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Title : On the rank of quadratic equations of projective varieties

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Example : (1) rank$(x_0 x_1) = 2$

$$\text{since } x_0 x_1 = \left(\frac{x_0 + x_1}{2}\right)^2 + \left(i\frac{x_0 - x_1}{2}\right)^2.$$

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Remark : (1) $1 \leq \textbf{rank}(Q) \leq r+1$

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$$M_d := \{\boldsymbol{x}^I = x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r} \in S \mid I \in A_d\}$$ be the set of all monomials of degree d in S

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Indeed,

$\{z_I z_J - z_K z_L \mid I, J, K, L \in A_d,\ I + J = K + L\}$ generates the homogeneous ideal $I(\nu_d(\mathbb{P}^r))$.

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- $I\big(\nu_\ell(\mathbb{P}^r)\big)$ is generated by quadrics of rank ≤ 4.
- $\nu_\ell(X)$ is a set-theoretic linear section of $\nu_\ell(\mathbb{P}^r)$

  since $X \subset \mathbb{P}^r$ is cut out by forms of degree $\leq d$.

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Theorem (Han-Lee-Moon-Park, 2019) : Suppose that char$(K) \neq 2,3$. Then

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Corollary 1. : For $X \subset \mathbb{P}^r$, let m be an integer such that

X is j-normal for all $j \geq m$ and $I(X) = \langle I(X)_{\leq m} \rangle$ (e.g., $m = \mathrm{reg}(X)$).

Then for all $\ell \geq m$,

the ℓth Veronese variety $\nu_\ell(X)$ of X is ideal-theoretically a linear section of $\nu_\ell(\mathbb{P}^r)$.

In particular, $I(\nu_\ell(X))$ is generated by quadrics of rank 3.

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the minimal free resolution of $I(X)$ is of the form

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(3) We say that (X, L) satisfies property $QR(k)$ if

$I(X)$ can be generated by quadratic equations of rank $\leq k$.

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This gives us the following quadratic equation of C.

$$Q := (x_0+x_1)(x_2+x_3) - (x_1+x_2)^2 \in I(C)_2.$$

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$$C = \{[s^3 : s^2t : st^2 : t^3] \mid [s:t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$$

(1) Let $\Omega = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$. Then $I(C) = I(\Omega, 2) = \langle Q_1 = x_0x_2 - x_1^2,\ Q_2 = x_0x_3 - x_1x_2,\ Q_3 = x_1x_3 - x_2^2 \rangle$.

Also the minimal free resolution of $I(C)$ is $0 \to S(-3)^2 \to S(-2)^3 \to I(C) \to 0$.

Thus $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ satisfies property N_2.

(2) rank(Q_1) =rank$(Q_3) = 3$ and rank$(Q_2) = 4$. Thus $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ satisfies property $QR(4)$.

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Theorem : (Varieties defined by Quadratic Equations, David Mumford, 1969)

Let $X \subset \mathbb{P}^r$ be a nondegenerate irreducible projective variety of degree d. Then for all $\ell \geq d$,

the ℓth Veronese variety $\nu_\ell(X)$ of X is a set-theoretic linear section of $\nu_\ell(\mathbb{P}^r)$.

In particular, $\nu_\ell(X)$ is set-theoretically cut out by quadrics of rank ≤ 4.

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Remark : If (X, L) is determinantally presented, then it satisfies property $QR(4)$.

Example :

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More Examples : (1) Rational Normal Scrolls

(2) Segre Embedding $\sigma(\mathbb{P}^a \times \mathbb{P}^b) \subset \mathbb{P}^{ab+a+b}$

Theorem (Eisenbud-Koh-Stillman, 1988) : Let C be an integral curve of arithmetic genus g. If $\mathscr{L}$ is a line bundle on C of degree $\geq 4g+2$, then $(C, \mathscr{L})$ is determinantally presented.

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That is, there exists a line bundle A on X such that

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Theorem (M. Pucci, 1998) : For the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$, it holds that

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Step 4. : Double induction on (n,d) + $Aut\left(\nu_d(\mathbb{P}^n), \mathbb{P}^N\right)$ ■

Example : The ideal of the rational normal curve $\nu_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is equal to $I(2, \Omega)$ where

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Then $\{F_{i,i+1} \mid 0 \le i \le d-1\} \cup \{G_{ij} \mid 0 \le i < j \le d-2\}$ is a basis for $I\big(\nu_d(\mathbb{P}^1)\big)_2$.

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$$\begin{array}{ccc} \nu_\ell(\mathbb{P}^r) & \subset & \mathbb{P}^{\binom{r+\ell}{r}-1} \\ \cup & & \cup \\ \nu_\ell(X) & \subset & \langle \nu_\ell(X) \rangle \end{array} \quad \Longrightarrow \quad \nu_\ell(X) = \nu_\ell(\mathbb{P}^r) \cap \langle \nu_\ell(X) \rangle$$

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Example : Let $X = Gr(\ell, k^n)$ be the Grassmannian manifold of k-dimensional subspaces of k^n.

Let L be the generator of Pic(X) which defines the Plücker embedding of X.

When $n \geq 3$ and $1 \leq \ell \leq n-2$,

(X, L) fails to satisfy property $QR(5)$ and (X, L^d) satisfies property $QR(3)$ for all $d \geq 2$.

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Corollary 2. : Let L be a very ample line bundle on a projective variety X such that (X, L) satisfies property N_1. Then (X, L^d) satisfies property $QR(3)$ for all $d \geq 2$.

Corollary 3. : Let A be an ample line bundle on a projective variety X. Then there is a positive integer d_0 such that (X, A^d) satisfies property $QR(3)$ for all even $d \geq d_0$.

◎ Curves of High Degree

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Theorem (Park, 2019) : If $g = 0,1$ and $d \geq 2g+2$ or $g \geq 2$ and $d \geq 4g+4$, then

$(C, \mathscr{L})$ satisfies property $QR(3)$.