Rigorous derivation of a macroscopic model for the spatially-extended FitzHugh-Nagumo system

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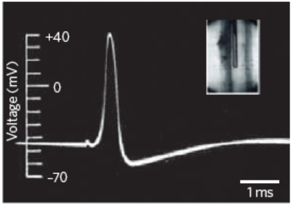
Neuron model

The FitzHugh-Nagumo model for one neuron

We consider:

$$\begin{cases} \dot{v} & = & N(v) - w + I_{ext}, \\ \dot{w} & = & \tau\,(v - \gamma\, w), \end{cases} \qquad (1)$$



Hodgkin & Huxley, '39.

- $v \in \mathbb{R}$: membrane potential of the neuron,
- $w \in \mathbb{R}$: adaptation variable,
- I_{ext}: input current received from the environment,
- $N(v) = v\,(1 - v)\,(v - \theta), \quad \theta \in (0, 1)$,
- $\tau \geq 0$, $\gamma \geq 0$: given constants.

References

Hodgkin & Huxley '52 , FitzHugh '61 , Nagumo, Arimoto & Yoshizawa '62

Neural network model

The FitzHugh-Nagumo model for a network of n neurons:

For $i \in \{1, ..., n\}$, we define:

- $\mathbf{x}_i \in \mathbb{R}^d$: spatial position of the neuron i, $d \in \{1, 2, 3\}$,
- $v_i \in \mathbb{R}$: membrane potential,
- $w_i \in \mathbb{R}$: adaptation variable.

We consider:

$$\begin{cases} \dot{\mathbf{x}}_i & = & 0, \\ \dot{v}_i & = & N(v_i) - w_i - \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} \Phi(\|\mathbf{x}_i - \mathbf{x}_j\|)\,(v_i - v_j), \\ \dot{w}_i & = & \tau\,(v_i - \gamma\, w_i), \end{cases} \tag{2}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a connectivity kernel.

Purpose

We want to find a macroscopic description of the FitzHugh-Nagumo model in the limit $n \to +\infty$, taking into account interactions between neurons. We define the macroscopic quantities:

- $\rho_0(\mathbf{x}) \geq 0$ the neuron density in the network at position $\mathbf{x}$,
- $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$ the average values of potential and adaptation variable at time t and position $\mathbf{x}$.

FHN reaction-diffusion system

$$\begin{cases} \partial_t V(t,\mathbf{x}) &= \sigma \left[\rho_0 \, \Delta_{\mathbf{x}} V(t,\mathbf{x}) + 2 \, \nabla_{\mathbf{x}} \rho_0 \cdot \nabla_{\mathbf{x}} V(t,\mathbf{x})\right] + N(V(t,\mathbf{x})) - W(t,\mathbf{x}), \\ \partial_t W(t,\mathbf{x}) &= \tau \left(V(t,\mathbf{x}) - \gamma \, W(t,\mathbf{x})\right), \end{cases} \tag{3}$$

where $\sigma > 0$ depends on the interaction kernel.

Strategy

Derivation of an intermediary mean-field equation.

Neural network model

References

Baladron, Fasoli, Faugeras & Touboul '12 , Bossy, Faugeras & Talay '15:

- Mean-field limit of Hodgkin-Huxley and FitzHugh-Nagumo systems with noise and a conductance-based connectivity kernel,
- Numerical simulations.

Luçon & Stannat '14:

- Mean-field limit of FitzHugh-Nagumo-like equations with noise and a compactly supported singular connectivity kernel.

Mischler, Quiñinao & Touboul '15:

- Mean-field limit of the FitzHugh-Nagumo system with noise and a constant connectivity map,
- Existence and stability of a stationary state of the FitzHugh-Nagumo system.

Our framework

- We neglect the noise from the environment, so our model is deterministic,
- the connectivity between neurons is weighted only by the distance,
- the support of the connectivity kernel can be unbounded.

Mean-field limit

For all $n \in \mathbb{N}$, $(\mathbf{x}_j, v_j, w_j)_{1 \leq j \leq n}$ is the solution to the FitzHugh-Nagumo system, and we define the empirical measure:

$$\mu_n(t) := \frac{1}{n} \sum_{j=1}^{n} \delta_{(\mathbf{x}_j(t); v_j(t); w_j(t))}.$$

Assumption: $\Phi \in W^{1,\infty}(\mathbb{R}^d)$,
Purpose: prove that $\mu_n \to f$ as $n \to \infty$, where f satisfies the kinetic equation:

Nonlocal transport equation

$$\partial_t f + \partial_v \left[f \left(N(v) - w - \mathcal{K}_\Phi[f] \right) \right] + \partial_w \left[f \, \tau \left(v - \gamma \, w \right) \right] = 0, \tag{4}$$

where

$$\mathcal{K}_\Phi[f](t, \mathbf{x}, v) := \int_{\mathbb{R}^{d+2}} \Phi(\|\mathbf{x} - \mathbf{x}'\|) \, (v - v') \, f(t, \mathbf{x}', v', w') \, \mathrm{d}\mathbf{x}' \mathrm{d}v' \mathrm{d}w',$$

References

- Crevat '19,
- Bolley, Cañizo and Carrillo '11.

Regime of strong local interactions

We consider the regime of strong local interactions:

$$\Phi(\|\mathbf{x}\|) = \frac{1}{\varepsilon^{d+2}} \Psi\left(\frac{\|\mathbf{x}\|}{\varepsilon}\right),$$

and we investigate the limit $\varepsilon \to 0$.

Assumption: $\Psi > 0, \quad \Psi \in L^1(\mathbb{R}^d), \quad \sigma = \int_{\mathbb{R}^d} \Psi(\|\mathbf{x}\|) \frac{\|\mathbf{x}\|^2}{2} \, \mathrm{d}\mathbf{x} < \infty.$

Nonlocal transport equation:

$$\begin{cases} \partial_t f^\varepsilon + \partial_v \left[f^\varepsilon \left(N(v) - w - \mathcal{K}_\varepsilon[f^\varepsilon] \right) \right] + \partial_w \left[f^\varepsilon \, \tau \, (v - \gamma \, w) \right] = 0, \\ \mathcal{K}_\varepsilon[f^\varepsilon] = \dfrac{1}{\varepsilon^{d+2}} \displaystyle\int_{\mathbb{R}^{d+2}} \Psi\left(\frac{\|\mathbf{x} - \mathbf{x}'\|}{\varepsilon} \right) (v - v') \, f^\varepsilon(t, \mathbf{x}', v', w') \, \mathrm{d}\mathbf{x}' \, \mathrm{d}v' \, \mathrm{d}w'. \end{cases} \tag{5}$$

where we define the macroscopic quantities:

$$\begin{cases} \rho^\varepsilon(t, \mathbf{x}) = \rho_0^\varepsilon(\mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} f_0^\varepsilon(\mathbf{x}, v, w) \, \mathrm{d}v \, \mathrm{d}w, \\ \rho_0^\varepsilon(\mathbf{x}) \, V^\varepsilon(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} f^\varepsilon(t, \mathbf{x}, v, w) \, v \, \mathrm{d}v \, \mathrm{d}w, \\ \rho_0^\varepsilon(\mathbf{x}) \, W^\varepsilon(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} f^\varepsilon(t, \mathbf{x}, v, w) \, w \, \mathrm{d}v \, \mathrm{d}w. \end{cases}$$

Macroscopic model

The macroscopic quantities derived from f^ε satisfy the following system:

$$\left\{\begin{array}{lcl} \rho_0^\varepsilon\,\partial_t V^\varepsilon \;-\; \rho_0^\varepsilon\,\mathcal{L}_{\rho_0^\varepsilon}(V^\varepsilon) & = & \rho_0^\varepsilon\,N(V^\varepsilon) - \rho_0^\varepsilon\,W^\varepsilon \\ & & + \underbrace{\int f^\varepsilon\,(N(v) - N(V^\varepsilon))\,\mathrm{d}v\,\mathrm{d}w}_{:=\mathcal{E}(f^\varepsilon)}, \\ \rho_0^\varepsilon\,\partial_t W^\varepsilon & = & \tau\,\rho_0^\varepsilon\,(V^\varepsilon + a - b\,W^\varepsilon)\,, \end{array}\right. \tag{6}$$

where

$$\mathcal{L}_{\rho_0^\varepsilon}(V^\varepsilon)(t,\mathbf{x}) := \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} \Psi\left(\frac{\|\mathbf{x}-\mathbf{x}'\|}{\varepsilon}\right) \rho_0^\varepsilon(\mathbf{x}')\,\left(V^\varepsilon(t,\mathbf{x}') - V^\varepsilon(t,\mathbf{x})\right)\,\mathrm{d}\mathbf{x}'.$$

- The non local operator $\mathcal{L}_{\rho_0^\varepsilon}(V^\varepsilon)$ plays the role of diffusion in this system,
- To get a macroscopic FitzHugh-Nagumo model, we have to prove that as $\varepsilon \to 0$:
  - $\mathcal{E}(f^\varepsilon) \to 0$,
  - $\mathcal{L}_{\rho_0^\varepsilon}(V^\varepsilon)$ converges towards the local diffusion operator.
- This equation is not well-defined for $\mathbf{x} \in \mathbb{R}^d$ such that $\rho_0^\varepsilon(\mathbf{x}) = 0$.

Diffusive limit

Theorem: Diffusive limit (Crevat '19)

Assume that there exists a positive constant C such that for all $\varepsilon > 0$:

$$\|\rho_0^\varepsilon\|_{L^1} + \|\rho_0^\varepsilon\|_{L^\infty} \leq C, \quad \int_{\mathbb{R}^{d+2}} \left(\|\mathbf{x}\|^4 + |v|^4 + |w|^4\right) f_0^\varepsilon(\mathbf{x}, v, w)\,\mathrm{d}\mathbf{x}\,\mathrm{d}v\,\mathrm{d}w \leq C.$$

We choose a well-prepared initial data (ρ_0, V_0, W_0) such that

$$\rho_0 \geq 0, \quad \|\rho_0\|_{L^1} = 1, \quad \rho_0 \in H^2(\mathbb{R}^d), \quad \rho_0 \in \mathcal{C}_b^3(\mathbb{R}^d), \quad V_0,\ W_0 \in H^2(\mathbb{R}^d),$$

$$\frac{1}{\varepsilon^2}\|\rho_0^\varepsilon - \rho_0\|_{L^2}^2 + \int_{\mathbb{R}^d} \rho_0^\varepsilon(\mathbf{x})\left[|V_0^\varepsilon(\mathbf{x}) - V_0(\mathbf{x})|^2 + |W_0^\varepsilon(\mathbf{x}) - W_0(\mathbf{x})|^2\right]\mathrm{d}\mathbf{x} \to 0.$$

Then for all $t \in [0; T]$:

$$\int_{\mathbb{R}^d} \rho_0^\varepsilon(\mathbf{x})\,\frac{|V - V^\varepsilon|^2 + |W - W^\varepsilon|^2}{2}(t, \mathbf{x})\,\mathrm{d}\mathbf{x} \to 0,$$

where

$$V, W \in L^\infty\left([0, T], H^2(\mathbb{R}^d)\right) \cap \mathcal{C}^0\left([0, T], H^1(\mathbb{R}^d)\right)$$

is the solution to the macroscopic reaction-diffusion system, and $(\rho_0^\varepsilon, V^\varepsilon, W^\varepsilon)$ are the macroscopic quantities computed from the solution f^ε of the kinetic equation.

References

References: Relative entropy method for macroscopic limits

- Di Perna '79, Dafermos '79 : hyperbolic conservation laws
- Kang & Vasseur '15 : Vlasov-type equations under strong local alignment regime
- Karper, Mellet & Trivisa '12, Figalli & Kang '19 : kinetic Cucker-Smale system under strong local alignment regime,
- Crevat, Faye & Filbet '19 : kinetic FHN equation in a different regime of strong interactions.

Main steps of the proof

- Estimate of moments of f^ε to control a kinetic dissipation,
- Convergence of the macroscopic quantities using a relative entropy argument.

Difficulty

- Show that there exists a positive constant $C > 0$ such that:

$$\left| \int_0^T \int_{\mathbb{R}^d} \left(V^\varepsilon(t) - V(t)\right) \left(\int_{\mathbb{R}^2} f^\varepsilon(t) \left[N(v) - N(V^\varepsilon(t))\right] \mathrm{d}v\, \mathrm{d}w \right) \mathrm{d}\mathbf{x}\, \mathrm{d}t \right| \leq C \left(\int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t) \left|V^\varepsilon(t) - v\right|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}t \right)^{1/2}.$$

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Numerical simulations

Purpose:

- Reproduction of qualitative behaviours measured *in vivo*,
- Macroscopic behaviours with a numerical scheme for a kinetic model.

Some remarks on the numerical scheme:

- Discretization of (V, W): particle method,
- Discretization of space: spectral method,
- Discretization of time: semi-implicit numerical scheme of order 1.

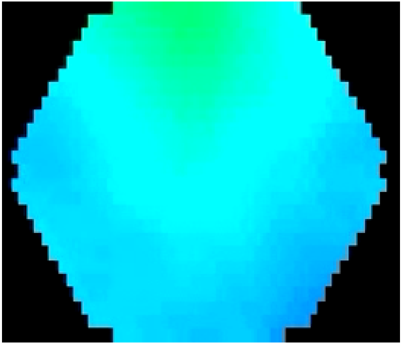
We simulate $V^{\varepsilon}(t, \mathbf{x}) = \int v\, f^{\varepsilon}(t, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w$.

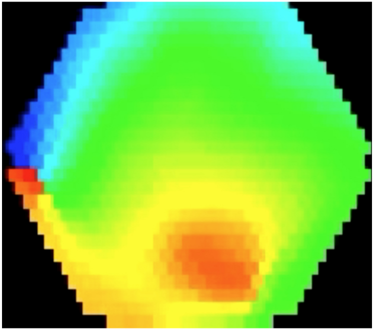
We consider the connectivity kernel

$$\Psi(\|\mathbf{x}\|) = \frac{n_0}{\sqrt{2\,\pi\, T_0}} \exp\left(-\frac{\|\mathbf{x}\|^2}{2\, T_0}\right),$$

with $n_0 = 0.1$ and $T_0 = 0.05$.

A variety of cortical waves

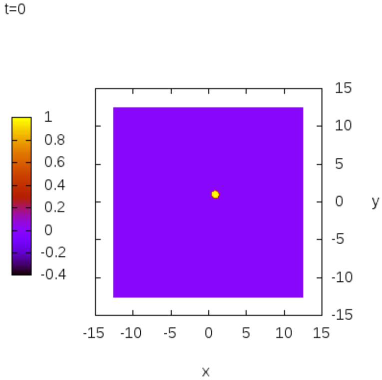




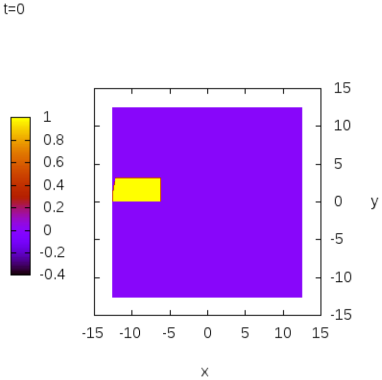
Huang *et al.* , Neuron, '10.

In vivo spiral waves in the neo-cortex.

A variety of cortical waves



A radially-propagating pulse



A rotating spiral wave

$$\varepsilon = 0.1,\ \tau = 0.002,\ \gamma = 1,\ \theta = 0.1$$

Conclusion

Conclusion:
We have rigorously established:

- a link between the FitzHugh-Nagumo system and the kinetic model, derived as the mean-field limit as $n \to \infty$,
- a link between the mean-field model of FitzHugh-Nagumo type and a macroscopic reaction-diffusion system, with an estimate of the error with respect to the parameter ε, using a relative entropy argument.

Work in progress:

- Development of a numerical approximation stable and consistent in ε: Asymptotic-Preserving scheme.

Perspectives:

- Analysis of macroscopic models (*e.g.* traveling wave solutions in heterogeneous and periodic media).

Thank you for your attention.