From Newton to Boltzmann

Lanford's theorem in a domain with boundary condition

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Statistical mechanics : the description of the matter at a mesoscopic level

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But since the number of particles in a single cubic meter of air is of order 10^{25}, one cannot describe explicitly the movement of any of its particles.

The fluid will be described by the quantity $f(t,x,v)$, the density of particles lying at time t at point x and moving with velocity v.

f is called the *one-particle density function in the phase space*.

Choosing the model for the dynamics of the particles : the hard spheres, with specular reflexion

One assumes that the gas is monoatomic and electrically neutral. The gas is composed of spherical particles of diameter ε, which evolve outside of an obstacle Ω of the Euclidean space $\mathbb{R}^d$ ($d \geq 2$). The position of the particle i at time t will be denoted $x_i(t)$, and its velocity at time t $v_i(t)$.

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Far enough from the obstacle (i.e. when $d(\Omega, x_i(t)) > \varepsilon/2$) and from the other particles (i.e. when $d(x_i(t), x_j(t)) > \varepsilon$ for $j \neq i$), the particles move in straight lines, with constant velocity :

$$\dot{v_i}(t) = 0$$

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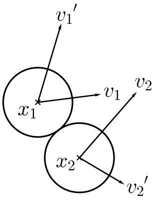

$$\begin{cases} v_1' &= v_1 - (\omega \cdot (v_1 - v_2))\omega, \\ v_2' &= v_2 + (\omega \cdot (v_1 - v_2))\omega, \end{cases}$$

$$\text{with } \omega = \frac{x_2 - x_1}{|x_2 - x_1|}$$

Figure: Collision between two particles : $|x_1 - x_2| = \varepsilon$

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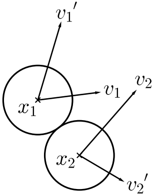

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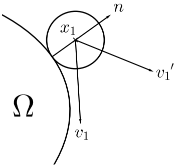

$$v_1' = v_1 - 2(v_1 \cdot n)n$$

Figure: Bouncing against the obstacle : $d(x_1, \Omega) = \varepsilon/2$

Formal way to obtain the Boltzmann equation

Equation satisfied by f ?

$$\partial_t f + v \cdot \nabla_x f = 0$$

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$$\partial_t f + v \cdot \nabla_x f = Q(t,x,v)$$

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$$\partial_t f + v \cdot \nabla_x f = Q\Big(f^{(2)}\Big)(t,x,v)$$

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$$\partial_t f + v \cdot \nabla_x f = \int_{v_*,v',v'_*} \Big[P\Big((v',v'_*) \to (v,v_*)\Big) f^{(2)}(t,x,v',x,v'_*) - P\Big((v,v_*) \to (v',v'_*)\Big) f^{(2)}(t,x,v,x,v_*) \Big] \, \mathrm{d}v'_* \, \mathrm{d}v' \, \mathrm{d}v_*$$

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$$\times \mathbb{1}_{v + v_* = v' + v'_*} \mathbb{1}_{\frac{|v|^2}{2} + \frac{|v_*|^2}{2} = \frac{|v'|^2}{2} + \frac{|v'_*|^2}{2}} \, \mathrm{d}v'_* \, \mathrm{d}v' \, \mathrm{d}v_*$$

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$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d_{v_*}} \int_{\mathbb{S}^{d-1}_\omega} B(v - v_*, \omega) \Big[f(t,x,v') f(t,x,v'_*) - f(t,x,v) f(t,x,v_*) \Big] \, \mathrm{d}\omega \, \mathrm{d}v_*$$

This is the Boltzmann equation (1872).

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For f a solution of the Boltzmann equation, the following quantities are conserved :

$$\int_x \int_v \begin{pmatrix} 1 \\ v_i \\ \frac{|v|^2}{2} \end{pmatrix} f(t,x,v) \, \mathrm{d}x \, \mathrm{d}v.$$

The stationnary solutions of the Boltzmann equation are exactly the Maxwellian functions :

$$M(v) = \lambda \exp(b \cdot v + c|v|^2), \text{ with } b \in \mathbb{R}^d, \ \lambda \geq 0, c < 0.$$

H-theorem and Loschmidt's paradox

For a solution f of the Boltzmann equation, if one considers the *entropy* :

$$H(f)(t) = \int_x \int_v f(t,x,v) \ln f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x,$$

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one can prove that, if f is not an equilibrium (i.e. a Maxwellian), then :

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f)(t) = -\frac{1}{4} \int_x \int_v \int_{v_*} \int_\omega B(v - v_*, \omega) \big(f(v') f(v'_*) - f(v) f(v_*) \big) \times \ln \left(\frac{f(v') f(v'_*)}{f(v) f(v_*)} \right) \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x < 0.$$

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Introducing the BBGKY hierarchy

One studies the system of N hard spheres evolving outside of the obstacle Ω, described by the configuration Z_N and the evolution of the distribution function f_N of the system in the *phase space* $\mathcal{D}_N^\varepsilon$.
One denotes :

$$Z_N = (x_1, v_1, \dots, x_N, v_N) = (z_1, \dots, z_N) \in \mathbb{R}^{2dN},$$

with $z_i = (x_i, v_i) \in \mathbb{R}^{2d}$, and

$$\mathcal{D}_N^\varepsilon = \Big\{ Z_N \in \Big(\big(\Omega + B(0, \varepsilon/2) \big)^c \times \mathbb{R}^d \Big)^N \ / \\ \forall i \neq j,\ |x_i - x_j| > \varepsilon \Big\}.$$

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when $|x_i - x_j| = \varepsilon$ and $(x_i - x_j) \cdot (v_i - v_j) > 0$.

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Introducing the marginals $f_N^{(s)}$ of the distribution function :

$$f_N^{(s)}(Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N) \mathbb{1}_{\mathcal{D}_N^\varepsilon} \, \mathrm{d}z_{s+1} \dots \mathrm{d}z_N,$$

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one can show that each marginal satisfies the equation (for $1 \leq s \leq N-1$):

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where $\mathcal{C}_{s,s+1}^{N,\varepsilon}$ is the collision term, which writes :

$$\mathcal{C}_{s,s+1}^{N,\varepsilon} f^{(s+1)} = \sum_{i=1}^{s} (N-s) \varepsilon^{d-1} \int_{\mathbb{S}_\omega^{d-1}} \int_{\mathbb{R}^d_{v_{s+1}}} \omega \cdot (v_{s+1} - v_i) \times f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \, \mathrm{d}\omega \, \mathrm{d}v_{s+1}.$$

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Those N equations constitute the BBGKY hierarchy.

The Boltzmann-Grad limit, and the Boltzmann hierarchy

So far, no link was given between the number N of particles of the system, and the radius $\varepsilon/2$ of those particles.

The Boltzmann-Grad limit, and the Boltzmann hierarchy

One will consider the *Boltzmann-Grad* limit :

$$N\varepsilon^{d-1} = 1.$$

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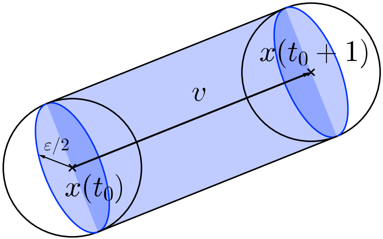

Figure: Volume covered by a particle of radius $\varepsilon/2$, traveling with a normalized velocity, during a time 1

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Decomposing the collision term $\mathcal{C}^{N,\varepsilon}_{s,s+1}$:

$$\sum_{i=1}^{s}(N-s)\varepsilon^{d-1}\int_{\mathbb{S}^{d-1}_{\omega}}\int_{\mathbb{R}^d_{v_{s+1}}}\omega\cdot(v_{s+1}-v_i)f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

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$$=\sum_{i=1}^{s}(N-s)\varepsilon^{d-1}\int_{\mathbb{S}^{d-1}_{\omega}}\int_{\mathbb{R}^d_{v_{s+1}}}\big[\omega\cdot(v_{s+1}-v_i)\big]_+f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

$$-\sum_{i=1}^{s}(N-s)\varepsilon^{d-1}\int_{\mathbb{S}^{d-1}_{\omega}}\int_{\mathbb{R}^d_{v_{s+1}}}\big[\omega\cdot(v_{s+1}-v_i)\big]_-f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

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Using the boundary condition for the incoming configurations :

$$\sum_{i=1}^{s}(N-s)\varepsilon^{d-1}\int_{\mathbb{S}_\omega^{d-1}}\int_{\mathbb{R}^d_{v_{s+1}}}\omega\cdot(v_{s+1}-v_i)f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

$$= \sum_{i=1}^{s}(N-s)\varepsilon^{d-1}\int_{\mathbb{S}_\omega^{d-1}}\int_{\mathbb{R}^d_{v_{s+1}}}\big[\omega\cdot(v_{s+1}-v_i)\big]_+$$

$$\times f_N^{(s+1)}(t,x_1,v_1,\dots,x_i,v_i',\dots,x_i+\varepsilon\omega,v_{s+1}')\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

$$- \sum_{i=1}^{s}(N-s)\varepsilon^{d-1}\int_{\mathbb{S}_\omega^{d-1}}\int_{\mathbb{R}^d_{v_{s+1}}}\big[\omega\cdot(v_{s+1}-v_i)\big]_- f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

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Performing the change of variables $\omega \to -\omega$ in the second term :

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And finally taking the limit $\varepsilon \to 0$, $N\varepsilon^{d-1} = 1$, the collision term becomes (formally) :

$$\sum_{i=1}^{s} \int_{\mathbb{S}^{d-1}_{\omega}} \int_{\mathbb{R}^d_{v_{s+1}}} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ f_N^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i', \dots, x_i, v_{s+1}') \, \mathrm{d}\omega \, \mathrm{d}v_{s+1}$$
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One will consider the *Boltzmann-Grad* limit :

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One defines the Boltzmann hierarchy as the *infinite* sequence of equations:

$$\forall s \geq 1,\ \partial_t f^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f^{(s)} = \mathcal{C}^0_{s,s+1} f^{(s+1)},$$

with $\mathcal{C}^0_{s,s+1} f^{(s+1)}$ denoting

$$\sum_{i=1}^{s} \int_{\mathbb{S}^{d-1}_{\omega}} \int_{\mathbb{R}^d_{v_{s+1}}} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ \big(f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i', \dots, x_i, v_{s+1}') \\ - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \big) \, \mathrm{d}v_{s+1} \, \mathrm{d}\omega.$$

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Considering the first equation of the Boltzmann hierarchy ($s = 1$):

$$\partial_t f^{(1)} + v_1 \cdot \nabla_{x_1} f^{(1)} = \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^d_{v_2}} \big[\omega \cdot (v_2 - v_1)\big]_+ \big(f^{(2)}(t, x_1, v_1', x_1, v_2') - f^{(2)}(t, x_1, v_1, x_1, v_2) \big) \, \mathrm{d}v_2 \, \mathrm{d}\omega,$$

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if one assumes in addition that the second marginal is tensorized :

$$f^{(2)}(t, x_1, v_1, x_2, v_2) = f^{(1)}(t, x_1, v_1) f^{(1)}(t, x_2, v_2),$$

the equation writes :

The Boltzmann-Grad limit, and the Boltzmann hierarchy

What's the link between the Boltzmann hierarchy and the Boltzmann equation ? Considering the first equation of the Boltzmann hierarchy ($s = 1$):

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Goal: proving the convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy.

The rigorous definition of the collision term

One considers the integrated in time versions of the hierarchies.

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$$f_N^{(s)}(t, Z_s) = f_{N,0}^{(s)}\big(T_{-t}^{s,\varepsilon}(Z_s)\big) + \int_0^t \mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)}\big(u, T_{u-t}^{s,\varepsilon}(Z_s)\big)\,\mathrm{d}u,$$

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For the Boltzmann hierarchy, the free transport with boundary condition $T^{s,0}$ preserves the continuity.

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One considers the integrated in time versions of the hierarchies. For the case of the Boltzmann hierarchy:

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For the Boltzmann hierarchy, the free transport with boundary condition $T^{s,0}$ preserves the continuity.

$\Rightarrow$ The Boltzmann hierarchy makes sense on continuous functions, decreasing sufficiently fast in the velocity variable.

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One considers the integrated in time versions of the hierarchies. For the case of the BBGKY hierarchy:

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Problem: the hard sphere transport $T^{s,\varepsilon}$ is only defined almost everywhere. One cannot work with continuous functions for the BBGKY hierarchy.

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Sense of the collision term ?

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Sense of the collision term ?

$$\int_{\mathbb{S}_\omega^{d-1}} \int_{\mathbb{R}_{v_{s+1}}^d} \omega \cdot (v_{s+1} - v_i) f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

Integral on a manifold of nonzero codimension.

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Idea [Gallagher, Saint-Raymond, Texier 2014]: integrating with respect to the remaining variables, to use the Fubini theorem.

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$$\underbrace{\int_{\mathcal{D}^{\varepsilon}_{s,Z_s}}}_{2ds} \int_{\mathbb{S}^{d-1}_{\omega}} \int_{\mathbb{R}^d_{v_{s+1}}} \omega \cdot (v_{s+1} - v_i) f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \, \mathrm{d}Z_s$$

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A dimension is still missing.

The rigorous definition of the collision term

Idea [Gallagher, Saint-Raymond, Texier 2014]: integrating with respect to the remaining variables, to use the Fubini theorem.

Adding the last missing dimension using the transport, acting on the time variable.

$$\int_0^t \int_{\mathcal{D}^{\varepsilon}_{s,Z_s}} \int_{\mathbb{S}^{d-1}_{\omega}} \int_{\mathbb{R}^d_{v_{s+1}}} \omega \cdot (v_{s+1} - v_i) h_N^{(s+1)} \big(T_{-t}^{s+1,\varepsilon}(Z_s, x_i + \varepsilon\omega, v_{s+1})\big) \,\mathrm{d}\omega \,\mathrm{d}v_{s+1} \,\mathrm{d}Z_s \,\mathrm{d}t$$

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One finally adds a cut-off $\mathbb{1}_{\mathfrak{D}}$ such that the hard sphere transport coincides with

$$(Z_s, t, \omega, v_{s+1}) \mapsto (X_s - tV_s, V_s, x_i + \varepsilon\omega - tv_{s+1}, v_{s+1}),$$

of Jacobian determinant $|\omega \cdot (v_{s+1} - v_i)|$.

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$$= \int_{\mathcal{D}^\varepsilon_{s+1}} h_N^{(s+1)} \, \mathrm{d}Z_{s+1}.$$

The rigorous definition of the collision term

Theorem [Gallagher, Saint-Raymond, Texier 2014], [D. 2019]

Let T be a positive number. Let $g_{s+1} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $r \in \mathbb{R}_+$, the function $t \mapsto g_{s+1}(t,r)$ is increasing, and

$$\left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R} |V_{s+1}| g_{s+1}(t, |V_{s+1}|) \, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T], L^\infty(\mathbb{R}^{ds}))} \underset{R \to +\infty}{\longrightarrow} 0.$$

Then for every function $h^{(s+1)} \in \mathcal{C}([0,T], L^\infty(\mathcal{D}^\varepsilon_{s+1}))$ such that

$$|h^{(s+1)}(t, Z_{s+1})| \leq \lambda g_{s+1}(t, |V_{s+1}|),$$

the transport-collision operator $\mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}$ belongs to $L^\infty([0,T] \times \mathcal{D}^\varepsilon_s)$ and satisfies

$$\left| \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t, Z_s) \right| \leq \lambda \sum_{i=1}^{s} (N-s) \varepsilon^{d-1} \frac{|\mathbb{S}^{d-1}|}{2} \int_{\mathbb{R}^d} (|v_i| + |v_{s+1}|) g_{s+1}(t, |V_{s+1}|) \, \mathrm{d}v_{s+1}.$$

Existence and uniqueness of the solutions of the hierarchies

Introducing the following functional spaces:
one considers a positive number β and the following norm:

$$\left|f^{(s)}\right|_{s,\beta} = \sup_{Z_s} \left[\left|f^{(s)}(Z_s)\right| \exp\left(\frac{\beta}{2} \sum_{i=1}^{s} |v_i|^2 \right) \right],$$

Existence and uniqueness of the solutions of the hierarchies

Introducing the following functional spaces:
one considers a positive integer s, a positive number β and the following norm:

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and the space $X_{0,s,\beta}$:

$$X_{0,s,\beta} = \left\{f^{(s)} \in \mathcal{C}_0\left(\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s\right)\ /\ \left|f^{(s)}\right|_{0,s,\beta} < +\infty\right\}$$

satisfying the specular boundary condition in the case in the particles lie outside of an obstacle.

Existence and uniqueness of the solutions of the hierarchies

Introducing the following functional spaces:
one considers a real number μ and the following norm:

$$\left\| \left(f^{(s)} \right)_{s \geq 1} \right\|_{\beta,\mu} = \sup_{s \geq 1} \left(\left| f^{(s)} \right|_{s,\beta} \exp(s\mu) \right),$$

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and the space $\mathbf{X}_{0,\beta,\mu}$:

$$\mathbf{X}_{0,\beta,\mu} = \left\{\left(f^{(s)}\right)_{s\geq 1} \; / \; \left\|\left(f^{(s)}\right)\right\|_{\beta,\mu} < +\infty\right\}.$$

Existence and uniqueness of the solutions of the hierarchies

Introducing the following functional spaces:
one considers a positive number T, two non increasing functions $t \mapsto \widetilde{\beta}(t) > 0$ and $t \mapsto \widetilde{\mu}(t) \in \mathbb{R}$, and the following norm:

$$\left|\!\left|\!\left| t \mapsto \left(f^{(s)}\right)_{s\geq 1} \right|\!\right|\!\right|_{\widetilde{\beta},\widetilde{\mu}} = \sup_{0\leq t\leq T} \left\| \left(f^{(s)}(t)\right)_{s\geq 1} \right\|_{\widetilde{\beta}(t),\widetilde{\mu}(t)}.$$

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and the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$:

$$\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}} = \left\{ t \mapsto \left(f^{(s)}\right)_{s\geq 1} \ / \ \left|\!\left|\!\left| t \mapsto \left(f^{(s)}\right)_{s\geq 1} \right|\!\right|\!\right|_{\widetilde{\beta},\widetilde{\mu}} < +\infty \right\},$$

satisfying the continuity in time condition:

$$\forall t \in \,]0,T], \ \forall s \geq 1, \ \lim_{u\to t^-} \left| f^{(s)}(t) - f^{(s)}(u) \right|_{0,s,\widetilde{\beta}(t)} = 0.$$

Existence and uniqueness of the solutions of the hierarchies

Introducing the following functional spaces:
one obtains the existence and uniqueness of solutions to the hierarchies in the spaces $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$.

Theorem ([Ukai 2001], [Gallagher, Saint-Raymond, Texier 2014])

Let β_0 be a strictly positive number, and μ_0 be a real number. There exist a time $T > 0$, a strictly positive decreasing function $\widetilde{\beta}$ and a decreasing function $\widetilde{\mu}$ defined on $[0, T]$ such that :

$$\widetilde{\beta}(0) = \beta_0,\ \widetilde{\mu}(0) = \mu_0,$$

and such that for any positive integer N in the Boltzmann-Grad limit $N\varepsilon^{d-1} = 1$, any pair of sequences of initial data $F_{N,0} \in \mathbf{X}_{N,\varepsilon,\beta_0,\mu_0}$ and $F_0 \in \mathbf{X}_{0,\beta_0,\mu_0}$ give rise respectively to unique solutions in $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$ to the BBGKY and the Boltzmann hierarchies.

Existence and uniqueness of the solutions of the hierarchies

Introducing the following functional spaces:
one obtains the existence and uniqueness of solutions to the hierarchies in the spaces $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$.
It is even possible to give explicit expressions of the solutions.

$$\begin{aligned} f^{(s)}(t,Z_s) = \ & \mathcal{T}^{s,0}_t f^{(s)}_0(Z_s) \\ & + \sum_{k=1}^{+\infty} \int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1} \int_0^{t_1} \mathcal{T}^{s,0}_{t_1-t_2} \mathcal{C}^0_{s+1,s+2} \cdots \\ & \qquad \int_0^{t_{k-1}} \mathcal{T}^{s+k-1,0}_{t_{k-1}-t_k} \mathcal{C}^0_{s+k-1,s+k} f^{(s+k)}(t_k,Z_s) \,\mathrm{d}t_k \ldots \mathrm{d}t_2 \,\mathrm{d}t_1. \end{aligned}$$

Existence and uniqueness of the solutions of the hierarchies

Introducing the following functional spaces:
one obtains the existence and uniqueness of solutions to the hierarchies in the spaces $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$.
It is even possible to give explicit expressions of the solutions ($k = 1$).

Considering for example the second term of the decomposition:

$$\sum_{j=1}^{s} \int_0^t \int_\omega \int_{v_{s+1}} \left[\omega \cdot \left(v_{s+1} - (T^{s,0}_{t_1-t}(Z_s))^{V,j}\right)\right]_+ \times \Big[f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(\big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j}, v_{s+1}\big)'_{j,s+1}\Big)\Big) - f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j}, v_{s+1}\Big)\Big)\Big] \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \, \mathrm{d}t_1,$$

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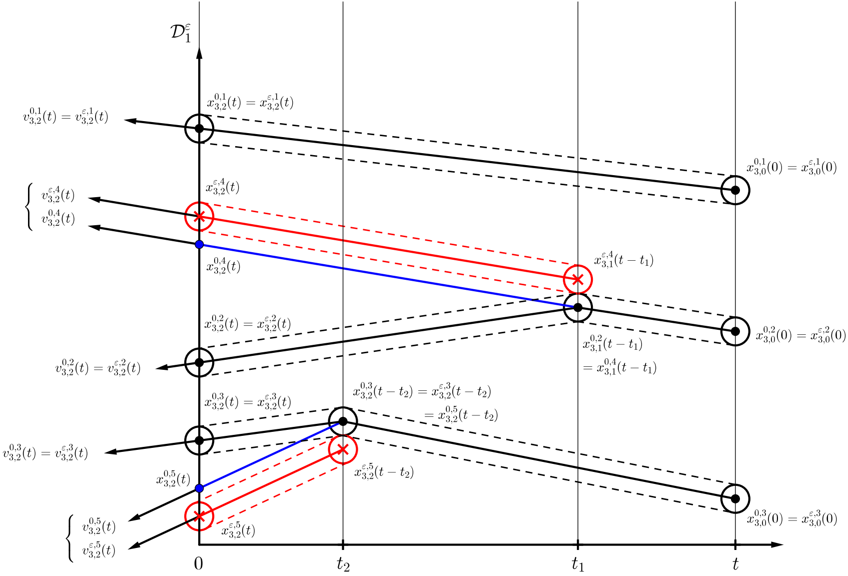
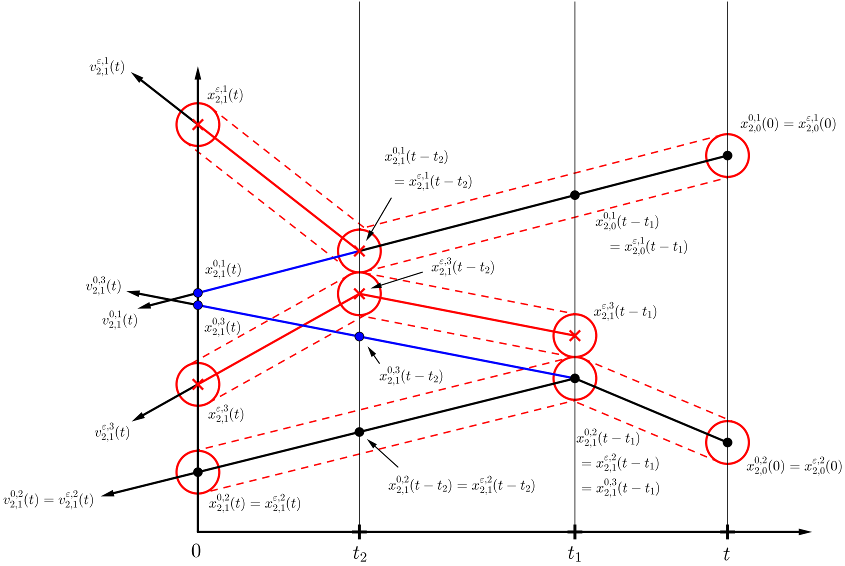
one is naturally led to consider pseudo-trajectories.

The convergence of the solutions, case without an obstacle



The convergence of the solutions, case without an obstacle



The convergence of the solutions, case without an obstacle

Theorem [Lanford 1975], [Gallagher, Saint-Raymond, Texier 2014]

Let $f_0 : \mathbb{R}^{2d} \to \mathbb{R}_+$ be a continuous density of probability such that

$$\left\| f_0(x,v) \exp\left(\frac{\beta}{2}|v|^2\right) \right\|_{L^\infty(\mathbb{R}^{2d})} < +\infty$$

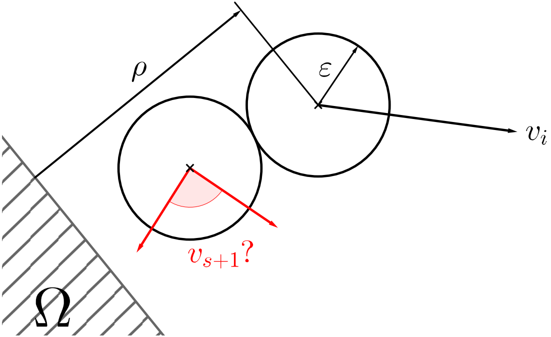
for some $\beta > 0$. Consider the system of N hard spheres of diameter ε, initially distributed according to f_0 and independent. Then, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, its distribution function $f_N^{(1)}$ converges to the solution of the Boltzmann equation f with the cross section $b(v,\omega) = (v \cdot \omega)_+$ and with initial data f_0, in the following sense:

$$\left\| \mathbb{1}_K(x) \int_{\mathbb{R}^d_v} \varphi(v) \big(f_N^{(1)} - f\big)(x,v) \,\mathrm{d}v \right\|_{L^\infty([0,T]\times\mathbb{R}^d_x)} \underset{N\to+\infty}{\longrightarrow} 0.$$

If in addition f_0 is Lispchitz, the rate of convergence is $O(\varepsilon^a)$ with $a < \frac{d-1}{d+1}$.

The convergence of the solutions in the half-space

In the case when there is an obstacle, one has to introduce a cut-off on the proximity between the obstacle and the particle undergoing an adjunction.



The convergence of the solutions in the half-space

Lanford's theorem in the half-space with specular reflexion, [D. 2019]

Let $f_0 : \{x \in \mathbb{R}^d\} \times \mathbb{R}^d \to \mathbb{R}_+$ be a continuous density of probability such that

$$f(x,v) \underset{|(x,v)| \to +\infty}{\longrightarrow} 0 \text{ and } \left\| f_0(x,v) \exp\left(\frac{\beta}{2}|v|^2\right) \right\|_{L^\infty(\mathbb{R}^{2d})} < +\infty$$

for some $\beta > 0$. Consider the system of N hard spheres of diameter ε inside the half-space with specular reflexion, initially distributed according to f_0 and independent. Then, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, its distribution function $f_N^{(1)}$ converges to the solution of the Boltzmann equation f with the cross section $b(v,\omega) = (v \cdot \omega)_+$, with specular reflexion and with initial data f_0, in the following sense:

$$\left\| \mathbb{1}_K(x,v)\big(f_N^{(1)} - f\big)(x,v) \right\|_{L^\infty([0,T]\times\{x\cdot e_1 > 0\}\times\{v \cdot e_1 \neq 0\})} \underset{N \to +\infty}{\longrightarrow} 0.$$

If in addition the square root of the initial datum $\sqrt{f_0}$ is Lipschitz with respect to the position variable uniformly in the velocity variable, the rate of convergence is $O(\varepsilon^a)$ with $a < 13/128$.