Boundary Regularity for the Steady Shear Thickening Flow

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Inaugural France-Korea Conference on

Algebraic Geometry, Number Theory, and Partial Differential Equations

Nov. 24-27, 2019

Outline

Stokes type non-Newtonian system

$$-\nabla \cdot \boldsymbol{\sigma} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.2}$$

$$\text{BC:} \qquad \boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \tag{1.3}$$

- $\mathbf{u} = (u^1, u^2, u^3)^\top$: unknown velocity
- p: uinknown pressure
- $\mathbf{f}$ given external force
- $\boldsymbol{\sigma} = (\sigma_{ij})$ shear stress;

$$\sigma_{ij} = |\mathbf{D}(\mathbf{u})|^{q-2} D_{ij}(\mathbf{u}), \qquad \mathbf{D}(\mathbf{u}) = \frac{1}{2}(D\mathbf{u} + (D\mathbf{u})^T)$$

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1. Introduction
   - Power-law non-Newtonian system
   - Review of Incompressible Fluids
   - Known Facts
2. Main Result
   - Definitions of Weak and Strong Solutions
   - Main Theorem
3. Weighted Sobolev Embedding
4. Main Proof: Gradient estimates of stress near boundary
   - Estimation of tangential derivatives
   - Estimation of derivative in normal direction y_3
   - Higher L^p estimate of $\nabla \boldsymbol{u}$ via weighted Sobolev embedding
   - $\varepsilon \to 0$
   - $N \to \infty$
   - Proof of Main Theorem

Motivation

(a) Shear thickening fluid (oobleck) (b) Shear thinning fluid (whipped cream)

Figure: Movie for non-Newtonian Fluids

Nonsteady Incompressible Fluids

$$
\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\boldsymbol{\sigma} + \nabla p = \mathbf{f},\\
&\operatorname{div}\mathbf{u} = 0,\\
&\text{IC}: \quad \mathbf{u}|_{t=0} = \mathbf{a},\\
&\text{BC}: \quad \mathbf{u}|_{\partial\Omega} = 0.
\end{aligned}
$$

a : given initial data

Two type of power-law models

$$\boldsymbol{\sigma} = 2|D|^{q-2}D, \tag{1.4}$$

$$\boldsymbol{\sigma} = (\mu_0 + \mu_1|D|^{q-2})D, \tag{1.5}$$

- For $\mu_1 = 0$ or $q = 2$, Navier-Stokes equations.
- the non-Newtonian fluids: nonlinear relation between the shear stress and the strain rate
  — some fluids such as ketchup, starch, blood
- The case $q > 2$ describes dilatant (shear thickening) fluids whose viscosity increases with the rate of shear. (movie (a))
- pseudo-plastic (shear thinning) fluids correspond to the case $1 < q < 2$, where viscosity decreases with increasing rate of shear. (movie (b))

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$$\begin{cases} \text{Newtonian} & \text{for } \mu_0 > 0,\ \mu_1 = 0, \\ \text{Rabinowitsch} & \text{for } \mu_0, \mu_1 > 0,\ \text{and } q = 4, \\ \text{Smagorinsky} & \text{for } \mu_0, \mu_1 > 0,\ \text{and } q = 3, n = 3 \\ \text{Ellis} & \text{for } \mu_0, \mu_1 > 0,\ \text{and } q > 2, \\ \text{Ostwald-de Waele} & \text{for } \mu_0 = 0,\ \mu_1 > 0,\ \text{and } q > 1, \\ \text{Bingham} & \text{for } \mu_0, \mu_1 > 0,\ \text{and } q = 1. \end{cases}$$

◦ paper pulp $\mu_1 = 0.418$, $q = 1.575$
◦ carboxymethyl cellulose(CMC) in water $\mu_1 = 0.194$, $q = 1.566$.

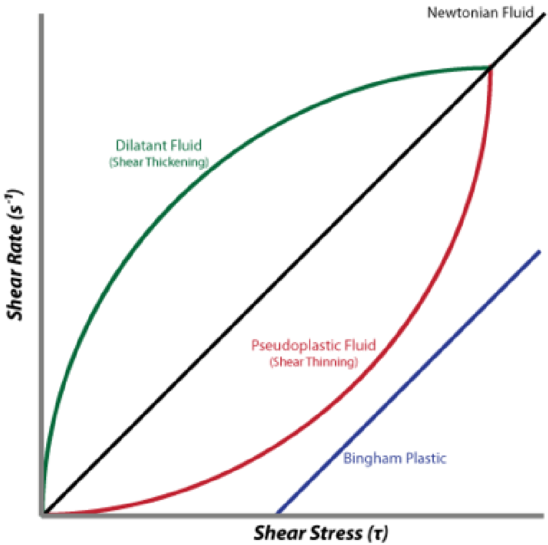

Figure: non-Newtonian fluids (From Wikipedia...)

Non-Newtonian, viscoelastic properties

Viscoelastic	Kelvin Material, Maxwell material	"Parallel" linearstic combination of elastic and viscous effects	Some lubricants, whipped cream, Silly Putty
Time dependent viscosity	Rheopecty	Apparent viscosity increases with duration of stress	printer ink, gypsum paste
	Trixotropic	Apparent viscosity decreases with duration of stress	Yogurt, xanthan gum solutions, aqueous iron oxide gels, gelatin gels, pectin gels, synovial fluid, hydrogenated castor oil, some clays (bentonite, montmorillonite), carbon black suspension in molten tire rubber, some drilling muds, many paints, many floc suspensions, many colloidal suspensions
Time independent viscosity	shear thickening (dilatant)	Apparent viscosity increases with increased stress	Suspensions of corn starch in water(oobleck), sand in water
	shear thinning (pseudoplastic)	Apparent viscosity decreases with increased stress	nail polish, whipped cream, ketchup, molasses, syrups, paper pulp in water, latex paint, ice, blood, some silicone oils, some silicone coatings
	generalized Newtonian fluids	Viscosity is constant. Stress depends on normal and shear strain rates and also pressure applied on it	blood plazma, custard, water

(from Wikipedia)

COMMENTS

◦ Similar models:

$$T_{ij} = \left(\mu_0 + \mu_1 |D|^2\right)^{(q-2)/2} D_{ij} \qquad (1.6)$$

$$T_{ij} = \left(\mu_0 + \mu_1 |D|\right)^{q-2} D_{ij}$$

◦ p-Laplacian equations
◦ Korns' inequality
◦ asymptotic behaviors for $q > 2$, $q = 2$ and $q < 2$.

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Outline

Periodic Domain

- Existence of space-periodic weak solution for $q > 1$ if $n = 2$ and $q > \frac{6}{5}$ if $n = 3$.
- Space-periodic weak solution is strong and unique for $q > 1$ if $n = 2$ and for $q > \frac{11}{5}$ if $n = 3$.
- Existence of space-periodic strong solution for small data is known for $q > \frac{5}{3}$ if $n = 3$.

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No Slip BC

- With no slip BC, the existence of weak solution for $q > 1$ if $n = 2$ and $q > \frac{6}{5}$ if $n = 3$
  (Wolf 2007, Diening,Ruzicka,Wolf 2010)
- Weak solution is strong for $q > \frac{9}{4}$ if $n = 3$ for (1.5).

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Steady Case

Apushkinskaya, Bildhauer, Fuchs (2005) $C^{1,\alpha}$ regularity for (1.6) when $\Omega \in \mathbb{R}^2$.
Beirao da Veiga (2009) boundary regularity for (1.5)

Steady Stokes type shear thickening viscosity flow: Wolf-B. (2015), Boundary Holder regularity

Outline

$\boldsymbol{C}^{\infty}_{0,\sigma}(\Omega)$: the space of all solenoidal smooth vector fields in $\mathbb{R}^3$ having its support in Ω.

$\boldsymbol{L}^s(\Omega)$, $\boldsymbol{W}^{1,s}_0(\Omega)$: usual Sobolev spaces

$$\boldsymbol{L}^s_\sigma(\Omega) = \textit{completion of } \boldsymbol{C}^{\infty}_{0,\sigma}(\Omega) \textit{ in } \boldsymbol{L}^s(\Omega),$$
$$\widehat{\boldsymbol{W}}^{1,s}_{0,\sigma}(\Omega) = \textit{completion of } \boldsymbol{C}^{\infty}_{0,\sigma}(\Omega) \textit{ in } \boldsymbol{W}^{1,s}_0(\Omega) \textit{ with } \|\nabla \boldsymbol{u}\|_{L^s}$$

Definition 2.1

Let $\boldsymbol{f} \in \boldsymbol{L}^1(\Omega)$. $\boldsymbol{u} \in \widehat{\boldsymbol{W}}^{1,q}_{0,\sigma}(\Omega)$ *a weak solution of* (1.1)-(1.3), *if*

$$\int_\Omega |\boldsymbol{D}(\boldsymbol{u})|^{q-2}\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\varphi)dx - \boldsymbol{u} \otimes \boldsymbol{u} : \nabla\varphi = \int_\Omega \boldsymbol{f} \cdot \varphi dx \quad \forall \varphi \in C^\infty_{0,\sigma}(\Omega). \tag{2.1}$$

Definition 2.2

Weak solution $(\boldsymbol{u}, p) \in \widehat{\boldsymbol{W}}^{1,q}_{0,\sigma}(\Omega) \times L^{q'}_{loc}(\bar{\Omega})$ *is called a strong solution if*

$$\sigma_{ij} \in \boldsymbol{W}^{1,s}(\Omega), \quad D_i p \in L^s(\Omega) \text{ for some } s \geq 1.$$

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   - Proof of Main Theorem

Main Theorem I

Assume $2 < q < 3$. Let $\boldsymbol{f} \in \boldsymbol{L}^{3q/(4q-3)}(\Omega) \cap \boldsymbol{W}^{1,q'}(\Omega)$. Then for every weak solution $\boldsymbol{u} \in \widehat{\boldsymbol{W}}^{1,q}_{0,\sigma}(\Omega)$ to (1.1)–(1.3) there holds

$$\int_\Omega (|D'\boldsymbol{\sigma}|^{q'} + |D'p|^{q'} + |D'|\boldsymbol{D}(\boldsymbol{u})|^{q/2}|^2)dx \le c(1 + |||\boldsymbol{f}|||^{\frac{q'}{q-2}}), \qquad (2.2)$$

$$\int_\Omega \left(|D_3\boldsymbol{\sigma}|^{q'} + |D_3 p|^{q'} + (D_3|\boldsymbol{D}(\boldsymbol{u})|^{q/2})^2\right)(x_3^\alpha \wedge 1)dx \le c(1 + |||\boldsymbol{f}|||^{\frac{q'}{q-2}}), \qquad (2.3)$$

($\alpha > \frac{1}{2}$), where

$$|||\boldsymbol{f}||| := \left(\|\boldsymbol{f}\|^{q'}_{\boldsymbol{W}^{1,q'}} + \|\boldsymbol{f}\|^{q'}_{\boldsymbol{L}^{3q/(4q-3)}}\right)^{1/q'}.$$

Main Theorem II

In addition, we have

$$\nabla \boldsymbol{u} \in \boldsymbol{L}^s(\Omega) \qquad \forall\, 1 \le s < \frac{5q}{2}, \tag{2.4}$$

$$D_3\sigma_{ij}, D_3 p \in L^s(\Omega) \qquad \forall\, 1 \le s < \frac{2q'}{3} \quad (i,j = 1,2,3). \tag{2.5}$$

In particular, we conclude that $(\boldsymbol{u}, p)$ is a strong solution to (1.1)–(1.3) and Hölder continuous,

$$\boldsymbol{u} \in \boldsymbol{C}^{\gamma}(\overline{\Omega}) \quad \forall\, 0 \le \gamma < 1 - \frac{6}{5q}$$

by Morrey's inequality.

IDEA of Proof of Main Theorem

Approximate Equation by adding $\varepsilon\Delta$

$\rightarrow$

Tangential Derivative Estimation

+ Normal Derivative Estimation with Weights near Boundary

+ Weighted Estimate near boundary (Proposition 3.2)

$\Rightarrow$

Boundary regularity

+ Morrey's inequality

$\rightarrow$

Hölder continuity

Weighted Inequality

Lemma 3.1

Let $\omega \in C([0,1])$ *be a positive function with* $\omega^{-1} \in L^1_{loc}([0,1]) \cap L^s_{loc}((0,1])$*.*
Then for all $f \in C^1([0,1])$ *with* $f(1) = 0$*, we have*

$$|f(x)| \leq \Big(\int_x^1 \omega^{-1} dy \Big)^{\frac{1}{s'}} \Big(\int_x^1 \omega^{s-1} |f'|^s dy \Big)^{\frac{1}{s}} \qquad \forall x \in (0,1). \tag{3.1}$$

Weighted Sobolev space

$$B_r' := B_r'(x_0) := \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - x_{0,1})^2 + (x_2 - x_{0,2})^2 < r^2\}$$
$$U_r := B_r' \times (-r, r)$$
$$U_r^+ := B_r' \times (0, r)$$

$$W_\Gamma^{1,2}(U^+; \omega) := \Big\{\phi \in W_{loc}^{1,2}(U^+ \cup (\partial U)^+) \,;\, \|\phi\|_{W_\Gamma^{1,2}(U^+;\omega(y_3))} < \infty, \quad \phi\big|_{\partial U \cap \mathbb{R}^+} = 0\Big\}$$

$$\|\varphi\|_{W_\Gamma^{1,2}(U_1^+;\omega(y_3))} = \|\nabla'\varphi\|_{2;U^+} + \left(\int_{U^+} [\varphi_{y_3}]^2 \omega(y_3) dy\right)^{\frac{1}{2}}$$

Proposition 3.2

Let $\omega \in C([0,1])$ be a weight function such that $\omega^{-1} \in L^1(0,1)$. Then, the space $W_\Gamma^{1,2}(U^+;\omega^\alpha)$ is continuously embedded into $L^{2(3-\alpha)}(U^+)$.
In addition,

$$\int_{U^+} \varphi^{2(3-\alpha)} dy \le c\left(\int_{U^+} |\nabla'\varphi|^2 dy\right)^{2-\alpha} \int_{U^+} [\varphi_{y_3}]^2 \omega^\alpha(y_3) dy \qquad (3.2)$$

for all $\varphi \in W_\Gamma^{1,2}(U^+;\omega^\alpha)$.

Later, for the main proof, we take $\varphi = |\boldsymbol{Du}|^{\frac{q}{2}}$.

Pressure Estimate

Theorem 3.3 (Wolf-B. 2015)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\omega \in A_{q'}$. Let $g_{ij}, p \in L^{q'}(\Omega;\omega)$ $(i,j = 1,\ldots,n)$, such that

$$\sum_{i,j=1}^{n} \int_{\Omega} g_{ij} D_j \varphi^i dx = \int_{\Omega} p \,\mathrm{div}\, \varphi dx \quad \forall \varphi \in \boldsymbol{C}_0^{\infty}(\Omega).$$

Then, the following estimate holds true

$$\|p - p_{\Omega}\|_{L^{q'}(\Omega;\omega)} \le c_{\omega} \|\boldsymbol{g}\|_{\boldsymbol{L}^{q'}(\Omega;\omega)}.$$

For pressure estimates, we need the following lemma.

Lemma 3.4

Let $\boldsymbol{A} \in L^s(U_r^+;\mathbb{R}^9), \boldsymbol{f} \in L^s(U_r^+;\mathbb{R}^9)$ *and* $p \in L^s(U_r^+)$ *with* $\int_{U_r^+} p dx = 0$ $(1 < s < \infty)$ *such that*

$$\int_{U_r^+} \boldsymbol{A} : \nabla\varphi - \boldsymbol{f}\cdot\varphi dx = \int_{U_r^+} p \,\mathrm{div}\,\varphi dx \quad \forall\,\varphi \in \boldsymbol{W}_0^{1,s'}(U_{2r}^+). \tag{3.3}$$

Then there exists $c_1 > 0$*, such that* $\forall t \in \left[\max\{1, \frac{3s}{s+3}\}, s\right]$,

$$\int_{U_r^+} |p|^s dx \le c_1 \int_{U_r^+} |\boldsymbol{A}|^s dx + c_1 r^{3+s-\frac{3s}{t}} \left(\int_{U_r^+} |\boldsymbol{f}|^t dx \right)^{\frac{s}{t}}. \tag{3.4}$$

Lemma 3.5

Assume that $D_k\boldsymbol{A} \in L^s(U_r^+;\mathbb{R}^9)$ *for* $k \in \{1,2\}$.
Then $D_k p \in L^s(U_\rho^+)$ *for all* $0 < \rho < r$.
Furthermore, there holds

$$\int_{U_r^+} \phi^s |D_k p|^s dx \leq c_2 \int_{U_r^+} (\phi^s |D_k \boldsymbol{A}|^s + r^{-s}|\boldsymbol{A}|^s + \phi^s|\boldsymbol{f}|^s)dx + c_2 r^{3-\frac{3s}{t}} \left(\int_{U_r^+} |\boldsymbol{f}|^t dx \right)^{\frac{s}{t}} \tag{3.5}$$

$\forall \phi \in C_0^\infty(U_r)$ *such that* $0 \leq \phi \leq 1$ *and* $|\nabla\phi| \leq cr^{-1}$ *in* U_r.

Main Proof Starts I

Approximate system

$\boldsymbol{u}, p$ weak solution, $x_0 \in \partial\Omega$

We estimate tangential and normal derivatives of $|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^{\frac{q}{2}}$ of the stress $\boldsymbol{\sigma}_\varepsilon$ and the pressure p_ε

$$\nabla \cdot \boldsymbol{w}_{\varepsilon,N} = 0 \quad \text{in} \quad U_{2r}^+, \tag{4.1a}$$

$$-\nabla \cdot \boldsymbol{\sigma}^{\varepsilon,N} + \nabla p_{\varepsilon,N} = -\boldsymbol{U}_N \cdot \nabla(\boldsymbol{U}_N + \boldsymbol{w}_{\varepsilon,N}) + \boldsymbol{f} \quad \text{in} \quad U_{2r}^+, \tag{4.1b}$$

$$\boldsymbol{w}_{\varepsilon,N} = 0 \quad \text{on} \quad \partial U_{2r}^+, \tag{4.1c}$$

where

$$\begin{cases} \boldsymbol{\sigma}^{\varepsilon,N} = 2\varepsilon \boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N}) + |\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})|^{q-2}\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N}), \\ \boldsymbol{u}_{\varepsilon,N} := \boldsymbol{U}_N + \boldsymbol{w}_{\varepsilon,N}, \end{cases} \tag{4.2}$$

Main Proof Starts II

Approximate system

$\boldsymbol{U}_N$: mollification of $\boldsymbol{u}$

$$\boldsymbol{U}_N(x) := \frac{1}{N^3} \int \psi(Ny)\tilde{\boldsymbol{u}}(x-y)dy,$$

$\tilde{\boldsymbol{u}}$: zero-extension of $\boldsymbol{u}$
$\boldsymbol{U}_N$ is zero on the boundary $x_3 = 0$

Existence, Uniqueness guaranteed by the monotone operator theory

Let $\boldsymbol{u}_\varepsilon \in \boldsymbol{W}^{1,q}_{0,\sigma}(\Omega)$.

cut-off functions:
$\phi \in C_0^\infty(U_{2r})$: $0 \le \phi \le 1$ and $|\nabla\phi| \le cr^{-1}$ in U_{2r}
$\eta \in W^{1,\infty}(0, 2r)$ with $\eta(0) = 0$

Tangential derivative

Test function $-D_k\left(\phi D_k \boldsymbol{u}_{\varepsilon,N}\right)$, $k = 1, 2$,

$$
\begin{aligned}
&2\varepsilon \int_{U_{2r}^+} \phi(x)|\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})(x)|^2 dx \\
&\qquad + \int_{U_{2r}^+} \phi(x)|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x)|^{q-2}|\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})(x)|^2 dx \\
&\leq -\sum_{i,j=1}^{3} \int_{U_{2r}^+} D_j\phi(x) D_k \sigma_{ij}^{\varepsilon,N}(x) D_k u^i_{\varepsilon,N}(x) dx \\
&\qquad + \sum_{i=1}^{3} \int_{U_{2r}^+} D_i\phi(x) D_k p_{\varepsilon,N}(x) D_k u^i_{\varepsilon.N}(x) dx \\
&\qquad + \int_{U_{2r}^+} \phi(x)(D_k \boldsymbol{f}_{\varepsilon,N}(x)) \cdot D_k \boldsymbol{u}_{\varepsilon,N}(x) dx \\
&\qquad + \int_{U_{2r}^+} (\boldsymbol{U}_N \cdot \nabla \boldsymbol{u}_{\varepsilon,N}(x)) \cdot D_k\left(\phi(x) D_k \boldsymbol{u}_{\varepsilon,N}(x)\right) dx \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

Normal derivative

Test function $\phi(x)\eta(x_3)D_3D_3\boldsymbol{u}_{\varepsilon,N}(x)$

$$\begin{aligned}
&\int_{U_{2r}^+}(2\varepsilon+|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})|^{q-2})\phi(x)\eta(x_3)|\boldsymbol{D}(D_3\boldsymbol{u}_{\varepsilon,N})|^2dx\\
&\leq -\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{U_{2r}^+}D_j\phi(x)\eta(x_3)D_3\sigma_{ij}^{\varepsilon,N}(x)D_3u^i_{\varepsilon,N}(x)dx\\
&\quad+\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{U_{2r}^+}(D_3\phi(x)\eta(x_3)+\phi(x)\eta'(x_3))D_j\sigma_{ij}^{\varepsilon,N}(x)D_3u^i_{\varepsilon,N}(x)dx\\
&\quad+\sum_{i=1}^{2}\int_{U_{2r}^+}D_i\phi(x)\eta(x_3)D_3p_{\varepsilon,N}(x)D_3u^i_{\varepsilon,N}(x)dx\\
&\quad-\sum_{i=1}^{2}\int_{U_{2r}^+}(D_3\phi(x)\eta(x_3)+\phi(x)\eta'(x_3))D_ip_{\varepsilon,N}(x)D_3u^i_{\varepsilon,N}(x)dx.\\
&\quad-\int_{U_{2r}^+}\phi(x)\eta(x_3)\boldsymbol{f}_{\varepsilon,N}(x)\cdot D_3D_3\boldsymbol{u}_{\varepsilon,N}(x)dx\\
&\quad+\int_{U_{2r}^+}\phi(x)\eta(x_3)(\boldsymbol{U}_N\cdot\nabla\boldsymbol{u}_{\varepsilon,N}(x))\cdot D_3D_3\boldsymbol{u}_{\varepsilon,N}(x)dx\\
&=I\!I_1+I\!I_2+I\!I_3+I\!I_4+I\!I_5+I\!I_6. \qquad (4.3)
\end{aligned}$$

Outline

$$V_{\varepsilon,N}(x) := |\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x)|^{\frac{q}{2}}$$

$$|D_k \boldsymbol{\sigma}^{\varepsilon,N}| \leq 2\varepsilon |\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})| + (q-1)|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})|^{q-2}|\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})|, \tag{4.4}$$

$$(D_k V_{\varepsilon,N})^2 \leq \frac{q^2}{4}|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})|^{q-2}|\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})|^2, \tag{4.5}$$

$$D_k \boldsymbol{\sigma}^{\varepsilon,N} : \boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N}) \geq 2\varepsilon |\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})| + \frac{4}{q^2}(D_k V_{\varepsilon,N})^2$$

$$M_{\varepsilon,N}(r) := \int_{U_r^+(x_0)} (\varepsilon |\nabla \boldsymbol{u}_{\varepsilon,N}|^2 + |\nabla \boldsymbol{u}_{\varepsilon,N}|^q + |\nabla \boldsymbol{f}|^{q'} + |\boldsymbol{f}|^{q'})dx.$$

$$I_1 \leq \frac{\varepsilon}{4} \int_{U_{2r}^+} \zeta^q |\nabla D_k \boldsymbol{u}_{\varepsilon,N}|^2 dx + \frac{1}{4} \int_{U_{2r}^+} \zeta^q |\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})|^{q-2} |\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})|^2 dx + cM_{\varepsilon,N}(2r)$$

Applying (3.5) in Lemma 3.5
with $\boldsymbol{A} = \boldsymbol{\sigma}^{\varepsilon,N}$, $s = q'$, $t = \frac{3q}{6-q}$, $\phi = \zeta^{q-1}$, replacing $\boldsymbol{f}$ by $\boldsymbol{f} - \boldsymbol{U}_N \cdot \nabla \boldsymbol{u}_{\varepsilon,N}$,
$\Longrightarrow$

$$\int_{U_{2r}^+} \zeta^q |D_k p_{\varepsilon,N}|^{q'} dx \leq c \int_{U_{2r}^+} (\zeta^q |D_k \boldsymbol{\sigma}^{\varepsilon,N}|^{q'} + |\boldsymbol{\sigma}^{\varepsilon,N}|^{q'} + |\boldsymbol{f}|^{q'} + \zeta^q |\boldsymbol{U}_N \cdot \nabla \boldsymbol{u}_{\varepsilon,N}|^{q'}) dx + c\Big(\int_{U_{2r}^+} (|\boldsymbol{U}_N||\nabla \boldsymbol{u}_{\varepsilon,N}|)^{\frac{3q}{6-q}} dx \Big)^{\frac{6-q}{3q-3}}$$

Hölder's inequality, Sobolev-Poincaré inequality, Young's inequality $\Longrightarrow$

$$
\begin{aligned}
I_2 \le\ & \frac{\varepsilon}{4}\int_{U_{2r}^+}\zeta^q|\nabla D_k \boldsymbol{u}_{\varepsilon,N}|^2 dx + \frac{1}{4}\int_{U_{2r}^+}\zeta^q|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})|^{q-2}|\boldsymbol{D}(D_k\boldsymbol{u}_{\varepsilon,N})|^2 dx \\
& + c\varepsilon M_{\varepsilon,N}(2r)^{2/q} + cM_{\varepsilon,N}(2r) + \int_{U_{2r}^+}\zeta^q|\boldsymbol{U}_N\cdot\nabla\boldsymbol{u}_{\varepsilon,N}|^{q'}dx \\
& + c\|\nabla\boldsymbol{U}_N\|_{\boldsymbol{L}^q(U_{2r}^+)}^{q'}\|\nabla\boldsymbol{u}_{\varepsilon,N}\|_{\boldsymbol{L}^q(U_{2r}^+)}^{q'}, \\
I_3 \le\ & \int_{U_{2r}^+}(|D_k\boldsymbol{u}_{\varepsilon,N}|^q + |\nabla\boldsymbol{f}|^{q'})dx \le M_{\varepsilon,N}(2r), \\
I_4 \le\ & c\int_{U_{2r}^+}\zeta^q|\boldsymbol{U}_N|^{q'}\,|D_k\boldsymbol{u}_{\varepsilon,N}|^{q'}dx + c\int_{U_{2r}^+}\zeta^q|\nabla\boldsymbol{U}_N|\,|\nabla\boldsymbol{u}_{\varepsilon,N}|^2 dx + M_{\varepsilon,N}(2r)
\end{aligned}
$$

Estimates of $I_1, I_2, I_3, I_4 \Rightarrow$

$$\begin{aligned}
&\varepsilon \int_{U_{2r}^+} \zeta^q |\nabla D_k \boldsymbol{u}_{\varepsilon,N}|^2 dx + \int_{U_{2r}^+} \zeta^q |\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})|^{q-2} |\boldsymbol{D}(D_k \boldsymbol{u}_{\varepsilon,N})|^2 dx \\
&\quad \leq c\Big(\varepsilon M_{\varepsilon,N}(2r)^{2/q} + M_{\varepsilon,N}(2r)\Big) \\
&\qquad + c \int_{U_{2r}^+} \zeta^q |D_k \boldsymbol{U}_N| \, |\nabla \boldsymbol{u}_{\varepsilon,N}|^2 + \zeta^q |\boldsymbol{U}_N|^{q'} |\nabla \boldsymbol{u}_{\varepsilon,N}|^{q'} dx \\
&\qquad + c \|\nabla \boldsymbol{U}_N\|^{q'}_{\boldsymbol{L}^q(U_{2r}^+)} \|\nabla \boldsymbol{u}_{\varepsilon,N}\|^{q'}_{\boldsymbol{L}^q(U_{2r}^+)}.
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
K_{\varepsilon,N}(2r) :=& \Big(\varepsilon M_{\varepsilon,N}(2r)^{2/q} + M_{\varepsilon,N}(2r)\Big) \\
&+ \int_{U_{2r}^+} \zeta^q |\nabla \boldsymbol{U}_N| \, |\nabla \boldsymbol{u}_{\varepsilon,N}|^2 + \zeta^q |\boldsymbol{U}_N|^{q'} |\nabla \boldsymbol{u}_{\varepsilon,N}|^{q'} dx \\
&+ \|\nabla \boldsymbol{U}_N\|^{q'}_{\boldsymbol{L}^q(U_{2r}^+)} \|\nabla \boldsymbol{u}_{\varepsilon,N}\|^{q'}_{\boldsymbol{L}^q(U_{2r}^+)}.
\end{aligned}$$

Estimate (4.6) yields

$$\int_{U_{2r}^+} \zeta^q (D_k V_{\varepsilon,N})^2 dx \leq cK_{\varepsilon,N}(2r), \tag{4.7}$$

$$\int_{U_{2r}^+} \zeta^q |D_k \boldsymbol{\sigma}^{\varepsilon,N}|^{q'} + \zeta^q |D_k p_{\varepsilon,N}|^{q'} dx \leq cK_{\varepsilon,N}(2r), \tag{4.8}$$

$$\int_{U_{2r}^+} \zeta^{q-1} (|D_k \boldsymbol{\sigma}^{\varepsilon,N}| + |D_k p_{\varepsilon,N}|) |\nabla \boldsymbol{u}_{\varepsilon,N}| dx \leq cK_{\varepsilon,N}(2r) \tag{4.9}$$

Outline

Normal Gradient I

Let $r > 1$fixed.
Into (4.3) we put $\phi = \zeta^q$, $\zeta \in C_0^\infty(U_{2r})$ cut-off

$$II_1 \le \frac{\varepsilon}{2}\int_{U_{2r}^+} \zeta^q(x)\eta(x_3)|\boldsymbol{D}(D_3\boldsymbol{u}_{\varepsilon,N})(x)|^2 dx$$
$$+ \frac{1}{4}\int_{U_{2r}^+} \zeta^q(x)\eta(x_3)|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x)|^{q-2}|\boldsymbol{D}(D_3\boldsymbol{u}_{\varepsilon,N})(x)|^2 dx + cM_{\varepsilon,N}(2r)$$

(4.9) $\Rightarrow$

$$II_2 \le c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r)$$

Normal Gradient II

Third equation of (4.1b) $\Rightarrow$

$$
\begin{aligned}
II_3 = & \sum_{j=1}^{2}\sum_{i=1}^{2}\int_{U_{2r}^{+}} q\zeta^{q-1}(x)D_i\zeta(x)\eta(x_3)D_j\sigma_{3j}^{\varepsilon,N}(x)D_3u_{\varepsilon,N}^{i}(x)dx \\
& + \sum_{i=1}^{2}\int_{U_{2r}^{+}} q\zeta^{q-1}(x)D_i\zeta(x)\eta(x_3)D_3\sigma_{33}^{\varepsilon,N}(x)D_3u_{\varepsilon,N}^{i}(x)dx \\
& + \sum_{i=1}^{2}\int_{U_{2r}^{+}} q\zeta^{q-1}(x)D_i\zeta(x)\eta(x_3)f^3(x)D_3u_{\varepsilon,N}^{i}(x)dx \\
& + \sum_{i=1}^{2}\int_{U_{2r}^{+}} q\zeta^{q-1}(x)D_i\zeta(x)\eta(x_3)U_N\cdot\nabla u_{\varepsilon,N}^{3}(x)D_3u_{\varepsilon,N}^{i}(x)dx.
\end{aligned}
$$

Normal Gradient III

First term on the right like II_1,
Second on the right like II_2,
Third and Fourth term by Hölder's inequality
$\Longrightarrow$

$$II_3 \le \frac{\varepsilon}{2}\int_{U_{2r}^+}\zeta^q(x)\eta(x_3)|\boldsymbol{D}(D_3\boldsymbol{u}_{\varepsilon,N})(x)|^2dx$$
$$+\frac{1}{4}\int_{U_{2r}^+}\zeta^q(x)\eta(x_3)|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x)|^{q-2}|\boldsymbol{D}(D_3\boldsymbol{u}_{\varepsilon,N})(x)|^2dx + cK_{\varepsilon,N}(2r).$$

Like II_2, (4.9) $\Longrightarrow$

$$II_4 \le c(1+\|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r).$$
$$II_5 \le K_{\varepsilon,N}(2r)$$

Normal Gradient IV

Finally, like I_4, we obtain

$$II_6 \leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r).$$

$II_1 - II_6 \implies$

$$\int_{U_{2r}^+} (2\varepsilon + |\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x)|^{q-2})\zeta^q(x)\eta(x_3)|\boldsymbol{D}(D_3\boldsymbol{u}_{\varepsilon,N})(x)|^2 dx$$
$$\leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r). \tag{4.10}$$

In particular,

$$\int_{U_{2r}^+} \zeta^q(x)\eta(x_3)(D_3 V_{\varepsilon,N}(x))^2 dx \leq c(1 + \|\eta'\|_{L^\infty})K_{\varepsilon,N}(2r). \tag{4.11}$$

Outline

Tangential derivative estimate (4.7)

$\Rightarrow$

$$\int_{U_{2r}^+} \zeta^q (D' V_{\varepsilon,N})^2 dx \leq cK_{\varepsilon,N}(2r)$$

Normal derivative estimate (4.11) with $\eta(t) = t$

$\Rightarrow$

$$\int_{U_{2r}^+} \zeta^q (D_3 V_{\varepsilon,N})^2 x_3 dx \leq cK_{\varepsilon,N}(2r).$$

$$\int_{U_{2r}^+} \zeta^{3q/2} V_{\varepsilon,N}^3 dx \le cK_{\varepsilon,N}(2r)^{3/2} \tag{4.12}$$

(4.12), (4.6), (4.8)
$\Rightarrow$

$$\begin{aligned} II_2 + II_4 + II_6 &\le cK_{\varepsilon,N}(2r) + \|\eta'\|_{L^3}^{1/q} K_{\varepsilon,N}(2r)^{1/q'} \|\zeta \nabla \boldsymbol{u}_{\varepsilon,N}\|_{\boldsymbol{L}^{3q/2}(U_{2r}^+)} \\ &\le c(1 + \|\eta'\|_{L^3}^{1/q}) K_{\varepsilon,N}(2r). \end{aligned}$$

$\Longrightarrow$

$$\int_{U_{2r}^+} \zeta^q(x)\eta(x_3)(D_3 V_{\varepsilon,N}(x))^2 dx \le c(1 + \|\eta'\|_{L^3}^{1/q}) K_{\varepsilon,N}(2r) \tag{4.13}$$

for all $\eta \in W^{1,3}(0,2r)$ with $\eta(0) = 0$ and $\eta \ge 0$ in $(0,2r)$.

For $\eta(t) = t^{8/9}$, $V_{\varepsilon,N} \in W^{1,2}(U_{2r}^+; x_3^{8/9})$.

Lemma 3.1 (Weighted Inequality) $\Rightarrow$

$$\int_{B'_{2r}} \zeta^q(x', x_3)|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x', x_3)|^q dx' \le K_{\varepsilon,N}(2r) \tag{4.14}$$

$\Rightarrow$

$$\int_{U_{2r}^+} \zeta^q(x)|\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x)|^q x_3^{-\alpha} dx \le c(1-\alpha)^{-1} K_{\varepsilon,N}(2r) \tag{4.15}$$

Sobolev-Poincaré inequality and (4.12) implies

$$\|\zeta(\boldsymbol{u}_\varepsilon - (\boldsymbol{u}_{\varepsilon,N})_{U_{2r}^+})\|^q_{\boldsymbol{L}^\infty(U_{2r}^+)} \leq K_{\varepsilon,N}(2r) \tag{4.16}$$

$\Rightarrow \zeta \boldsymbol{u}_{\varepsilon,N} \in \boldsymbol{L}^q(U_{2r}^+; x_3^{-\alpha}) \qquad \forall \alpha \in (0,1)$

Weighted Korn's inequality, (4.15) and (4.16) $\Rightarrow$

$$\int_{U_{2r}^+} \zeta^q(x)|\nabla \boldsymbol{u}_{\varepsilon,N}(x)|^q x_3^{-\alpha} dx \leq K_{\varepsilon,N}(2r) \tag{4.17}$$

Estimate on $D_k\boldsymbol{\sigma}^{\varepsilon,N}$ $(k = 1, 2)$ I

(4.4), Hölder's inequality, Young's inequality,

$$\int_0^{2r} \left(\int_{B'_{2r}} \zeta^q(x', x_3) |D_k \boldsymbol{\sigma}^{\varepsilon,N}(x', x_3)|^{q'} dx' \right)^{2/q'} dx_3 \leq cK_{\varepsilon,N}(2r)^{2/q'}$$

$$\int_{U^+_{2r}} \zeta^q(x) |D_k \boldsymbol{\sigma}^{\varepsilon,N}(x)|^{q'} x_3^{-\beta} dx' \leq cK_{\varepsilon,N}(2r) \tag{4.18}$$

for all $\beta \in \left(0, 1 - \frac{q'}{2}\right)$, with $c > 0$ depending only on β and q.

Estimate on $D_k p^{\varepsilon,N}\,(k=1,2)$ I

D_k to (4.1b), multiply the result by $\zeta^q \to$

$$\sum_{i,j=1}^{3}\int_{U_{2r}^+} g_{ij} D_j \varphi^i dx = \int_{U_{2r}^+} \zeta^q D_k p_{\varepsilon,N}\,\mathrm{div}\,\varphi dx \quad \forall\,\boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(U_{2r}^+),$$

where

$$\begin{aligned} g_{ij}(x) = {} & \zeta^q(x) D_k \sigma_{ij}^{\varepsilon,N}(x) - \delta_{jk}\zeta^q(x)\boldsymbol{U}_N(x)\cdot\nabla u^i_{\varepsilon,N}(x) \\ & - q\delta_{3j}\int_0^{x_3} D_l\zeta(x',t)\zeta^{q-1}(x',t) D_k\sigma_{il}^{\varepsilon,N}(x',t)dt \\ & + q\delta_{3j}\int_0^{x_3} D_i\zeta(x',t)\zeta^{q-1}(x',t) D_k p_{\varepsilon,N}(x',t)dt \\ & + \delta_{3j}\int_0^{x_3} \zeta^q(x',t) D_k f^i(x',t)dt \\ & - q\delta_{3j}\int_0^{x_3} D_k\zeta(x',t)\zeta^{q-1}(x',t)\boldsymbol{U}_N(x',t)\cdot\nabla u^i_{\varepsilon,N}(x',t)dt \end{aligned}$$

Estimate on $D_k p^{\varepsilon,N}$ $(k = 1, 2)$ II

(4.18) and (4.6) $\Rightarrow$

$$\int_{U_{2r}^+} |\boldsymbol{g}(x)|^{q'} x_3^{-\beta} dx \leq cK_{\varepsilon,N}(2r) + c\|\nabla \boldsymbol{U}_N\|_{\boldsymbol{L}^q(U_{2r}^+)}^{q'} K_{\varepsilon,N}(2r)^{\frac{1}{q-1}},$$

Theorem 3.3(pressure esitmate), (4.8) $\Rightarrow$

$$\int_{U_{2r}^+} \zeta^q(x)|D_k p_{\varepsilon,N}(x)|^{q'} x_3^{-\beta} dx \leq cK_{\varepsilon,N}(2r) + c\|\nabla \boldsymbol{U}_N\|_{\boldsymbol{L}^q(U_{2r}^+)}^{q'} K_{\varepsilon,N}(2r)^{\frac{1}{q-1}} \quad (4.19)$$

Estimates of $I\!I_2$ and $I\!I_4$

$\lambda \in \left(\frac{q'-1}{2}, \frac{1}{2}\right)$ arbitrarily chosen

$$\alpha := \frac{1}{2} + \lambda, \qquad \beta := \frac{1}{2} + \lambda - \frac{q'}{2}, \qquad \frac{\alpha}{q} + \frac{\beta}{q'} = \lambda.$$

Test funtion: $\phi(x) = \zeta^4(x)$, $\eta = \eta_\theta(t) := t^{1-\lambda}$, $t \in [0, 2r]$
$\Rightarrow$

$$I\!I_2 \le cK_{\varepsilon,N}(2r),$$
$$I\!I_4 \le cK_{\varepsilon,N}(2r) + c\|\nabla \boldsymbol{U}_N\|_{\boldsymbol{L}^q(U_r^+)} K_{\varepsilon,N}(2r)^{\frac{2}{q}}$$

Bound of Approximation Solution

$$\begin{aligned}
&\int_{U_{2r}^+} \zeta^q |\boldsymbol{D}(\boldsymbol{u}_{\varepsilon,N})(x)|^{q-2} |\boldsymbol{D}(D_3 \boldsymbol{u}_{\varepsilon,N})(x)|^2 x_3^{1-\lambda} dx \\
&\qquad + \int_{U_{2r}^+} \zeta^q (D_3 V_{\varepsilon,N}(x))^2 x_3^{1-\lambda} dx \\
&\leq cK_{\varepsilon,N}(2r) + c\|\nabla \boldsymbol{U}_N\|_{\boldsymbol{L}^q(U_r^+)} K_{\varepsilon,N}(2r)^{\frac{2}{q}},
\end{aligned} \tag{4.20}$$

Outline

$\varepsilon \to 0$ I

$(\boldsymbol{w}_{\varepsilon,N}, p_{\varepsilon,N}) \in \boldsymbol{W}^{1,q}_{0,\sigma}(U^+_{2r}) \times L^{q'}(U^+_{2r})$: weak solution to (4.1)

By means of reflexivity, Minty-trick,
there exist $\boldsymbol{w}_N \in \boldsymbol{W}^{1,q}_{0,\sigma}(U^+_{2r})$, and a sequence $\varepsilon_m \to 0$:
$\Rightarrow$
$\boldsymbol{w}_N$ is a weak solution to

$$\nabla \cdot \boldsymbol{w}_N = 0 \quad \text{in} \quad U^+_{2r}, \tag{4.21a}$$

$$-\nabla \cdot \boldsymbol{\sigma}^N + \nabla p_N = -\boldsymbol{U}_N \cdot \nabla \boldsymbol{u}_N + \boldsymbol{f} \quad \text{in} \quad U^+_{2r}, \tag{4.21b}$$

$$\boldsymbol{w}_N = 0 \quad \text{on} \quad \partial U^+_{2r}. \tag{4.21c}$$

$\varepsilon \to 0$ II

Testing (4.21b) with $\boldsymbol{w}_N$,

$$\int_{U_{2r}^+} \boldsymbol{\sigma}_N : \boldsymbol{D}(\boldsymbol{w}_N)dx = \int_{U_{2r}^+} \boldsymbol{U}_N \otimes \boldsymbol{U}_N : \nabla \boldsymbol{w}_N + \boldsymbol{f} \cdot \boldsymbol{w}_N dx. \tag{4.22}$$

Observing

$$\boldsymbol{\sigma}_N : \boldsymbol{D}(\boldsymbol{w}_N) \geq |\boldsymbol{D}(\boldsymbol{w}_N)|^q - |\boldsymbol{D}(\boldsymbol{U}_N)|^{q-2}\boldsymbol{D}(\boldsymbol{U}_N) : \boldsymbol{D}(\boldsymbol{w}_N),$$

Hölder's inequality, Sobolev-Poincaré inequality, Young's inequality, (4.22)

$\Rightarrow$

$$\int_{U_{2r}^+} |\boldsymbol{D}(\boldsymbol{w}_N)|^q dx \leq c\Big(\|\boldsymbol{D}(\boldsymbol{U}_N)\|^q_{\boldsymbol{L}^q(U_{2r}^+)} + r^{\frac{5q-9}{q-1}} \|\nabla \boldsymbol{U}_N\|^{\frac{2q}{q-1}}_{\boldsymbol{L}^q(U_{2r}^+)} + \|\boldsymbol{f}\|^{q'}_{\boldsymbol{L}^{q'}(U_{2r}^+)}\Big)$$

$\varepsilon \to 0$ III

Lower semi-continuity, (4.8), (4.7)

$\Rightarrow$

$$\int_{U_{2r}^+} \zeta^q |D' p_N|^{q'} + \zeta^q |D' \boldsymbol{\sigma}_N|^{q'} + \zeta^q |D' V_N|^2 dx \le c K_N(2r),$$

$$\int_{U_{2r}^+} (\zeta^q |D_3 \boldsymbol{\sigma}_N|^{q'} + \zeta^q (D_3 V_N)^2) x_3^\alpha dx \le c K_N(2r)$$

for all $\alpha > \frac{1}{2}$, where

$$V_N(x) = |\boldsymbol{D}(\boldsymbol{u}_N)(x)|^{q/2}.$$

$\varepsilon \to 0$ IV

$\omega(t) = t^{\frac{2\alpha+1}{4\alpha}}, \qquad (t \in [0,1])$
$\zeta^{q/2} V \in W^{1,2}_{\Gamma}(U^+_{2r}; \omega^\alpha(x_3)) \cap L^{2(3-\alpha)}(U^+_1)$

By Proposition 3.2 (Weighted Sobolev inequality),

$$\int_{U^+_{2r}} (\zeta^{q/2} V_N)^{2(3-\alpha)} dx \le c K_N(2r)^{3-\alpha},$$

$$K_N(2r) \le c M_N(2r) + c \|\nabla \boldsymbol{U}_N\|^{q/(q-2)}_{\boldsymbol{L}^q(U^+_{2r})}.$$

Outline

$(\boldsymbol{w}_N, p_N) \to (\boldsymbol{u}, p)$

$$M_N(2r) + \|\nabla \boldsymbol{U}_N\|^{q/(q-2)}_{\boldsymbol{L}^q(U^+_{2r})}$$
$$\to K(2r) := \int_{U^+_{2r}} |\nabla \boldsymbol{u}|^q + |\nabla \boldsymbol{f}|^{q'} + |\boldsymbol{f}|^{q'} dx + \|\nabla \boldsymbol{u}\|^{q/(q-2)}_{L^q(U^+_{2r})}$$

$$\int_{U^+_{2r}} \zeta^{q'} |D' p|^{q'} + \zeta^{q'} |D' \boldsymbol{\sigma}|^{q'} + \zeta^q |D' V|^2 dx \le cK(2r), \tag{4.23}$$
$$\int_{U^+_{2r}} (\zeta^{q'} |D_3 \boldsymbol{\sigma}_N|^{q'} + \zeta^q (D_3 V)^2) x_3^\alpha dx \le cK(2r) \tag{4.24}$$

for all $\alpha \in \left(\frac{1}{2}, 1\right)$, where $V(x) = |\boldsymbol{D}(\boldsymbol{u})(x)|^{q/2}$.

$$\int_{U^+_{2r}} \zeta^{q(3-\alpha)} |\nabla \boldsymbol{u}|^{q(3-\alpha)} dx \le cK(2r)^{3-\alpha}. \tag{4.25}$$

Outline

(4.25) and standard covering argument, for all $\alpha \in \left(\frac{1}{2}, 1\right)$

$$\int_\Omega |\nabla \boldsymbol{u}|^{q(3-\alpha)} dx \leq c\Big(\|\nabla \boldsymbol{u}\|^q_{\boldsymbol{L}^q(\Omega)} + \|\nabla \boldsymbol{u}\|^{q/(q-2)}_{\boldsymbol{L}^q(\Omega)} + \|\boldsymbol{f}\|^{q'}_{\boldsymbol{W}^{1,q'}(\Omega)} \Big)^{3-\alpha}$$
$$\leq c(1 + \|\|\boldsymbol{f}\|\|^{q'/(q-2)})^{3-\alpha},$$

where $\|\|\boldsymbol{f}\|\| := \left(\|\boldsymbol{f}\|^{q'}_{\boldsymbol{W}^{1,\,q'}} + \|\boldsymbol{f}\|^{q'}_{\boldsymbol{L}^{3q/(4q-3)}} \right)^{1/q'}$.
$\Longrightarrow$ (2.4), (2.5),

(4.23), (4.24) $\Longrightarrow$

$$\int_\Omega |D'p|^{q'} + |D'\boldsymbol{\sigma}|^{q'} + |D'V|^2 dx \leq c(1 + \|\|\boldsymbol{f}\|\|^{q'/(q-2)}),$$
$$\int_\Omega (|D_3 \boldsymbol{\sigma}_N|^{q'} + (D_3 V)^2) \min\{x_3^\alpha, 1\} dx \leq c(1 + \|\|\boldsymbol{f}\|\|^{q'/(q-2)})$$

$\Longrightarrow$ (2.2) and (2.3).
This completes the proof of Theorem