Three cohomology classes on $\overline{\mathcal{M}}_{g,n}$

France – Korea conference

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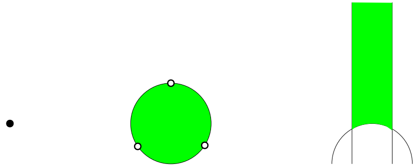
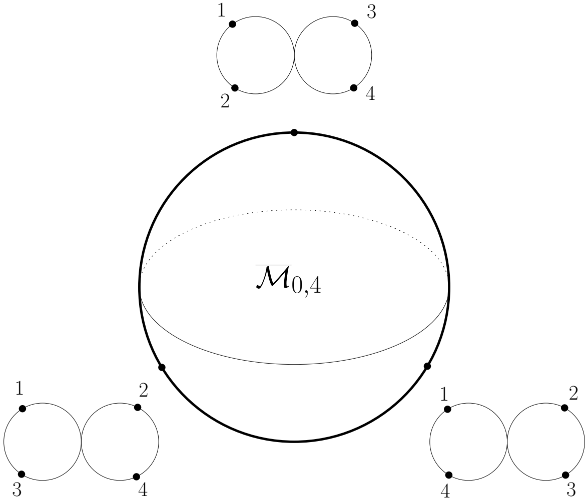


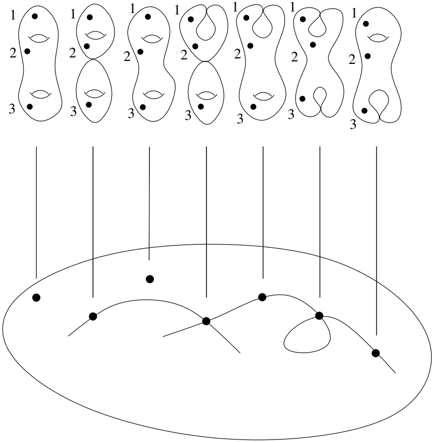
$\mathcal{M}_{0,3}$ $\mathcal{M}_{0,4}$ $\mathcal{M}_{1,1}$

Moduli spaces

$\overline{\mathcal{M}}_{g,n}$ = Deligne-Mumford compactification of $\mathcal{M}_{g,n}$

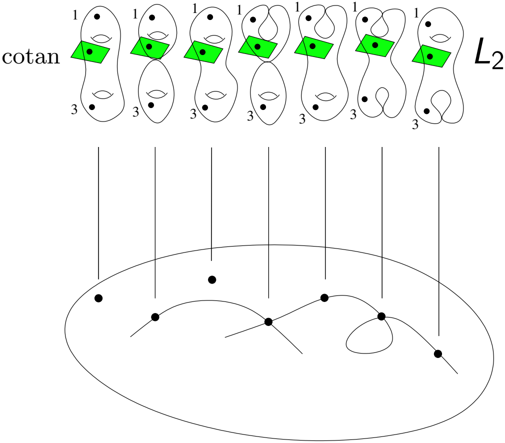


Universal curve

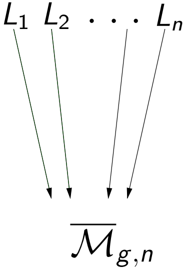


$$\overline{\mathcal{C}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

Tautological classes



$L_2 = s_2^* \omega_{\mathrm{rel}}$



$\psi_i = c_1(L_i)$ $\psi_1, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n})$

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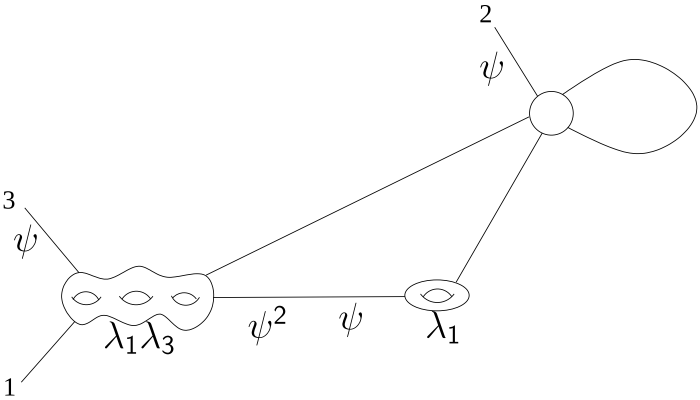
- Hodge bundle $E \to \overline{\mathcal{M}}_{g,n}$:
  fiber $E_p = \{\text{holomorphic 1-forms on } C_p\}$.

  $\lambda_m := c_m(E)$ $\qquad\qquad$ $\lambda_m \in H^{2m}(\overline{\mathcal{M}}_{g,n})$

- Boundary strata

General tautological class



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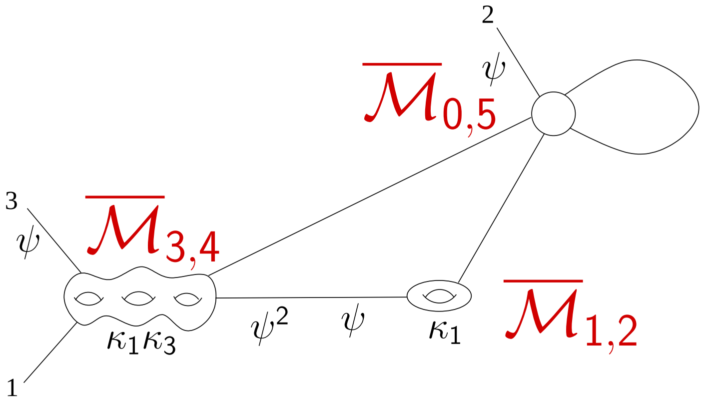

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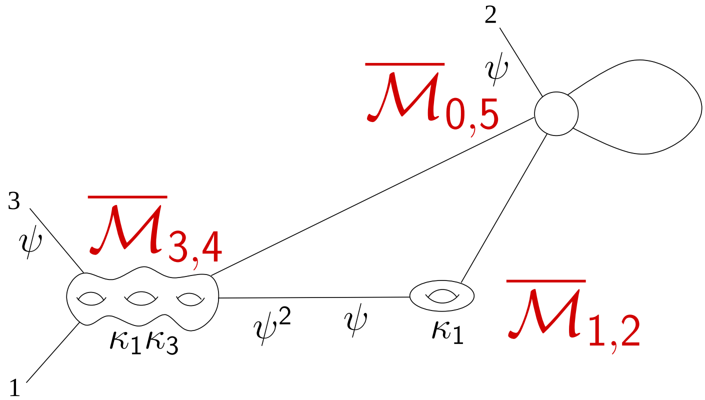

$$j : \overline{\mathcal{M}}_{3,4} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{6,3}$$

ch(Verlinde bundle) [Marian, Oprea, Pandharipande, Pixton, Z]

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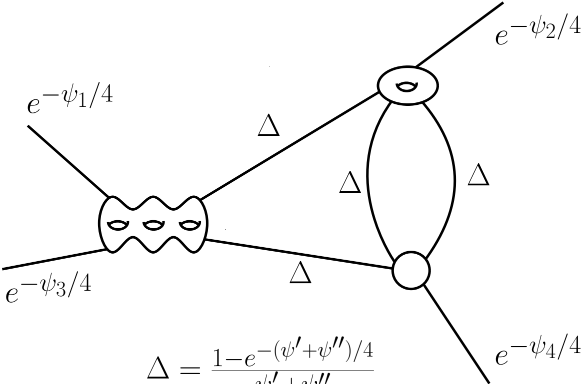
$\longrightarrow$ Verlinde bundle $V_{g,n}(\mu_1, \ldots, \mu_n)$,
μ_i = representation of level k

ch(Verlinde bundle)

$$\mathrm{ch}\, V_{g,n}(\square, \ldots, \square) = e^{-\lambda_1/2} \sum_{\substack{\text{stable graphs } \Gamma \text{ with} \\ \text{even degree vertices}}} \frac{2^{g-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}$$



$$\Delta = \frac{1-e^{-(\psi'+\psi'')/4}}{\psi_1'+\psi''}$$

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$$\mathrm{DR}_{g,n}(a_1, \ldots, a_n) \in H^{2g}(\overline{\mathcal{M}}_{g,n})$$

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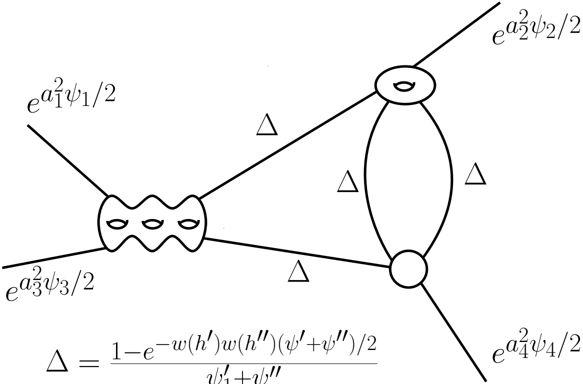
- $\sum_{\text{around vertex}} w(h) = 0 \mod r$.

Double ramification cycle

$$\sum_{\substack{\text{stable graphs } \Gamma \text{ with} \\ \text{weighting } w}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}$$



$$\Delta = \frac{1-e^{-w(h')w(h'')(\psi'+\psi'')/2}}{\psi_1'+\psi''}$$

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$\longrightarrow$ Plug $r = 0$ and take deg $2g$ part. Get $\mathrm{DR}_{g,n}(a_1, \ldots, a_n)$.

Gromov-Witten theory of $\mathbb{C}P^1$ [Rosset]

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$$GW_{g,n} = \left\{(C, x_1, \ldots, x_n) \in \overline{\mathcal{M}}_{g,n} \;\middle|\; \exists f : C \to \mathbb{CP}^1,\ f(x_i) = 0\right\}$$

Sum over deg f $\longrightarrow$ mixed-degree cohomology class.

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$$A(z) = \sum_{m \geq 0} \frac{(2m)!}{(m!)^3} \left(-\frac{z}{2^6}\right)^m,$$

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Vertex $v \longrightarrow$ variable ζ_v, $\zeta_v^2 = 1$.

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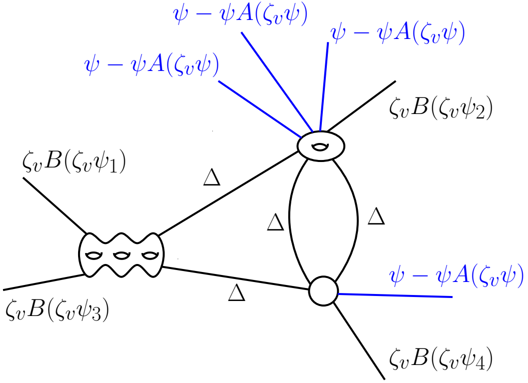
$$\sum_{\text{stable graphs } \Gamma} \frac{2^{g-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}$$



$$\Delta = \frac{\zeta' + \zeta'' - A(\zeta'\psi')\zeta'' B(\zeta''\psi'') - \zeta' B(\zeta'\psi') A(\zeta''\psi'')}{\psi_1' + \psi''}$$

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$\longrightarrow$ Get $\mathrm{GW}_{g,n}$.


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