0 + \mu_1 |D|)^{q-2} D_{ij} \]

○ \( p \)-Laplacian equations

○ Korns’ inequality

○ asymptotic behaviors for \( q > 2 \), \( q = 2 \) and \( q < 2 \).
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Periodic Domain

- Existence of space-periodic weak solution for $q > 1$ if $n = 2$ and $q > \frac{6}{5}$ if $n = 3$.
- Space-periodic weak solution is strong and unique for $q > 1$ if $n = 2$ and for $q > \frac{11}{5}$ if $n = 3$.
- Existence of space-periodic strong solution for small data is known for $q > \frac{5}{3}$ if $n = 3$. 
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- Existence of space-periodic strong solution for small data is known for $q > \frac{5}{3}$ if $n = 3$. 
With no slip BC, the existence of weak solution for $q > 1$ if $n = 2$ and $q > \frac{6}{5}$ if $n = 3$ 
(Wolf 2007, Diening, Ruzicka, Wolf 2010)

Weak solution is strong for $q > \frac{9}{4}$ if $n = 3$ for (1.5).

There are not many results for the of non-Newtonian fluid of the form (1.4).

- Short time regularity : Berselli ($q < 2$), Wolf-B. ($q > 2$)
No Slip BC

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Steady Case

Apushkinskaya, Bildhauer, Fuchs (2005) $C^{1,\alpha}$ regularity for (1.6) when $\Omega \in \mathbb{R}^2$.

Beirao da Veiga (2009) boundary regularity for (1.5)

Steady Stokes type shear thickening viscosity flow: Wolf-B. (2015), Boundary Holder regularity
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$C_{0,\sigma}^\infty(\Omega)$: the space of all solenoidal smooth vector fields in $\mathbb{R}^3$ having its support in $\Omega$.

$L^s(\Omega), \ W_{0,\sigma}^{1,s}(\Omega)$: usual Sobolev spaces

\[
L^s(\Omega) = \text{completion of } C_{0,\sigma}^\infty(\Omega) \text{ in } L^s(\Omega),
\]

\[
\widehat{W}_{0,\sigma}^{1,s}(\Omega) = \text{completion of } C_{0,\sigma}^\infty(\Omega) \text{ in } W_{0,\sigma}^{1,s}(\Omega) \text{ with } \|\nabla u\|_{L^s}
\]
**Definition 2.1**

Let $f \in L^1(\Omega)$. $u \in \widehat{W}^{1,q}_{0,\sigma}(\Omega)$ a weak solution of (1.1)-(1.3), if

$$\int_{\Omega} |D(u)|^{q-2} D(u) : D(\varphi) \, dx - u \otimes u : \nabla \varphi = \int_{\Omega} f \cdot \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

(2.1)

**Definition 2.2**

Weak solution $(u, p) \in \widehat{W}^{1,q}_{0,\sigma}(\Omega) \times L^{q'}_{loc}(\bar{\Omega})$ is called a strong solution if

$$\sigma_{ij} \in W^{1,s}(\Omega), \quad D_i p \in L^s(\Omega) \text{ for some } s \geq 1.$$
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Main Theorem I

Assume $2 < q < 3$. Let $f \in L^{3q/(4q-3)}(\Omega) \cap W^{1,q}(\Omega)$. Then for every weak solution $u \in \overset{\wedge}{W}^{1,q}_0(\Omega)$ to (1.1)–(1.3) there holds

$$
\int_{\Omega} (|D^\prime \sigma|^{q'} + |D^\prime p|^{q'} + |D^\prime D(u)|^{q/2}^2) \, dx \leq c(1 + \|f\|^{q'}_{q'-2}), \quad (2.2)
$$

$$
\int_{\Omega} \left( |D_3 \sigma|^{q'} + |D_3 p|^{q'} + (D_3 |D(u)|^{q/2})^2 \right) (x_3^\alpha \wedge 1) \, dx \leq c(1 + \|f\|^{q'}_{q'-2}), \quad (2.3)
$$

($\alpha > \frac{1}{2}$), where

$$
\|f\| := \left( \|f\|_{W^{1,q'}}^{q'} + \|f\|_{L^{3q/(4q-3)}}^{q'} \right)^{1/q'}.
$$
In addition, we have

\[ \nabla u \in L^s(\Omega) \quad \forall \, 1 \leq s < \frac{5q}{2}, \quad (2.4) \]

\[ D_3\sigma_{ij}, D_3p \in L^s(\Omega) \quad \forall \, 1 \leq s < \frac{2q'}{3} \quad (i, j = 1, 2, 3). \quad (2.5) \]

In particular, we conclude that \((u, p)\) is a strong solution to (1.1)–(1.3) and Hölder continuous,

\[ u \in C^\gamma(\Omega) \quad \forall \, 0 \leq \gamma < 1 - \frac{6}{5q} \]

by Morrey’s inequality.
IDEA of Proof of Main Theorem

Approximate Equation by adding $\varepsilon \Delta$

$\rightarrow$

Tangential Derivative Estimation

$+$ Normal Derivative Estimation with Weights near Boundary

$+$ Weighted Estimate near boundary (Proposition 3.2)

$\Rightarrow$

Boundary regularity

$+$ Morrey’s inequality

$\rightarrow$

Hölder continuity
Lemma 3.1

Let $\omega \in C([0, 1])$ be a positive function with
$\omega^{-1} \in L^1_{loc}([0, 1]) \cap L^s_{loc}((0, 1])$.

Then for all $f \in C^1([0, 1])$ with $f(1) = 0$, we have

$$|f(x)| \leq \left( \int_x^1 \omega^{-1} \, dy \right)^{\frac{1}{s'}} \left( \int_x^1 \omega^{s-1} |f'|^s \, dy \right)^{\frac{1}{s}} \quad \forall x \in (0, 1). \quad (3.1)$$
Weighted Sobolev Embedding

Weighted Sobolev space

\[ B'_r := B'_r(x_0) := \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - x_{0,1})^2 + (x_2 - x_{0,2})^2 < r^2 \} \]

\[ U_r := B'_r \times (-r, r) \]

\[ U^+_r := B'_r \times (0, r) \]

\[ W^{1,2}_r(U^+; \omega) := \left\{ \phi \in W^{1,2}_{loc}(U^+ \cup (\partial U)^+) ; \| \phi \|_{W^{1,2}_r(U^+; \omega(y_3))} < \infty, \quad \phi \big|_{\partial U \cap \mathbb{R}^+} = 0 \right\} \]

\[ \| \varphi \|_{W^{1,2}_r(U^+_1; \omega(y_3))} = \| \nabla' \varphi \|_{2; U^+} + \left( \int_{U^+} [\varphi y_3]^2 \omega(y_3) dy \right)^{\frac{1}{2}} \]
Proposition 3.2

Let $\omega \in C([0, 1])$ be a weight function such that $\omega^{-1} \in L^1(0, 1)$. Then, the space $W^{1, 2}_\Gamma(U^+; \omega^\alpha)$ is continuously embedded into $L^{2(3-\alpha)}(U^+)$. In addition,

$$\int_{U^+} \varphi^{2(3-\alpha)} \, dy \leq c \left( \int_{U^+} |\nabla' \varphi|^2 \, dy \right)^{2-\alpha} \int_{U^+} [\varphi_{y_3}]^2 \omega^{\alpha(y_3)} \, dy$$

(3.2)

for all $\varphi \in W^{1, 2}_\Gamma(U^+; \omega^\alpha)$. 

Later, for the main proof, we take $\varphi = |Du|^2 \frac{q}{2}$.
Theorem 3.3 (Wolf-B. 2015)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\omega \in A_{q'}$. Let $g_{ij}, p \in L^{q'}(\Omega; \omega)$ ($i, j = 1, \ldots, n$), such that

$$\sum_{i,j=1}^{n} \int_{\Omega} g_{ij} D_j \varphi^i \, dx = \int_{\Omega} p \, \text{div} \, \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Then, the following estimate holds true

$$\|p - p_\Omega\|_{L^{q'}(\Omega; \omega)} \leq c_\omega \|g\|_{L^{q'}(\Omega; \omega)}.$$
For pressure estimates, we need the following lemma.

**Lemma 3.4**

Let \( A \in L^s(U^+_r; \mathbb{R}^9) \), \( f \in L^s(U^+_r; \mathbb{R}^9) \) and \( p \in L^s(U^+_r) \) with \( \int_{U^+_r} p \, dx = 0 \) (\( 1 < s < \infty \)) such that

\[
\int_{U^+_r} A : \nabla \varphi - f \cdot \varphi \, dx = \int_{U^+_r} p \, \text{div} \, \varphi \, dx \quad \forall \, \varphi \in W^{1,s'}_{0}(U^+_{2r}). \tag{3.3}
\]

Then there exists \( c_1 > 0 \), such that \( \forall \, t \in \left[ \max\{1, \frac{3s}{s+3}\}, s \right] \),

\[
\int_{U^+_r} |p|^s \, dx \leq c_1 \int_{U^+_r} |A|^s \, dx + c_1 r^{3+s-3s/t} \left( \int_{U^+_r} |f|^t \, dx \right)^{s/t}. \tag{3.4}
\]
Lemma 3.5

Assume that $D_k A \in L^s(U_r^+; \mathbb{R}^9)$ for $k \in \{1, 2\}$. Then $D_k p \in L^s(U^+_{\rho})$ for all $0 < \rho < r$. Furthermore, there holds

$$
\int_{U_r^+} \phi^s |D_k p|^s \, dx \leq c_2 \int_{U_r^+} (\phi^s |D_k A|^s + r^{-s} |A|^s + \phi^s |f|^s) \, dx \\
+ c_2 r^{3-\frac{3s}{t}} \left( \int_{U_r^+} |f|^t \, dx \right)^{\frac{s}{t}}
$$

(3.5)

$\forall \phi \in C_0^\infty(U_r)$ such that $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq cr^{-1}$ in $U_r$. 

H.-O.Bae (Ajou University)  Steady Shear Thickening Fluid  IBM, Bordeaux, France-Korea  28 / 63
Main Proof Starts I

Approximate system

\( u, p \) weak solution, \( \chi_0 \in \partial \Omega \)

We estimate tangential and normal derivatives of \( |D(u_\varepsilon)|^q \) of the stress \( \sigma_\varepsilon \) and the pressure \( p_\varepsilon \)

\[
\begin{align*}
\nabla \cdot w_{\varepsilon,N} &= 0 & \text{in } U_{2r}^+, & \quad (4.1a) \\
-\nabla \cdot \sigma_{\varepsilon,N} + \nabla p_{\varepsilon,N} &= -U_N \cdot \nabla (U_N + w_{\varepsilon,N}) + f & \text{in } U_{2r}^+, & \quad (4.1b) \\
w_{\varepsilon,N} &= 0 & \text{on } \partial U_{2r}^+, & \quad (4.1c)
\end{align*}
\]

where

\[
\left\{
\begin{array}{l}
\sigma_{\varepsilon,N} = 2\varepsilon D(u_{\varepsilon,N}) + |D(u_{\varepsilon,N})|^{q-2}D(u_{\varepsilon,N}), \\
u_{\varepsilon,N} := U_N + w_{\varepsilon,N},
\end{array}
\right.
\]
Main Proof Starts II
Approximate system

$U_N$: mollification of $u$

$$U_N(x) := \frac{1}{N^3} \int \psi(Ny) \tilde{u}(x - y) dy,$$

$\tilde{u}$: zero-extension of $u$

$U_N$ is zero on the boundary $x_3 = 0$

Existence, Uniqueness guaranteed by the monotone operator theory
Let \( u_\varepsilon \in W^{1,q}_{0,\sigma}(\Omega) \).

cut-off functions:

\( \phi \in C_0^\infty(U_{2r}) : 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq cr^{-1} \) in \( U_{2r} \)

\( \eta \in W^{1,\infty}(0, 2r) \) with \( \eta(0) = 0 \)
Tangential derivative

Test function $-D_k(\phi D_k u_{\varepsilon,N})$, $k = 1, 2,$

\[
2\varepsilon \int_{U_{2r}^+} \phi(x)|D(D_k u_{\varepsilon,N})(x)|^2 dx + \int_{U_{2r}^+} \phi(x)|D(u_{\varepsilon,N})(x)|^{q-2}|D(D_k u_{\varepsilon,N})(x)|^2 dx \\
\leq -\sum_{i,j=1}^{3} D_j \phi(x) D_k \sigma_{ij}^{\varepsilon,N}(x) D_k u_{\varepsilon,N}^i(x) dx + \sum_{i=1}^{3} \int_{U_{2r}^+} D_i \phi(x) D_k p_{\varepsilon,N}(x) D_k u_{\varepsilon,N}^i(x) dx \\
+ \int_{U_{2r}^+} \phi(x)(D_k f_{\varepsilon,N}(x)) \cdot D_k u_{\varepsilon,N}(x) dx + \int_{U_{2r}^+} (U_N \cdot \nabla u_{\varepsilon,N}(x)) \cdot D_k(\phi(x) D_k u_{\varepsilon,N}(x)) dx \\
= l_1 + l_2 + l_3 + l_4,
\]
Normal derivative

Test function $\phi(x) \eta(x_3) D_3 D_3 u_\varepsilon, N(x)$

\[
\int_{U_2^+} (2\varepsilon + |D(u_\varepsilon, N)|^{q-2}) \phi(x) \eta(x_3) |D(D_3 u_\varepsilon, N)|^2 dx \\
\leq - \sum_{j=1}^2 \sum_{i=1}^3 \int_{U_2^+} D_j \phi(x) \eta(x_3) D_3 \sigma_\varepsilon^{ij}, N(x) D_3 u_\varepsilon^j, N(x) dx \\
+ \sum_{j=1}^2 \sum_{i=1}^3 \int_{U_2^+} (D_3 \phi(x) \eta(x_3) + \phi(x) \eta'(x_3)) D_j \sigma_\varepsilon^{ij}, N(x) D_3 u_\varepsilon^j, N(x) dx \\
+ \sum_{i=1}^2 \int_{U_2^+} D_i \phi(x) \eta(x_3) D_3 p_\varepsilon, N(x) D_3 u_\varepsilon^i, N(x) dx \\
- \sum_{i=1}^2 \int_{U_2^+} (D_3 \phi(x) \eta(x_3) + \phi(x) \eta'(x_3)) D_i \varepsilon, N(x) D_3 u_\varepsilon^i, N(x) dx. \\
- \int_{U_2^+} \phi(x) \eta(x_3) f_\varepsilon, N(x) \cdot D_3 D_3 u_\varepsilon, N(x) dx \\
+ \int_{U_2^+} \phi(x) \eta(x_3) (U_N \cdot \nabla u_\varepsilon, N(x)) \cdot D_3 D_3 u_\varepsilon, N(x) dx \\
= \| I_1 \| + \| I_2 \| + \| I_3 \| + \| I_4 \| + \| I_5 \| + \| I_6 \|. 
\] 

(4.3)
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   - Estimation of tangential derivatives
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   - $N \to \infty$
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\[ V_{\varepsilon,N}(x) := \left| D(u_{\varepsilon,N})(x) \right|^q \]

\[ |D_k \sigma_{\varepsilon,N}| \leq 2\varepsilon |D(D_k u_{\varepsilon,N})| + (q - 1) |D(u_{\varepsilon,N})|^{q-2} |D(D_k u_{\varepsilon,N})|, \quad (4.4) \]

\[ (D_k V_{\varepsilon,N})^2 \leq \frac{q^2}{4} |D(u_{\varepsilon,N})|^{q-2} |D(D_k u_{\varepsilon,N})|^2, \quad (4.5) \]

\[ D_k \sigma_{\varepsilon,N} : D(D_k u_{\varepsilon,N}) \geq 2\varepsilon |D(D_k u_{\varepsilon,N})| + \frac{4}{q^2} (D_k V_{\varepsilon,N})^2 \]

\[ M_{\varepsilon,N}(r) := \int_{U_r^+(x_0)} (\varepsilon |\nabla u_{\varepsilon,N}|^2 + |\nabla u_{\varepsilon,N}|^q + |\nabla f|^{q'} + |f|^{q'}) \, dx. \]
\[ l_1 \leq \frac{\varepsilon}{4} \int_{U_{2r}^+} \zeta^q |\nabla D_k u_{\varepsilon,N}|^2 dx \]
\[ + \frac{1}{4} \int_{U_{2r}^+} \zeta^q |D(u_{\varepsilon,N})|^{q-2} |D(D_k u_{\varepsilon,N})|^2 dx + cM_{\varepsilon,N}(2r) \]

Applying (3.5) in Lemma 3.5 with \( A = \sigma^{\varepsilon,N} \), \( s = q' \), \( t = \frac{3q}{6-q} \), \( \phi = \zeta^{q-1} \), replacing \( f \) by \( f - U_N \cdot \nabla u_{\varepsilon,N} \), \( \implies \)

\[ \int_{U_{2r}^+} \zeta^q |D_k p_{\varepsilon,N}|^{q'} dx \leq c \int_{U_{2r}^+} (\zeta^q |D_k \sigma^{\varepsilon,N}|^{q'} + |\sigma^{\varepsilon,N}|^{q'} + |f|^{q'}) dx \]
\[ + \zeta^q |U_N \cdot \nabla u_{\varepsilon,N}|^{q'} dx + c \left( \int_{U_{2r}^+} (|U_N||\nabla u_{\varepsilon,N}|)^{\frac{3q}{6-q}} dx \right)^{\frac{6-q}{3q-3}} \]
Hölder’s inequality, Sobolev-Poincaré inequality, Young’s inequality

\[ I_2 \leq \frac{\varepsilon}{4} \int_{U^+_2r} \zeta^q |\nabla D_k u_{\varepsilon,N}|^2 dx + \frac{1}{4} \int_{U^+_2r} \zeta^q |D(u_{\varepsilon,N})|^{q-2} |D(D_k u_{\varepsilon,N})|^2 dx \]

\[ + c\varepsilon M_{\varepsilon,N}(2r)^{2/q} + cM_{\varepsilon,N}(2r) + \int_{U^+_2r} \zeta^q |U_N \cdot \nabla u_{\varepsilon,N}|^{q'} dx \]

\[ + c \|\nabla U_N\|_{L^q(U^+_2r)}^{q'} \|\nabla u_{\varepsilon,N}\|_{L^q(U^+_2r)}^{q'} , \]

\[ I_3 \leq \int_{U^+_2r} (|D_k u_{\varepsilon,N}|^q + |\nabla f|^q) dx \leq M_{\varepsilon,N}(2r) , \]

\[ I_4 \leq c \int_{U^+_2r} \zeta^q |U_N|^{q'} |D_k u_{\varepsilon,N}|^{q'} dx + c \int_{U^+_2r} \zeta^q |\nabla U_N| |\nabla u_{\varepsilon,N}|^2 dx + M_{\varepsilon,N}(2r) \]
Estimates of $l_1, l_2, l_3, l_4 \Rightarrow$

$$\varepsilon \int_{U_2^+} \zeta^q |\nabla D_k u_{\varepsilon,N}|^2 dx + \int_{U_2^+} \zeta^q |D(u_{\varepsilon,N})|^{q-2} |D(D_k u_{\varepsilon,N})|^2 dx$$

$$\leq c \left( \varepsilon M_{\varepsilon,N}(2r)^{2/q} + M_{\varepsilon,N}(2r) \right)$$

$$+ c \int_{U_2^+} \zeta^q |D_k U_N| |\nabla u_{\varepsilon,N}|^2 + \zeta^q |U_N|^{q'} |\nabla u_{\varepsilon,N}|^{q'} dx$$

$$+ c \|\nabla U_N\|_{L^q(U_2^+)}^{q'} \|\nabla u_{\varepsilon,N}\|_{L^q(U_2^+)^{'}}. \quad (4.6)$$

$$K_{\varepsilon,N}(2r) := \left( \varepsilon M_{\varepsilon,N}(2r)^{2/q} + M_{\varepsilon,N}(2r) \right)$$

$$+ \int_{U_2^+} \zeta^q |\nabla U_N| |\nabla u_{\varepsilon,N}|^2 + \zeta^q |U_N|^{q'} |\nabla u_{\varepsilon,N}|^{q'} dx$$

$$+ \|\nabla U_N\|_{L^q(U_2^+)}^{q'} \|\nabla u_{\varepsilon,N}\|_{L^q(U_2^+)^{'}}.$$
Estimate (4.6) yields

\[ \int_{U_{2r}^+} \zeta^q (D_k V_{\varepsilon,N})^2 \, dx \leq cK_{\varepsilon,N}(2r), \quad (4.7) \]

\[ \int_{U_{2r}^+} \zeta^q |D_k \sigma_{\varepsilon,N}| q' + \zeta^q |D_k p_{\varepsilon,N}| q' \, dx \leq cK_{\varepsilon,N}(2r), \quad (4.8) \]

\[ \int_{U_{2r}^+} \zeta^{q-1} (|D_k \sigma_{\varepsilon,N}| + |D_k p_{\varepsilon,N}|) |\nabla u_{\varepsilon,N}| \, dx \leq cK_{\varepsilon,N}(2r) \quad (4.9) \]
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H.-O. Bae (Ajou University) 

Steady Shear Thickening Fluid

IBM, Bordeaux, France-Korea
Normal Gradient I

Let \( r > 1 \) fixed.
Into (4.3) we put \( \phi = \zeta^q, \zeta \in C_0^\infty(U_{2r}) \) cut-off

\[
\|I\|_1 \leq \frac{\varepsilon}{2} \int_{U_{2r}^+} \zeta^q(x) \eta(x_3) |D(D_3 u_{\varepsilon,N})(x)|^2 dx \\
+ \frac{1}{4} \int_{U_{2r}^+} \zeta^q(x) \eta(x_3) |D(u_{\varepsilon,N})(x)|^{q-2} |D(D_3 u_{\varepsilon,N})(x)|^2 dx + cM_{\varepsilon,N}(2r)
\]

(4.9) \Rightarrow

\[
\|I\|_2 \leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r)
\]
Normal Gradient II

Third equation of (4.1b) \(\Rightarrow\)

\[ l_{3} = \sum_{j=1}^{2} \sum_{i=1}^{2} \int_{U_{2r}^{+}} q \zeta^{q-1}(x) D_{i} \zeta(x) \eta(x_{3}) D_{j} \sigma_{3j}^{\varepsilon,N}(x) D_{3} u_{\varepsilon,N}^{i}(x) dx \]

\[ + \sum_{i=1}^{2} \int_{U_{2r}^{+}} q \zeta^{q-1}(x) D_{i} \zeta(x) \eta(x_{3}) D_{3} \sigma_{33}^{\varepsilon,N}(x) D_{3} u_{\varepsilon,N}^{i}(x) dx \]

\[ + \sum_{i=1}^{2} \int_{U_{2r}^{+}} q \zeta^{q-1}(x) D_{i} \zeta(x) \eta(x_{3}) f^{3}(x) D_{3} u_{\varepsilon,N}^{i}(x) dx \]

\[ + \sum_{i=1}^{2} \int_{U_{2r}^{+}} q \zeta^{q-1}(x) D_{i} \zeta(x) \eta(x_{3}) U_{N} \cdot \nabla u_{\varepsilon,N}^{3}(x) D_{3} u_{\varepsilon,N}^{i}(x) dx. \]
Normal Gradient III

First term on the right like $\|I\|_1$,
Second on the right like $\|I\|_2$,
Third and Fourth term by Hölder’s inequality

$$\|I\|_3 \leq \frac{\varepsilon}{2} \int_{U^+_2} \zeta^q(x)\eta(x_3)|D(D_3u_{\varepsilon,N})(x)|^2 \, dx$$

$$+ \frac{1}{4} \int_{U^+_2} \zeta^q(x)\eta(x_3)|D(u_{\varepsilon,N})(x)|^{q-2}|D(D_3u_{\varepsilon,N})(x)|^2 \, dx + cK_{\varepsilon,N}(2r).$$

Like $\|I\|_2$, (4.9) $\implies$

$$\|I\|_4 \leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r).$$

$$\|I\|_5 \leq K_{\varepsilon,N}(2r).$$
Finally, like $I_4$, we obtain

$$\|I_6\| \leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r).$$

\(\|I_1-I_6\| \Rightarrow\)

$$\int_{U_{2r}^+} (2\varepsilon + |D(u_{\varepsilon,N}(x)|^{q-2}) \zeta^q(x)\eta(x_3)|D(D_3u_{\varepsilon,N}(x)|^2 dx$$

$$\leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r).$$  \hspace{1cm} (4.10)

In particular,

$$\int_{U_{2r}^+} \zeta^q(x)\eta(x_3)(D_3V_{\varepsilon,N}(x))^2 dx \leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r).$$  \hspace{1cm} (4.11)
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Tangential derivative estimate (4.7)

\[ \int_{U_{2r}^+} \zeta^q (D' V_{\epsilon,N})^2 \, dx \leq cK_{\epsilon,N}(2r) \]

Normal derivative estimate (4.11) with \( \eta(t) = t \)

\[ \int_{U_{2r}^+} \zeta^q (D_3 V_{\epsilon,N})^2 x_3 \, dx \leq cK_{\epsilon,N}(2r). \]
\[
\int_{U_{2r}^+} \zeta^{3q/2} V_{\varepsilon,N}^3 \, dx \leq cK_{\varepsilon,N}(2r)^{3/2} \tag{4.12}
\]

(4.12), (4.6), (4.8)
\[
\begin{align*}
\|\eta\|_{L^3}^{1/q} K_{\varepsilon,N}(2r)^{1/q'} & \|\zeta \nabla u_{\varepsilon,N}\|_{L^{3q/2}(U_{2r}^+)} \\
& \leq c(1 + \|\eta'\|_{L^3}^{1/q}) K_{\varepsilon,N}(2r). 
\end{align*}
\]

\[
\int_{U_{2r}^+} \zeta^q(x) \eta(x_3)(D_3 V_{\varepsilon,N}(x))^2 \, dx \leq c(1 + \|\eta'\|_{L^3}^{1/q}) K_{\varepsilon,N}(2r) \tag{4.13}
\]

for all \( \eta \in W^{1,3}(0, 2r) \) with \( \eta(0) = 0 \) and \( \eta \geq 0 \) in \( (0, 2r) \).
For $\eta(t) = t^{8/9}$, $V_{\varepsilon,N} \in W^{1,2}(U^+_r; \mathbb{R}^{8/9})$.

Lemma 3.1 (Weighted Inequality) ⇒

$$\int_{B'_{2r}} \zeta^q(x', x_3) |D(u_{\varepsilon,N})(x', x_3)|^q \, dx' \leq K_{\varepsilon,N}(2r) \quad (4.14)$$

⇒

$$\int_{U^+_r} \zeta^q(x) |D(u_{\varepsilon,N})(x)|^q x_3^{-\alpha} \, dx \leq c(1 - \alpha)^{-1} K_{\varepsilon,N}(2r) \quad (4.15)$$
Sobolev-Poincaré inequality and (4.12) implies

$$\left\| \zeta(u_\varepsilon - (u_\varepsilon,N)_{U_2^+}) \right\|_{L^\infty(U_2^+)}^q \leq K_{\varepsilon,N}(2r) \quad (4.16)$$

$$\Rightarrow \zeta u_\varepsilon,N \in L^q(U_2^+; x_3^{-\alpha}) \quad \forall \alpha \in (0, 1)$$

Weighted Korn’s inequality, (4.15) and (4.16) \Rightarrow

$$\int_{U_2^+} \zeta^q(x) \left| \nabla u_\varepsilon,N(x) \right|^q x_3^{-\alpha} dx \leq K_{\varepsilon,N}(2r) \quad (4.17)$$
Estimate on $D_k\sigma^{\varepsilon,N}(k = 1, 2)$

(4.4), Hölder's inequality, Young's inequality,

$$
\int_0^{2r} \left( \int_{B'_{2r}} \zeta^q(x', x_3) |D_k\sigma^{\varepsilon,N}(x', x_3)|^{q'} \, dx' \right)^{2/q'} \, dx_3 \leq cK_{\varepsilon,N}(2r)^{2/q'}
$$

$$
\int_{U^+_2} \zeta^q(x) |D_k\sigma^{\varepsilon,N}(x)|^{q'} x_3^{-\beta} \, dx' \leq cK_{\varepsilon,N}(2r) \tag{4.18}
$$

for all $\beta \in \left(0, 1 - \frac{q'}{2}\right)$, with $c > 0$ depending only on $\beta$ and $q$. 
Estimate on $D_k p^\varepsilon, N (k = 1, 2)$

$D_k$ to (4.1b), multiply the result by $\zeta^q$ →

$$
\sum_{i,j=1}^{3} \int_{U^+_2} g_{ij} D_j \varphi^i \, dx = \int_{U^+_2} \zeta^q D_k p^\varepsilon, N \text{ div } \varphi \, dx \quad \forall \varphi \in C^\infty_0 (U^+_2),
$$

where

$$
g_{ij}(x) = \zeta^q(x) D_k \sigma_{ij}^\varepsilon, N(x) - \delta_{jk} \zeta^q(x) U_N(x) \cdot \nabla u^i_{\varepsilon, N}(x)
$$

$$
- q \delta_{3j} \int_0^{x_3} D_i \zeta(x', t) \zeta^{q-1}(x', t) D_k \sigma_{ij}^\varepsilon, N(x', t) \, dt
$$

$$
+ q \delta_{3j} \int_0^{x_3} D_i \zeta(x', t) \zeta^{q-1}(x', t) D_k p^\varepsilon, N(x', t) \, dt
$$

$$
+ \delta_{3j} \int_0^{x_3} \zeta^q(x', t) D_k f^i(x', t) \, dt
$$

$$
- q \delta_{3j} \int_0^{x_3} D_k \zeta(x', t) \zeta^{q-1}(x', t) U_N(x', t) \cdot \nabla u^i_{\varepsilon, N}(x', t) \, dt
$$
Estimate on $Dkp^{\varepsilon,N}(k = 1, 2)$ II

(4.18) and (4.6) ⇒

$$
\int_{U_{2r}^+} |g(x)|^{q'} x_3^{-\beta} dx \leq cK_{\varepsilon,N}(2r) + c\|\nabla U_N\|_{L^q(U_{2r}^+)}^{q'} K_{\varepsilon,N}(2r)^{\frac{1}{q-1}},
$$

Theorem 3.3(pressure esitmate), (4.8) ⇒

$$
\int_{U_{2r}^+} \zeta^q(x)|Dkp^{\varepsilon,N}(x)|^{q'} x_3^{-\beta} dx
\leq cK_{\varepsilon,N}(2r) + c\|\nabla U_N\|_{L^q(U_{2r}^+)}^{q'} K_{\varepsilon,N}(2r)^{\frac{1}{q-1}}
$$

(4.19)
Estimates of $\|2\|$ and $\|4\|$ 

$\lambda \in \left( \frac{q' - 1}{2}, \frac{1}{2} \right)$ arbitrarily chosen

$$\alpha := \frac{1}{2} + \lambda, \quad \beta := \frac{1}{2} + \lambda - \frac{q'}{2}, \quad \frac{\alpha}{q} + \frac{\beta}{q'} = \lambda.$$ 

Test function: $\phi(x) = \zeta^4(x)$, $\eta = \eta_\theta(t) := t^{1-\lambda}$, $t \in [0, 2r]$

$$\|2\| \leq cK_{\varepsilon,N}(2r),$$

$$\|4\| \leq cK_{\varepsilon,N}(2r) + c\|\nabla U_N\|_{L^q(U^+_r)} K_{\varepsilon,N}(2r)^{\frac{2}{q}}$$
Bound of Approximation Solution

\[
\int_{U_2^+} \zeta^q |D(u_{\varepsilon,N})(x)|^{q-2} |D(D_3 u_{\varepsilon,N})(x)|^2 x_3^{1-\lambda} \, dx
\]
\[
+ \int_{U_2^+} \zeta^q (D_3 V_{\varepsilon,N}(x))^2 x_3^{1-\lambda} \, dx
\]
\[
\leq cK_{\varepsilon,N}(2r) + c \|\nabla U_N\|_{L^q(U_2^+)} K_{\varepsilon,N}(2r)^{\frac{2}{q}},
\]  
(4.20)
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By means of reflexivity, Minty-trick, there exist \( w_N \in W_{0,\sigma}^{1,q}(U_{2r}^+) \), and a sequence \( \varepsilon_m \to 0 \):

\[
\begin{align*}
\nabla \cdot w_N &= 0 \quad \text{in} \quad U_{2r}^+, \\
-\nabla \cdot \sigma^N + \nabla p_N &= -U_N \cdot \nabla u_N + f \quad \text{in} \quad U_{2r}^+, \\
w_N &= 0 \quad \text{on} \quad \partial U_{2r}^+.
\end{align*}
\]
Main Proof: Gradient estimates of stress near boundary

\( \varepsilon \to 0 \)

Testing (4.21b) with \( w_N \),

\[
\int_{U^+_2} \sigma_N : D(w_N) \, dx = \int_{U^+_2} U_N \otimes U_N : \nabla w_N + f \cdot w_N \, dx. \tag{4.22}
\]

Observing

\[
\sigma_N : D(w_N) \geq |D(w_N)|^q - |D(U_N)|^{q-2} D(U_N) : D(w_N),
\]

Hölder’s inequality, Sobolev-Poincaré inequality, Young’s inequality,

\( \Rightarrow \)

\[
\int_{U^+_2} |D(w_N)|^q \, dx \leq c \left( \|D(U_N)\|_{L^q(U^+_2)}^q + r^{\frac{5q-9}{q-1}} \|\nabla U_N\|_{L^q(U^+_2)}^{\frac{2q}{q-1}} + \|f\|_{L^{q'}(U^+_2)}^{q'} \right)
\]
Lower semi-continuity, (4.8), (4.7)

\[ \int_{U_{2r}^+} \zeta^q |D' \rho_N|^{q'} + \zeta^q |D' \sigma_N|^{q'} + \zeta^q |D' V_N|^2 \ dx \leq cK_N(2r), \]

\[ \int_{U_{2r}^+} (\zeta^q |D_3 \sigma_N|^{q'} + \zeta^q (D_3 V_N)^2) x_3^\alpha \ dx \leq cK_N(2r) \]

for all \( \alpha > \frac{1}{2} \), where

\[ V_N(x) = |D(u_N)(x)|^{q/2}. \]
\[ \omega(t) = t^{\frac{2\alpha+1}{4\alpha}}, \quad (t \in [0, 1]) \]
\[ \zeta^{q/2} V \in W^{1,2}_r(U_{2r}^+; \omega^\alpha(x_3)) \cap L^{2(3-\alpha)}(U_1^+) \]

By Proposition 3.2 (Weighted Sobolev inequality),

\[
\int_{U_{2r}^+} (\zeta^{q/2} V_N)^{2(3-\alpha)} \, dx \leq cK_N(2r)^{3-\alpha},
\]
\[
K_N(2r) \leq cM_N(2r) + c\|\nabla U_N\|_{L^q(U_{2r}^+)}^{q/(q-2)}.
\]
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Main Proof: Gradient estimates of stress near boundary

\((w_N, p_N) \to (u, p)\)

\[M_N(2r) + \|\nabla U_N\|_{L^q(U_{2r}^+)}^{q/(q-2)}\]

\[\to K(2r) := \int_{U_{2r}^+} |\nabla u|^q + |\nabla f|^q' + |f|^{q'} dx + \|\nabla u\|_{L^q(U_{2r}^+)}^{q/(q-2)}\]

\[\int_{U_{2r}^+} \zeta q' |D' p|^q' + \zeta q' |D' \sigma|^q' + \zeta q |D' V|^2 dx \leq cK(2r), \tag{4.23}\]

\[\int_{U_{2r}^+} (\zeta q' |D_3 \sigma_N|^q' + \zeta q (D_3 V)^2) x_3^\alpha dx \leq cK(2r) \tag{4.24}\]

for all \(\alpha \in \left(\frac{1}{2}, 1\right)\), where \(V(x) = |D(u)(x)|^{q/2}\).

\[\int_{U_{2r}^+} \zeta q(3-\alpha) |\nabla u|^{q(3-\alpha)} dx \leq cK(2r)^{3-\alpha}. \tag{4.25}\]
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(4.25) and standard covering argument, for all $\alpha \in \left(\frac{1}{2}, 1\right)$

\[
\int_{\Omega} |\nabla u|^{q(3-\alpha)} \, dx \leq c\left( \|\nabla u\|_{L^q(\Omega)}^q + \|\nabla u\|_{L^q(\Omega)}^{q/(q-2)} + \|f\|_{W^{1,q}(\Omega)}^{q'} \right)^{3-\alpha} \\
\leq c \left( 1 + \|f\|^{q/(q-2)} \right)^{3-\alpha},
\]

where $\|f\| := \left( \|f\|_{W^{1,q}}^{q'} + \|f\|_{L^{3q/(4q-3)}}^{q'} \right)^{1/q'}$.

$\Rightarrow$ (2.4), (2.5),

(4.23), (4.24) $\Rightarrow$

\[
\int_{\Omega} |D'p|^{q'} + |D'\sigma|^{q'} + |D'V|^2 \, dx \leq c \left( 1 + \|f\|^{q/(q-2)} \right), \\
\int_{\Omega} (|D_{3\sigma N}|^{q'} + (D_{3V})^2) \min\{x_3^\alpha, 1\} \, dx \leq c \left( 1 + \|f\|^{q/(q-2)} \right)
\]

$\Rightarrow$ (2.2) and (2.3).

This completes the proof of Theorem