Cauchy problem for the Hall-MHD system without resistivity: ill-posedness

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Outline

I. Intro. to Hall-MHD and main nonlinear results
II. Stationary solutions and main linear results
III. Formal discussions
IV. Ideas of the linear proof
V. Linear to nonlinear

I. Introduction

(1) The systems: Hall-MHD and electron-MHD

(2) Main results: ill-posedness vs. well-posedness

Magnetohydrodynamic (MHD) systems

- MHD = Euler/Navier-Stokes + Maxwell (Alfven 1942):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \mathbf{p} - \nu \Delta \mathbf{u} = \mathbf{J} \times \mathbf{B}, \\ \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0, \end{cases} \tag{MHD}$$

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- $\mathbf{u}(t) : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{p}(t) : \mathbb{R}^3 \to \mathbb{R}$ are the bulk plasma velocity field and pressure,
- $\mathbf{B}(t), \mathbf{E}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ are the magnetic and electric fields, and
- $\mathbf{J}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the current density.

The usual MHD system

- Close the system in terms of **u** and **B** with

$$\mathbf{J} = \nabla \times \mathbf{B} \qquad \text{(Ampere's law)}$$

and

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}, \qquad \text{(Ohm's law)}$$

where $\eta > 0$ is magnetic resistivity.

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- The resulting system:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \mathbf{p} - \nu \Delta \mathbf{u} = (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \epsilon \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) = \eta \Delta \mathbf{B}, \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0. \end{cases} \tag{Hall-MHD}$$

Electron-MHD system

- Formally take $\mathbf{u} = 0$:

$$\begin{cases} \partial_t \mathbf{B} + \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) = 0, \\ \nabla \cdot \mathbf{B} = 0. \end{cases} \tag{E-MHD}$$

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- Mathematical work: mostly in the resistive case (loss of one derivative due to the Hall term).
- Chae-Weng: finite time blow-up under LWP assumption.

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Theorem (Nonexistence)

For any $\epsilon > 0$ and $s > 3 + 1/2$, there is a data with compact support in $(\mathbf{u}_0, \mathbf{B}_0) \in H^{s-1} \times H^s(M)$ for which there is no solution in the space $(\mathbf{u}, \mathbf{B}) \in L^\infty([0, \delta]; H^{s-1} \times H^s(M))$ for any $\delta > 0$.

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- The situation is not better for data in C^∞ or even in analytic (any Gevrey) regularity.
- Domain: $M = \mathbb{R}^k \times \mathbb{T}^{3-k}$ (weaker result in the $\mathbb{T}^3$-case).
- Norm inflation for perturbations near *degenerate* stationary magnetic fields $\to$ Nonexistence by superposition.

II. Stationary solutions and main linear results

(1) Stationary solutions and linearized systems
(2) Main linear result

Basic properties of the system

- Energy is conserved: for a solution $(\mathbf{u}, \mathbf{B})$, we have *formally*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_M (|\mathbf{u}|^2 + |\mathbf{B}|^2)(t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\right) = -\nu \int_M |\nabla \mathbf{u}|^2(t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

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- Situation is different for higher norms: we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\partial^{(N)}\mathbf{B}|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= -\int_M (\nabla\times\partial^{(N)}\mathbf{B})\cdot((\nabla\times\mathbf{B})\times\partial^{(N)}\mathbf{B})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + O.K.$$

A class of stationary magnetic field

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- A time-independent magnetic field $\mathring{\mathbf{B}}$ defines a stationary solution (with zero velocity field) if $\mathrm{div}\mathring{\mathbf{B}} = 0$ and $\nabla \times (\nabla \times \mathring{\mathbf{B}})$ is a pure gradient.
- We impose further conditions on $\mathring{\mathbf{B}}$: assume *planarity* as well as invariance with respect to a 1-parameter family of isometries of the plane.
- Then, essentially we have

$$\mathring{\mathbf{B}} = f(y)\partial_x \quad \text{or} \quad g(r)\partial_\theta.$$

Energy identities for the linearization

- The linearization around $(0, \mathring{\mathbf{B}})$ takes the following form:

$$\begin{cases} \partial_t u - \nu \Delta u = \mathbb{P}((\nabla \times \mathring{\mathbf{B}}) \times b + (\nabla \times b) \times \mathring{\mathbf{B}}) \\ \partial_t b + \nabla \times (u \times \mathring{\mathbf{B}}) \\ \qquad + \nabla \times ((\nabla \times b) \times \mathring{\mathbf{B}}) + \nabla \times ((\nabla \times \mathring{\mathbf{B}}) \times b) = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \end{cases} \tag{Hall-MHD-lin}$$

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- Formally taking $u \equiv 0$, we obtain the linearization around $\mathring{\mathbf{B}}$ for the E-MHD system.

Energy identities for the linearization

- We have the following *linearized* energy identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_M |u|^2(t)+|b|^2(t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)+\nu\int_M |\nabla u|^2(t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_M ((b\cdot\nabla)\mathring{\mathbf{B}}_j)u^j-((u\cdot\nabla)\mathring{\mathbf{B}}_j)b^j\,\mathrm{d}x\mathrm{d}y\mathrm{d}z+\int_M((b\cdot\nabla)(\nabla\times\mathring{\mathbf{B}})_j)$$

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$$= \int_M ((b\cdot\nabla)\mathring{\mathbf{B}}_j)u^j - ((u\cdot\nabla)\mathring{\mathbf{B}}_j)b^j\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_M ((b\cdot\nabla)(\nabla\times\mathring{\mathbf{B}})_j)$$

- Gives an L^2 a priori estimate for the perturbation (u, b).

Main ill-posedness statement for the linearization

- (Translationally symmetric case.) Assume that

$$\mathring{\mathbf{B}} = f(y)\partial_x$$

with linearly *degenerate* profile:

$$\exists y_0, \quad f'(y_0) \neq 0,\ f(y_0) = 0.$$

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- Then, there exists a profile $\mathfrak{b}(x, y) \in C_c^\infty$ and $G(y) \in C^\infty$ such that with initial data

$$u_0 = 0, b_{(\lambda),0} = \mathrm{Re}(e^{i\lambda(x+G(y))}\mathfrak{b}(x, y)),$$

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any L^2-solution for the linearization satisfies the following norm growth:

$$\|b_{(\lambda)}(t)\|_{H^s(M)} \gtrsim_{s,\mathring{\mathbf{B}}} \lambda^s e^{|f'(y_0)|s\lambda t}\|b_{(\lambda),0}\|_{L^2}.$$

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- (Axi-symmetric case.) We assume that

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- Then, there exists a profile $\mathfrak{b}(r) \in C_c^\infty(0,\infty)$ and $G(r) \in C^\infty(0,\infty)$ such that with initial data

$$u_0 = 0, b_{(\lambda),0} = \mathrm{Re}(e^{i\lambda(\theta+G(r))}\mathfrak{b}),$$

any L^2-solution for the linearization satisfies:

$$\|b_{(\lambda)}(t)\|_{H^s(M)} \gtrsim_{s,\mathring{\mathbf{B}},r_0} \lambda^s e^{|g'(r_0)|s\lambda t}\|b_0\|_{L^2}.$$

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- Not simple amplitude growth in Fourier, but *transfer* of energy to higher Fourier modes with speed proportional to the initial frequency (contrast with backwards heat).
- Seems to be a general feature for *degenerate* dispersive equations. c.f. Craig-Goodman: ill-posedness for

$$\partial_t u \pm x \partial_x^3 u = 0.$$

III. Formal discussions

(1) Whistler waves
(2) Bicharacteristics
(3) A formal model equation

Linearization around a constant magnetic field

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- This system can be diagonalized;

$$\partial_t b_\pm \pm \bar{\mathbf{B}}\partial_x |\nabla| b_\pm = 0, \quad \omega = \bar{\mathbf{B}}\xi_x |\xi|,$$

where

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- Comparison with Alfven waves.

Linearization around a non-constant magnetic field

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- After diagonalizing the principal symbol $-(\mathring{\mathbf{B}} \cdot \xi)\xi\times$, the analogue of the group velocity is given by the Hamiltonian vector field

$$(\nabla_\xi p, -\nabla_x p) \text{ on } T^*M$$

with associated ODE

$$\dot{X} = \nabla_\xi p(X, \Xi)$$
$$\dot{\Xi} = -\nabla_x p(X, \Xi)$$

where $p = \pm\mathring{\mathbf{B}}(x) \cdot \xi|\xi|$.

Model example: bicharacteristics for $\mathring{\mathbf{B}} = y\partial_x$

- Conservation: Ξ_x and Ξ_z due to translation invariance, and $p(X, \Xi) = y(X)\Xi_x|\Xi|$ which is just the Hamiltonian.

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- That is, the Hamiltonian ODE is completely integrable.
- Take for instance $X(0) = (0, 1, 0)$ and $\Xi(0) = (\lambda, -\lambda, 0)$ for $\lambda > 0$; explicit integration gives

$$\Xi_y = -\lambda \sinh(\lambda t + \ln(1 + \sqrt{2})) \simeq \lambda e^{\lambda t}$$

$$y = \frac{\cosh(\ln(1 + \sqrt{2}))}{\cosh(\lambda t + \ln(1 + \sqrt{2}))} \simeq e^{-\lambda t}.$$

A formal model equation

- Take on $\mathbb{R}^2$ the following scalar equation:

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whose principal symbol is similar to that for linearized E-MHD.

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  whose principal symbol is similar to that for linearized E-MHD.
- Explicitly solvable: first separate x-dependence by taking the Fourier transform in x, and then change coordinates $\partial_\tau = \xi_x \partial_t$, $\partial_\eta = f(y)\partial_y$ to get $(\partial_\tau - \partial_\eta)\tilde{b} = 0$.

IV. Ideas of the proof

"Construction of approximate solutions + generalized energy estimate"

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(1) 2+1/2 dimensional reduction
(2) Degenerating wave packets
(3) Generalized energy identities
(4) Incorporating the velocity field

2+1/2 dimensional reduction

- We take advantage of the 2+1/2 d reduction (z-invariance): it is natural then to introduce ψ and ω by

$$(\nabla \times b)^z = -\Delta\psi, \quad (\nabla \times u)^z = \omega.$$

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- For $\mathring{\mathbf{B}} = f(y)\partial_x$, the linearized system in terms of (u^z, ω, b^z, ψ) is given by

$$\begin{cases} \partial_t u^z - f(y)\partial_x b^z - \nu\Delta u^z = 0, \\ \partial_t\omega - f''(y)\partial_x\psi + f(y)\partial_x\Delta\psi - \nu\Delta\omega = 0, \\ \partial_t b^z - f(y)\partial_x u^z + f''(y)\partial_x\psi - f(y)\partial_x\Delta\psi = 0, \\ \partial_t\psi - f(y)\partial_x(-\Delta)^{-1}\omega + f(y)\partial_x b^z = 0, \end{cases}$$

2+1/2 dimensional reduction

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- Near $\mathring{\mathbf{B}} = g(r)\partial_\theta$, the system is essentially the same:

$$\begin{cases} \partial_t b^z - g(r)\partial_\theta \Delta\psi + \left(g''(r) + \dfrac{3}{r} g'(r)\right) \partial_\theta\psi = 0, \\ \partial_t\psi + g(r)\partial_\theta b^z = 0. \end{cases}$$

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- Here we have a *gap*.

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- Pass to a second order system for ψ and write down the ansatz

$$\psi \approx \lambda^{-1} e^{i\lambda(x+G(\lambda t,y))} H(\lambda t, x, y)$$

(guided by the bicharacteristics).

Degenerating wave packets

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- To this end, consider

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and after conjugation $\varphi = f^{-\frac{1}{2}}\psi$, we obtain

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- In the case of $\mathbb{T}_x$, x-dependence can be separated and similarly θ-dependence in the axisymmetric case.

Degenerating wave packets

- Ansatz $\varphi = \lambda^{-1} e^{i\lambda(x+\Phi(\tau,\eta))} h(\tau, x, \eta)$ gives

$$e^{-i\lambda(x+\Phi)} \left[\partial_\tau^2 + (\lambda^{-1}\partial_x)^2\partial_\eta^2 + \lambda^2 f^2(\lambda^{-1}\partial_x)^4\right] (\lambda^{-1} e^{i\lambda(x+\Phi)} h)$$
$$= \lambda(-(\partial_\tau\Phi)^2 + (\partial_\eta\Phi)^2 + f^2)h$$
$$+ (2i\partial_\tau\Phi\partial_\tau + i\partial_\tau^2\Phi - i\partial_\eta^2\Phi - 2i\partial_\eta\Phi\partial_\eta - 2i(\partial_\eta\Phi)^2\partial_x - 4if^2\partial_x)h$$
$$+ \lambda^{-1}(\cdots)$$

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& = \lambda(-(\partial_\tau\Phi)^2 + (\partial_\eta\Phi)^2 + f^2)h \\
& \quad + (2i\partial_\tau\Phi\partial_\tau + i\partial_\tau^2\Phi - i\partial_\eta^2\Phi - 2i\partial_\eta\Phi\partial_\eta - 2i(\partial_\eta\Phi)^2\partial_x - 4if^2\partial_x)h \\
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- Obtain a hierarchy of equations (general rule).

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$$\max_{0 \le k, l \le m} \|\partial_\tau^k \partial_x^l \partial_\eta^{m-k-l} h(\tau)\|_{L^\infty_\tau L^2_{x,\eta}} \lesssim_m \|h_0\|_{H^m_{x,\eta}}$$

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- The error in the φ-equation:

$$\|\boldsymbol{e}_\varphi(\tau)\|_{L^2_{x,\eta}} \lesssim \lambda^{-1}\|h_0\|_{H^4}.$$

Degenerating wave packets

- Returning to the original coordinates, we obtain an approximate solution (for each $\lambda \in \mathbb{N}$)

$$\tilde{b} = (\nabla^{\perp}\tilde{\psi}, \tilde{b}^z),$$

satisfying

$$\|\tilde{b}\|_{L_t^\infty L_{x,y}^2} \approx 1,$$

$$\|\tilde{b}(t)\|_{L_x^2 L_y^1} \lesssim e^{-\frac{f'(0)}{2}\lambda t},$$

and

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- GEI: let b be a solution and $\tilde{b}$ be an approx. solution with $O(1)$ error, initially close to b_0 and L^2-normalized. Then,

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- But then, for some $t \in [0, T]$ we have

$$\|b\|_{L^2_x L^\infty_y} \|\tilde{b}\|_{L^2_x L^1_y} \geq \langle b, \tilde{b} \rangle > \frac{1}{2}$$

and degeneration of $\|\tilde{b}\|_{L^2_x L^1_y}$ gives growth for b.

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- Then we have a smoothing of order one: with $\tilde{u} = (\nabla^{\perp}(-\Delta^{-1})\tilde{\omega}, \tilde{u}^z)$,

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- We then proceed using the GEI. In the case $\nu > 0$, we also utilize the a priori bound for $\nu \|\nabla u\|_{L^2}$.

V. Linear to nonlinear

(1) Unboundedness of the solution operator

(2) Nonexistence

Unboundedness of the solution operator

Theorem

Near $\mathring{\mathbf{B}} = f(y)\partial_x$ *or* $g(r)\partial_\theta$ *(with degenerate profile), assume that the solution map is well-defined:*

$$\mathcal{B}_\epsilon((0, \mathring{\mathbf{B}}); H^r_{comp} \times H^s_{comp}) \to L^\infty_t([0,\delta]; H^{s_0-1}) \times L^\infty_t([0,\delta]; H^{s_0})$$

for some $\epsilon, \delta, r, s, s_0 > 0$*. Then this solution map is unbounded for* $s_0 \geq 3$*.*

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Proof.

Contradiction argument and use the energy to handle the nonlinearity: take GEI for $\frac{d}{dt}\langle b, \tilde{b}\rangle$ where b is now viewed as a *linear* approx. solution with the nonlinearity as the RHS. Then take $\lambda \to \infty$ to derive a contradiction. □

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- Localize the GEI to derive contradiction. Here a significant technical difference between $\mathbb{T}_y$ and $\mathbb{R}_y$.

Thanks!