Hypocoercive Techniques in Collisional Kinetic Theory

Marc Briant

Laboratoire MAP5, University of Paris - Paris Descartes (Paris 5)

France-Korea Conference on PDEs
Institute of Mathematics, University of Bordeaux
November 26^{th} 2019

Collisional Kinetic Theory

A probabilistic approach to dilute particle systems

- PHASE SPACE : Particles move in a bounded domain $\Omega \subset \mathbb{R}^3$ with velocities in $\mathbb{R}^3$;

- MESCOPIC VIEWPOINT : Focus on the mean behaviour of a random particle
- Study the equation of the resulting density function

$$F : \left| \begin{array}{ccc} [0,T] \times \Omega \times \mathbb{R}^d & \longrightarrow & \mathbb{R}^+ \\ (t,x,v) & \longmapsto & F(t,x,v) \end{array} \right.$$

- $F(t,x,v)dxdv$ represents the probability of having at time t a particle inside $B(x,dx)$ with a velocity in $B(v,dv)$

$\Rightarrow$ Minimal Requirement :

$$\forall t \in [0,T], \quad F(t,\cdot,\cdot) \in L^1_{loc}\left(\Omega, L^1_v\left(\mathbb{R}^d\right)\right)$$

The collisional Process

1. BINARY COLLISIONS : Two particles sufficiently closed are deviated
2. LOCALISED COLLISIONS : Trajectories changes are immediate and on the spot
3. ELASTIC COLLISIONS : One particle of mass m_i and one of mass m_j

$$m_i v' + m_j v'_* = m_i v + m_j v_*$$
$$m_i \left|v'\right|^2 + m_j \left|v'_*\right|^2 = m_i \left|v\right|^2 + m_j \left|v_*\right|^2$$

4. MICROREVERSIBLE COLLISIONS : microscopic dynamics are reversible in time
5. MOLECULAR CHAOS : particles evolve independently (before colliding)

Boltzmann equation

Only one species.

$$\forall t \geqslant 0,\ \forall (x,v) \in \Omega \times \mathbb{R}^d, \quad \partial_t F + v \cdot \nabla_x F = Q(F,F)$$

THE COLLISION OPERATOR :

$$Q(F,F) = \int_{\mathbb{S}^2 \times \mathbb{R}^3} b(\cos\theta)\,|v - v_*|^\gamma \left[F' F'_* - F F_*\right]\, d\sigma dv_*$$

- $\begin{cases} v' &= \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \\ v'_* &= \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \end{cases}$, and $\cos\theta = \langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle$.

BOUNDARY CONDITIONS : None (torus) or : Specular, Diffuse, Maxwell (convex combination)

Boltzmann equation for mixture

N species non chemically reacting with potentially different masses $(m_i)_{1\leqslant i\leqslant N}$.

$$\forall 1 \leqslant i \leqslant N, \forall t \geqslant 0, \forall (x,v) \in \Omega \times \mathbb{R}^d, \quad \partial_t F_i + v\cdot\nabla_x F_i = Q_i(\mathbf{F},\mathbf{F})$$

THE COLLISION OPERATORS : Cross-interactions play a central role

$$Q_i(F,F) = Q_{ii}(F_i,F_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} Q_{ij}(F_i,F_j)$$

Entropy and global equilibria

A priori properties of solutions

- Conservation laws : mass, momentum and energy
- Boltzmann's H-theorem : entropy
  $S(F)(t) = \int_{\mathbb{T}^d \times \mathbb{R}^d} F \ln(F) dxdv$

$$\frac{d}{dt} S(F) = -\int_{\mathbb{T}^d} D(F) dx \leqslant 0.$$

Entropy and global equilibria

A priori properties of solutions

- Conservation laws : mass, momentum and energy
- Boltzmann's H-theorem : entropy $S(F)(t) = \int_{\mathbb{T}^d \times \mathbb{R}^d} F \ln(F) dxdv$

$$\frac{d}{dt} S(F) = -\int_{\mathbb{T}^d} D(F) dx \leqslant 0.$$

Equilibrium state

- Equilibria : $M_{(\rho,u,T)}(t,x,v) = \frac{\rho(t,x)}{(2\pi T(t,x))^{\frac{d}{2}}} e^{-\frac{|v-u(t,x)|^2}{2T(t,x)}}$
- Under conditions on Ω, a unique stationary equilibrium

$$\mu(v) = M_{(1,0,1)} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}}$$

Quantifying the trend to equilibrium

Convergence to equilibrium

- weak convergence by compactness : Arkeryd, Lions (torus) ; Desvillettes, Arkeryd, Bose, Grzegorczyk, Nouri (bdd)
- Entropy dissipation inequalities : DiPerna, Lions, Carlen, Carvalho, Alexandre, Wennberg, **Desvillettes-Villani '05**

Quantifying the trend to equilibrium

Convergence to equilibrium

- weak convergence by compactness : Arkeryd, Lions (torus) ; Desvillettes, Arkeryd, Bose, Grzegorczyk, Nouri (bdd)
- Entropy dissipation inequalities : DiPerna, Lions, Carlen, Carvalho, Alexandre, Wennberg, **Desvillettes-Villani '05**

Perturbative setting

- Looking at convergence at a linearised level
- Construct solutions $F(t,x,v) = \mu(v) + f(t,x,v)$
- Perturbed equation

$$\partial_t f + v \cdot \nabla_x f = L[f] + Q(f,f)$$

A bit of litterature

Existing Cauchy theories for hard potentials with cutoff, small perturbations

- ON THE TORUS
  - Sobolev with expo. weights : Grad 1958, Ukai, Guo, Yu, Mouhot, Neumann, MB
  - Lebesgue with poly. weights : Gualdani-Mischler-Mouhot '17, Merino-Aceituno, MB
- WITH BOUNDARY CONDITIONS
  - L^∞ with expo. weights for specular and diffuse : Guo, Esposito, Kim, Marra, Tonon, Trescases, Lee
  - L^∞ with poly. weights for Maxwell : Guo, MB
- MULTI-SPECIES CASE ON THE TORUS
  - Lebesgue with ploy. weights : Daus, MB
  - Sobolvec with expo. weights (hydro. limit) : Bondesan, MB

The Linear equation and hypocoercivity

The linear collisional operator

$$\partial_t f + v \cdot \nabla_x f = L[f]$$

PROPERTIES OF THE LINEAR OPERATOR :

- Local in time and space
- L is unbounded and self-adjoint in $L^2_v(\mu^{-\frac{1}{2}})$
- $L = -\nu(v) + K$
  - $\nu(v) \sim 1 + |v|^\gamma$
  - K compact in L^2_v and kernel operator

The linear collisional operator

- Fluid part of the solution f
  - $\mathrm{Ker}(L) = \mathrm{Span}\left(1, v, |v|^2\right)\mu(v)$
  - Fluid part = projection onto $\mathrm{Ker}(L)$

$$\pi_L(f)(t,x,v) = \left(\rho(t,x) + u(t,x)\cdot v + e(t,x)\,|v|^2\right)\mu(v)$$

The linear collisional operator

- FLUID PART OF THE SOLUTION f
  - $\mathrm{Ker}(L) = \mathrm{Span}\left(1, v, |v|^2\right)\mu(v)$
  - Fluid part = projection onto $\mathrm{Ker}(L)$

$$\pi_L(f)(t,x,v) = \left(\rho(t,x) + u(t,x)\cdot v + e(t,x)\,|v|^2\right)\mu(v)$$

- MICROSCOPIC PART AND SPECTRAL GAP
  - $\pi_L^{\perp}(f) = f - \pi_L(f)$
  - Carleman, Grad, Bobylev, **Baranger, Mouhot**

$$\langle L[f], f\rangle_{L^2_v(\mu^{-\frac{1}{2}})} \leqslant -\lambda_L \left\|\pi_L^{\perp}(f)\right\|^2_{L^2_v(\nu\mu^{-\frac{1}{2}})}$$

[Same kind of properties for multi-species : Daus, Jüngel, Mouhot, Zamponi ($m_i = m_j$) ; Daus, MB (general)]

Call for hypocoercivity

- Natural space associated to L is $L^2_{x,v}\left(\mu^{-\frac{1}{2}}\right)$
- First a priori estimate

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_{x,v}(\mu^{-\frac{1}{2}})} \leqslant -\lambda_L \left\|\pi_L^\perp(f)\right\|^2_{L^2_{x,v}(\nu\mu^{-\frac{1}{2}})}$$

- Main issue
  - How to recover the full norm ?
  - Control of π_L by $\pi_L^\perp$ in the set of solutions ?

Hypocoercivity : abstract framework

- Parallel with hypoellipticity $\partial_t f + T[f] = D^* D[f]$
  - $D * D$ is elliptic degenerate differential operator
  - T is skew-symmetric
  - A fully elliptic operator can be recovered from the mixing effects between T and D : Hörmander $[D, T]^*[T, D] + D^* D$
- Hypocoercivity
  - L has a dissipative property but a large kernel
  - $T = v \cdot \nabla_x$ is skew-symmetric so non-dissipative but mixes position and velocity
  - Understand how the interactions between L and T generates a full coercivity

Hypocoercivity : our framework

- WHAT WE WANT $\|\pi_L(f)\|_{L^2_v(\mu^{-\frac{1}{2}})} \leqslant \|\pi_L^{\perp}(f)\|_{L^2_v(\mu^{-\frac{1}{2}})}$
- THREE DIFFERENT SPIRITS
  - Contradiction : Guo
  - Micro-Macro decomposition : Liu, Yu, Guo
  - Constructing new Lyapunov functionals : Mouhot, Neumann, Dolbeaut, Schmeiser

Hypocoercive techniques in the space of linearisation

The mixing of transport in Sobolev spaces

$$[v \cdot \nabla_x, \nabla_v] = -\nabla_x$$

- Spatial derivatives commute with L

The mixing of transport in Sobolev spaces

$$[v \cdot \nabla_x, \nabla_v] = -\nabla_x$$

- Spatial derivatives commute with L
- Velocity derivatives still generate a negative feedback

$$\langle \nabla_v L[f], \nabla_v f \rangle_{L^2_v(\mu^{-\frac{1}{2}}} \leqslant -\lambda \left\| f \right\|^2_{L^2_v(\nu\mu^{-\frac{1}{2}})} + C \left\| f \right\|^2_{L^2_v(\mu^{-\frac{1}{2}})}$$

The mixing of transport in Sobolev spaces

$$[v \cdot \nabla_x, \nabla_v] = -\nabla_x$$

- Spatial derivatives commute with L
- Velocity derivatives still generate a negative feedback

$$\langle \nabla_v L[f], \nabla_v f \rangle_{L^2_v(\mu^{-\frac{1}{2}})} \leqslant -\lambda \|f\|^2_{L^2_v(\nu\mu^{-\frac{1}{2}})} + C \|f\|^2_{L^2_v(\mu^{-\frac{1}{2}})}$$

- New H^1 functional

$$\|f\|^2_{\mathcal{H}^1_{x,v}} = a \|f\|^2_{L^2_\mu} + b \|\nabla_x f\|^2_{L^2_\mu} + c \|\nabla_v f\|^2_{L^2_\mu} + d \langle \nabla_x f, \nabla_v f \rangle_{L^2_\mu}$$

Closing estimates : Poincaré and conservation laws

- FULL ENERGY ESTIMATE

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\mathcal{H}^1_{x,v}} \leqslant -\lambda_L^- \left[\left\|\pi_L^\perp(f)\right\|^2_{L^2_{\nu\mu}} + \|\nabla_x f\|^2_{L^2_{\nu\mu}} + \|\nabla_v f\|^2_{L^2_{\nu\mu}}\right]$$

- MASS, MOMENTUM AND ENERGY PRESERVATION

$$\int_\Omega \pi_L(f)(t,x,v)dx = \int_\Omega \pi_L(f)(0,x,v)dx = 0$$

- POINCARÉ INEQUALITY

$$\left\|\pi_L^\perp(f)\right\|^2_{L^2_{\nu\mu}} \lesssim \left\|\nabla_x \pi_L^\perp(f)\right\|^2_{L^2_{\nu\mu}} \lesssim \|\nabla_x f\|^2_{L^2_{\nu\mu}}$$

Exponential convergence to equilibrium

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\mathcal{H}^1_{x,v}} \leqslant -\lambda_L^- \|f\|^2_{\mathcal{H}^1_{x,v}}$$

- GENERATION OF A C^0-SEMIGROUP WITH EXPONENTIAL DECAY
  - In $H^s(\mu^{-\frac{1}{2}})$: Mouhot-Neumann
  - In $H^s(\mu^{-\frac{1}{2}})$ with external force : Debussche, Vovelle, MB

Exponential convergence to equilibrium

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\mathcal{H}^1_{x,v}} \leqslant -\lambda_L^- \|f\|^2_{\mathcal{H}^1_{x,v}}$$

- GENERATION OF A C^0-SEMIGROUP WITH EXPONENTIAL DECAY
  - In $H^s(\mu^{-\frac{1}{2}})$: Mouhot-Neumann
  - In $H^s(\mu^{-\frac{1}{2}})$ with external force : Debussche, Vovelle, MB
- SIMILAR RESULT IN OTHER SETTINGS
  - In $H^s(\mu^{-\frac{1}{2}})$ in the incompressible Navier-Stokes limit : MB
  - In $H^s(\mu^{-\frac{1}{2}})$ for multi-species in the Maxwell-Stefan or Fick limit : Bondesan, Grec, MB

Exponential convergence to equilibrium

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\mathcal{H}^1_{x,v}} \leqslant -\lambda_L^- \|f\|^2_{\mathcal{H}^1_{x,v}}$$

- Generation of a C^0-semigroup with exponential decay
  - In $H^s(\mu^{-\frac{1}{2}})$: Mouhot-Neumann
  - In $H^s(\mu^{-\frac{1}{2}})$ with external force : Debussche, Vovelle, MB
- Similar result in other settings
  - In $H^s(\mu^{-\frac{1}{2}})$ in the incompressible Navier-Stokes limit : MB
  - In $H^s(\mu^{-\frac{1}{2}})$ for multi-species in the Maxwell-Stefan or Fick limit : Bondesan, Grec, MB
- With micro-macro decomposition
  - Recall $\pi_L(f) = \left(\rho(t,x) + u(t,x)\cdot v + e(t,x)\left|v\right|^2\right)\mu(v)$
  - Find PDE satisfied by ρ, u, e and $\pi_L^{\perp}(f)$ and close estimates
  - Same results mono species : Guo, Liu, Yu

Staying in Lebesgue space

$$\pi_L(f) = \left(\rho(t,x) + u(t,x)\cdot v + e(t,x)\,|v|^2\right)\mu(v)$$

- CONTROL MICRO-FLUID Elliptic regularity for ρ, u, e Guo

$$\Delta \pi_L(f) \sim \partial^2 \pi_L^{\perp}(f) + \text{h.o.t.}$$

- PROBLEMS IN BOUNDED DOMAIN
  - Usually no preservation of momentum nor energy (no Poincaré)
  - Appearance of singularities/discontinuities due to grazing set : Guo, Kim, Tonon, Trescases
  - No regularity higher than H^1 !

Staying in Lebesgue space

$$\pi_L(f) = \left(\rho(t,x) + u(t,x)\cdot v + e(t,x)\,|v|^2\right)\mu(v)$$

- CONTROL MICRO-FLUID Elliptic regularity for ρ, u, e Guo

$$\Delta\pi_L(f) \sim \partial^2\pi_L^{\perp}(f) + \text{h.o.t.}$$

- PROBLEMS IN BOUNDED DOMAIN
  - Usually no preservation of momentum nor energy (no Poincaré)
  - Appearance of singularities/discontinuities due to grazing set : Guo, Kim, Tonon, Trescases
  - No regularity higher than H^1 !

- Need of a micro-fluid control directly in $L^2_{x,v}(\mu^{-\frac{1}{2}})$.

Recovering the coercivity in L^2

- WEAKENING ELLIPTIC REGULARITY
  - Method introduced for diffuse b.c. :Esposito, Guo, Kim, Marra
  - Recovering estimates on ρ, u and e by integrating against test fonctions.

$$\psi_\rho(t,x,v) = \left(|v|^2 - \alpha_\rho\right)\sqrt{\mu}v \cdot \nabla_x \phi_\rho(t,x)$$

$$\text{where} \quad -\Delta_x \phi_\rho(t,x) = \rho(t,x); \quad \partial_n \phi_\rho|_{\partial\Omega} = 0$$

  - Laplacian is recovered *via* the transport operator
  - Need of elliptic estimates in negative Sobolev spaces

Recovering the coercivity in L^2

- **Weakening elliptic regularity**
  - Method introduced for diffuse b.c. :Esposito, Guo, Kim, Marra
  - Recovering estimates on ρ, u and e by integrating against test fonctions.

$$\psi_\rho(t,x,v) = \left(|v|^2 - \alpha_\rho\right)\sqrt{\mu} v \cdot \nabla_x \phi_\rho(t,x)$$

$$\text{where} \quad -\Delta_x \phi_\rho(t,x) = \rho(t,x); \quad \partial_n \phi_\rho|_{\partial\Omega} = 0$$

  - Laplacian is recovered *via* the transport operator
  - Need of elliptic estimates in negative Sobolev spaces
- **Generation of C^0-semigroup in $L^2_{x,v}(\mu^{-\frac{1}{2}})$**
  - Diffusive and Maxwell b.c. : Esposito, Guo, Kim, Marra, MB
  - Multi-species Boltzmann equation on torus : Daus, MB

Some details

We rewrite the solution $\tilde{f} = e^{\lambda t} f$: $\partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} = L[\tilde{f}] + \lambda \tilde{f}$

- ESTIMATE WITH SPECTRAL GAP :

$$\left\|\tilde{f}\right\|^2_{L^2_\mu} + \lambda_L \int_0^t \left\|\pi_L^\perp(\tilde{f})\right\|^2_{L^2_{\nu\mu}} ds \leqslant \left\|\tilde{f}(0)\right\|^2_{L^2_\mu} + \lambda \int_0^t \left\|\tilde{f}\right\|^2_{L^2_\mu} ds$$

- MICRO-FLUID CONTROL WITH WEAK REGULARITY

$$\int_0^t \left\|\pi_L(\tilde{f})\right\|^2_{L^2_\mu} ds \lesssim \left\|\tilde{f}\right\|^2_{L^2_\mu} - \left\|\tilde{f}(0)\right\|^2_{L^2_\mu} + \int_0^t \left\|\pi_L^\perp(\tilde{f})\right\|^2_{L^2_\mu} ds$$

- SUMMING WITH WEIGHTS : $\left\|\tilde{f}\right\|^2_{L^2_\mu}$ is bounded so $\|f\|^2_{L^2_\mu}$ decays exponentially.

Hypocoercive techniques to change functional space

A need to work outside $L^2_{x,v}(\mu^{-\frac{1}{2}})$

- MATHEMATICAL REASONS
  - Control the nonlinear remainder $Q(f,f,)$
  - Algebraic norms : $L^\infty_{x,v}$, $H^s_{x,v}$ for s large
  - Loss of weight : but gain of weight in spectral gap

A need to work outside $L^2_{x,v}(\mu^{-\frac{1}{2}})$

- MATHEMATICAL REASONS
  - Control the nonlinear remainder $Q(f, f,)$
  - Algebraic norms : $L^\infty_{x,v}$, $H^s_{x,v}$ for s large
  - Loss of weight : but gain of weight in spectral gap
- PHYSICAL PURPOSES
  - Larger spaces to obtain less regular solutions
  - Ultimately : $L^1_{x,v}(1 + |v|^2)$
  - Most optimal so far $L^1_v L^\infty_x(1 + |v|^{2+0})$ for Botlzmann : Gualdani-Mischler-Mouhot '17

A decomposition of L

- New framework : $L^2 - L^\infty$ theory "à la Guo"
  - Want to work in $L^\infty_{x,v}((1+|v|^\beta)\mu^{-\frac{1}{2}})$
  - Link with L^2-theory : $f \in L^\infty_{\beta,\mu} \Longrightarrow f(1+|v|)^{-\beta} \in L^2_\mu$
- Decomposition of L and collision frequency
  - $L = -\nu(v) + K$
  - ν positive multiplicative
  - K kernel operator with kernel $k(v, v_*)$

A decomposition of L

- New framework : $L^2 - L^\infty$ theory "à la Guo"
  - Want to work in $L^\infty_{x,v}((1+|v|^\beta)\mu^{-\frac{1}{2}})$
  - Link with L^2-theory : $f \in L^\infty_{\beta,\mu} \Longrightarrow f(1+|v|)^{-\beta} \in L^2_\mu$
- Decomposition of L and collision frequency
  - $L = -\nu(v) + K$
  - ν positive multiplicative
  - K kernel operator with kernel $k(v, v_*)$
- Collision frequency semigroup
  - $G_\nu = -\nu(v) - v \cdot \nabla_x$ generates a C^0 semigroup with expo decay
  - Not direct with b.c. : Guo (SR, MD), Guo-MB (Maxwell)

The key role of characteristics

- DUHAMEL FORM
  - $G = -\nu - v \cdot \nabla_x + K$
  - Write the semigroup as

$$S_G(t) = S_{G_\nu}(t) + \int_0^t S_{G_\nu}(t-s) \left(\int_{\mathbb{R}^d} k(v, v_*) S_G(s)(v_*) dv_* \right) ds$$

- $L^2 - L^\infty$ RELATIONSHIP
  - Characteristics variable inside the integral : $x - (t-s)v$
  - Change of variable $y = x - (t-s)v$ gives an integral over Ω
  - In real life : iterated Duhamel, not explicit characteristics...

The key role of characteristics

- **Duhamel form**
  - $G = -\nu - v \cdot \nabla_x + K$
  - Write the semigroup as

$$S_G(t) = S_{G_\nu}(t) + \int_0^t S_{G_\nu}(t-s)\left(\int_{\mathbb{R}^d} k(v,v_*)S_G(s)(v_*)dv_*\right) ds$$

- **$L^2 - L^\infty$ relationship**
  - Characteristics variable inside the integral : $x - (t-s)v$
  - Change of variable $y = x - (t-s)v$ gives an integral over Ω
  - In real life : iterated Duhamel, not explicit characteristics...

- **Results obtained with this method** C^0-semigroup with expo decay in $L^\infty_{\beta,\mu}$
  - Boltzmann with b.c. : Guo, Kim, Lee, MB
  - Boltzmann with non constant b.c. : Esposito, Guo, Kim, Marra
  - Multi-species Boltzmann on the torus : Daus, MB

From exponential Sobolev to polynomial Lebesgue

$$G = L - v \cdot \nabla_x$$

- ENLARGEMENT METHOD : abstract formalism from Gualdani-Mischler-Mouhot '17
  - G generates S_G with expo decay in E
  - We want to extend S_G to $\mathcal{E} \supset E$
  - Hierarchy of spaces $E = E_1 \subset \cdot \subset E_n = \mathcal{E}$
  - Decomposition $G = A + B$
    - B dissipative in every E_i
    - A regularises from E_{i+1} to E_i

From exponential Sobolev to polynomial Lebesgue

- ANALYTIC VIEWPOINT : Hierarchy of PDEs

$$\begin{cases} \partial_t f_1 = B(f_1) & \text{, in } \quad \mathcal{E} \\ \partial_t f_2 = B(f_2) + A(f_1) & \text{, in } \quad E_{n-1} \\ \vdots & \vdots \\ \partial_t f_n = G(f_n) + A(f_{n-1}) & \text{, in } \quad E \end{cases}$$

From exponential Sobolev to polynomial Lebesgue

- ANALYTIC VIEWPOINT : Hierarchy of PDEs

$$\begin{cases} \partial_t f_1 = B(f_1) & ,\text{ in } \quad \mathcal{E} \\ \partial_t f_2 = B(f_2) + A(f_1) & ,\text{ in } \quad E_{n-1} \\ \vdots & \vdots \\ \partial_t f_n = G(f_n) + A(f_{n-1}) & ,\text{ in } \quad E \end{cases}$$

- RESULTS OBTAINED WITH THIS METHOD :
  - Boltzmann on torus in $L^1_v L^\infty_x(1+|v|^{2+0})$: Gualdani-Mischler-Mouhot
  - Boltzmann with b.c. in $L^\infty_{x,v}(1+|v|^{5+\gamma+0})$
  - Multi-species Boltzmann on torus in $L^1_v L^\infty_x(1+|v|^{k_0+0})$ and $L^\infty_{x,v}(1+|v|^{k_1+0})$: Daus, MB

That's all folks ! !

THANK YOU FOR YOUR ATTENTION