Open 3-manifolds

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INSTITUT FOURIER

Outline

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  no incompressible tori in N_i.

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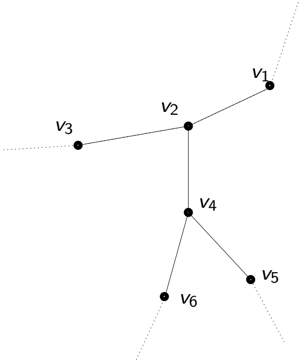

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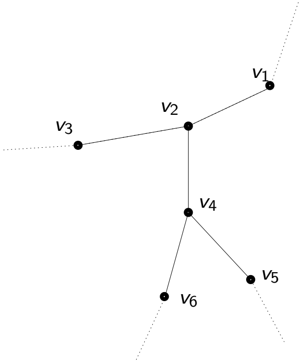

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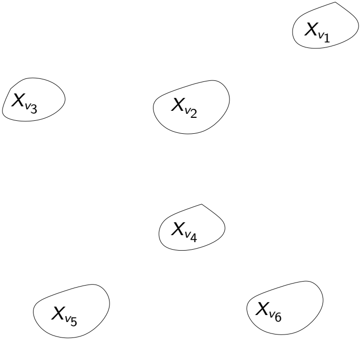

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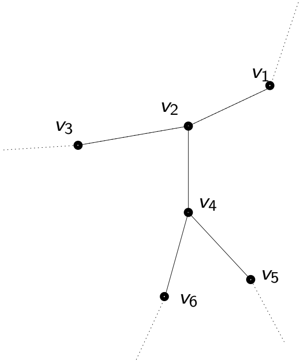

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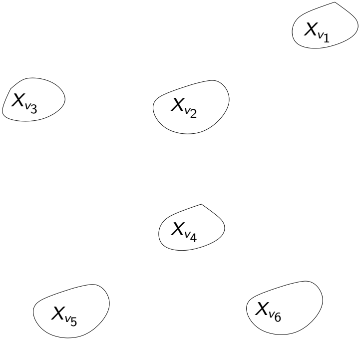

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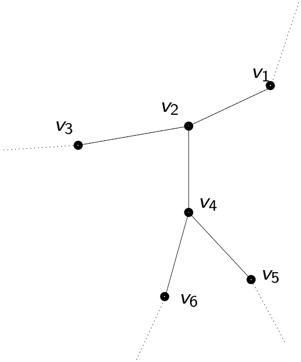

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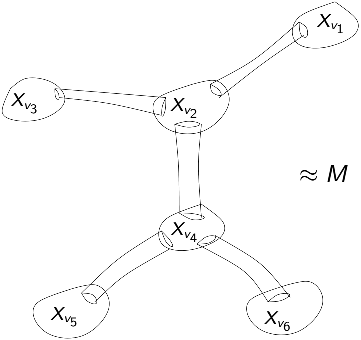

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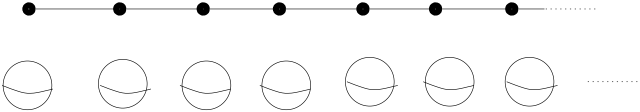

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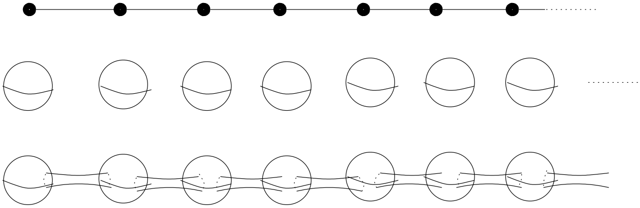

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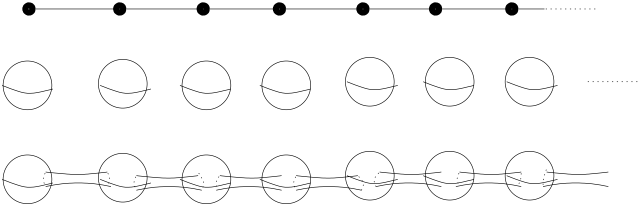

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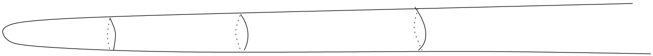

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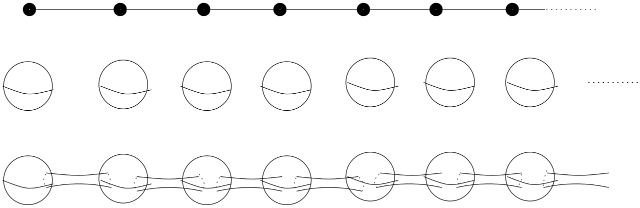

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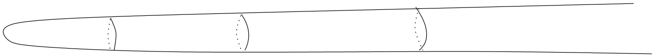

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- $\mathcal{X} = \{S^3\}$, $T =$ the line $\rightsquigarrow S^2 \times \mathbf{R}$

Graphs versus trees

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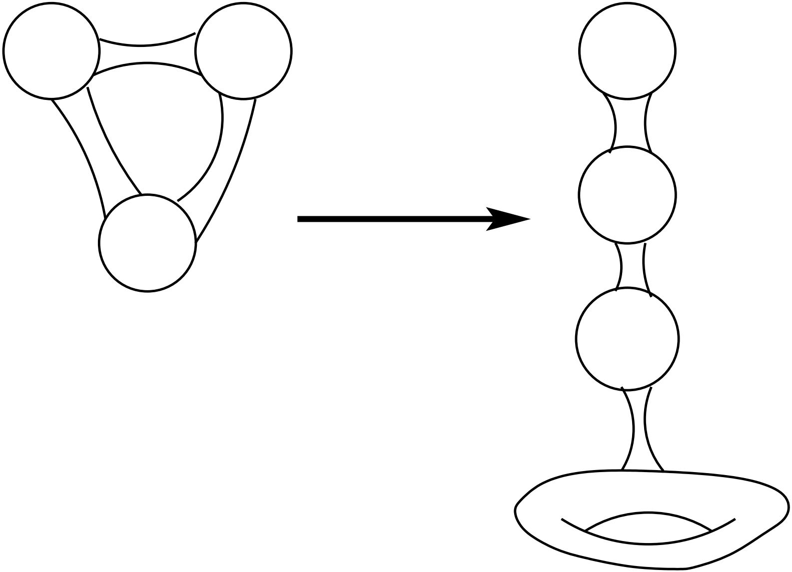

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Theorem (Bessières-B.-Maillot)

M has a complete metric of bounded geometry and $\operatorname{Scal} \geqslant 1$ iff there is a finite collection $\mathcal{F}$ of spherical manifolds such that M is a (maybe infinite) connected sum of copies of $S^2 \times S^1$ and members of $\mathcal{F}$.

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Take $T_1 \supset T_2 \supset T_3 \supset \ldots$ solid tori.

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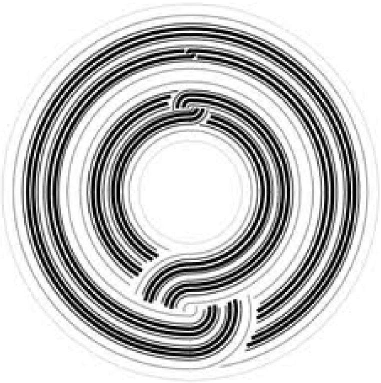

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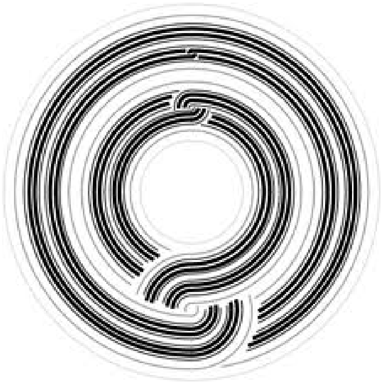

On the picture $T_{i+1} \subset T_i \subset T_{i-1}$.

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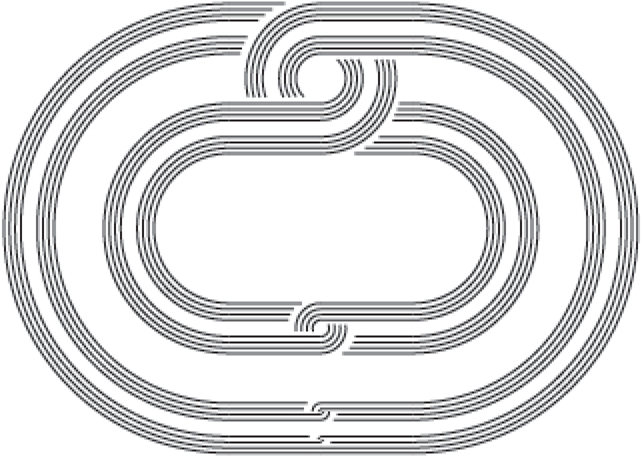
The idea is that the core of T_i and the meridian of T_{i-1} form the Whitehead link.

Whitehead continuum



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- Uncountably many examples which do not embed in S^3 (Kister-McMillan).
- Examples that cannot cover non-trivially any manifold (Myers).

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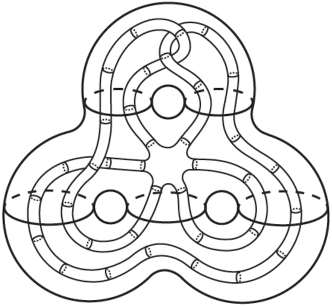

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- Same if π_1^∞ is trivial.

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What about higher dimension ? Exotic differential structure on $\mathbf{R}^4$?