Virtual Intersection Theories

Hyeonjun Park

Seoul National University

French-Korean Conference

November 26, 2019

Joint work with Young-Hoon Kiem

Based on arXiv:1908.03340

Overview

Virtual Fundamental Classes (Motivation)

Kontsevich's Hidden Smoothness Philosophy
"Many singular moduli spaces are truncations of smooth *derived* moduli spaces (in some sense)."

Virtual Fundamental Classes (Motivation)

Kontsevich's Hidden Smoothness Philosophy
"Many singular moduli spaces are truncations of smooth *derived* moduli spaces (in some sense)."

$\Longrightarrow$ These singular moduli spaces should carry *virtual fundamental classes*, which behaves like the fundamental classes of smooth schemes.

Kontsevich's Hidden Smoothness Philosophy
"Many singular moduli spaces are truncations of smooth *derived* moduli spaces (in some sense)."

$\Longrightarrow$ These singular moduli spaces should carry *virtual fundamental classes*, which behaves like the fundamental classes of smooth schemes.

Construction of Virtual Fundamental Classes
Rigorous mathematical definitions of virtual fundamental classes were introduced through the concept of *perfect obstruction theories*.

- [Behrend-Fantechi, Invent. Math. 1997]
- [Li-Tian, J. Amer. Math. Soc. 1998]

Virtual Fundamental Classes (Toy Model)

Consider a smooth scheme X and a Cartesian diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with smooth closed subschemes $Y, Z \subseteq X$.

Virtual Fundamental Classes (Toy Model)

Consider a smooth scheme X and a Cartesian diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with smooth closed subschemes $Y, Z \subseteq X$.

The *virtual fundamental class* of the intersection $W = Y \cap Z$ is given by the refined intersection product

$$[W]^{\mathrm{vir}} := [Y] \cdot [Z] = 0^!_{N_{Y/X}|_W}[C_{W/Z}] \in CH_*(W).$$

Virtual Fundamental Classes (Toy Model)

Consider a smooth scheme X and a Cartesian diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with smooth closed subschemes $Y, Z \subseteq X$.

The *virtual fundamental class* of the intersection $W = Y \cap Z$ is given by the refined intersection product

$$[W]^{\mathrm{vir}} := [Y] \cdot [Z] = 0^!_{N_{Y/X}|_W}[C_{W/Z}] \in CH_*(W).$$

- If Y and Z meet transversely, then $[W]^{\mathrm{vir}} = [W]$.

Virtual Fundamental Classes (Toy Model)

Consider a smooth scheme X and a Cartesian diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with smooth closed subschemes $Y, Z \subseteq X$.

The *virtual fundamental class* of the intersection $W = Y \cap Z$ is given by the refined intersection product

$$[W]^{\mathrm{vir}} := [Y] \cdot [Z] = 0^{!}_{N_{Y/X}|_W}[C_{W/Z}] \in CH_*(W).$$

- If Y and Z meet transversely, then $[W]^{\mathrm{vir}} = [W]$.
- If $Y = Z$, then $[W]^{\mathrm{vir}} = c_{\mathrm{top}}(N_{W/X})$.

Virtual Fundamental Classes (Toy Model)

Consider a smooth scheme X and a Cartesian diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with smooth closed subschemes $Y, Z \subseteq X$.

The *virtual fundamental class* of the intersection $W = Y \cap Z$ is given by the refined intersection product

$$[W]^{\mathrm{vir}} := [Y] \cdot [Z] = 0^!_{N_{Y/X}|_W}[C_{W/Z}] \in CH_*(W).$$

- If Y and Z meet transversely, then $[W]^{\mathrm{vir}} = [W]$.
- If $Y = Z$, then $[W]^{\mathrm{vir}} = c_{\mathrm{top}}(N_{W/X})$.

The virtual class $[W]^{\mathrm{vir}} \in CH_*(W)$ is not an *intrinsic* object. It depends on the additional information $C_{W/Z} \subseteq N_{Y/X}|_W$.

Virtual Fundamental Classes (Construction)

Intrinsic Normal Cone Let X be a quasi-projective scheme. Consider an embedding $X \hookrightarrow Y$ into a smooth scheme Y. The *intrinsic normal cone* of X is defined to be the quotient stack

$$\mathfrak{C}_X := [C_{X/Y}/\mathbb{T}_Y|_X],$$

which is independent of the choice of Y.

Intrinsic Normal Cone Let X be a quasi-projective scheme. Consider an embedding $X \hookrightarrow Y$ into a smooth scheme Y. The *intrinsic normal cone* of X is defined to be the quotient stack

$$\mathfrak{C}_X := [C_{X/Y}/\mathbb{T}_Y|_X],$$

which is independent of the choice of Y.

- The intrinsic normal cone $\mathfrak{C}_X$ is an intrinsic object which is "homotopically equivalent" to the normal cone $C_{X/Y}$ whenever we have an embedding $X \hookrightarrow Y$ into a smooth scheme Y.

Intrinsic Normal Cone Let X be a quasi-projective scheme. Consider an embedding $X \hookrightarrow Y$ into a smooth scheme Y. The *intrinsic normal cone* of X is defined to be the quotient stack

$$\mathfrak{C}_X := [C_{X/Y}/\mathbb{T}_Y|_X],$$

which is independent of the choice of Y.

- The intrinsic normal cone $\mathfrak{C}_X$ is an intrinsic object which is "homotopically equivalent" to the normal cone $C_{X/Y}$ whenever we have an embedding $X \hookrightarrow Y$ into a smooth scheme Y.

Perfect Obstruction Theory A *perfect obstruction theory* for X is a closed immersion

$$\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$$

of the intrinsic normal cone into a vector bundle stack $\mathfrak{E}$. A vector bundle stack is a quotient stack $[E_1/E_0]$ for some morphism $E_0 \to E_1$ of vector bundles on X.

Virtual Fundamental Classes (Construction)

Intrinsic Normal Cone Let X be a quasi-projective scheme. Consider an embedding $X \hookrightarrow Y$ into a smooth scheme Y. The *intrinsic normal cone* of X is defined to be the quotient stack

$$\mathfrak{C}_X := [C_{X/Y}/\mathbb{T}_Y|_X],$$

which is independent of the choice of Y.

- The intrinsic normal cone $\mathfrak{C}_X$ is an intrinsic object which is "homotopically equivalent" to the normal cone $C_{X/Y}$ whenever we have an embedding $X \hookrightarrow Y$ into a smooth scheme Y.

Perfect Obstruction Theory A *perfect obstruction theory* for X is a closed immersion

$$\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$$

of the intrinsic normal cone into a vector bundle stack $\mathfrak{E}$. A vector bundle stack is a quotient stack $[E_1/E_0]$ for some morphism $E_0 \to E_1$ of vector bundles on X.

- Many moduli spaces have natural perfect obstruction theories driven from their deformation theories (e.g. Hilbert scheme of curves on surfaces, moduli of stable maps, and moduli of stable sheaves on CY3).

Intrinsic Normal Cone Let X be a quasi-projective scheme. Consider an embedding $X \hookrightarrow Y$ into a smooth scheme Y. The *intrinsic normal cone* of X is defined to be the quotient stack

$$\mathfrak{C}_X := [C_{X/Y}/\mathbb{T}_Y|_X],$$

which is independent of the choice of Y.

- The intrinsic normal cone $\mathfrak{C}_X$ is an intrinsic object which is "homotopically equivalent" to the normal cone $C_{X/Y}$ whenever we have an embedding $X \hookrightarrow Y$ into a smooth scheme Y.

Perfect Obstruction Theory A *perfect obstruction theory* for X is a closed immersion

$$\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$$

of the intrinsic normal cone into a vector bundle stack $\mathfrak{E}$. A vector bundle stack is a quotient stack $[E_1/E_0]$ for some morphism $E_0 \to E_1$ of vector bundles on X.

- Many moduli spaces have natural perfect obstruction theories driven from their deformation theories (e.g. Hilbert scheme of curves on surfaces, moduli of stable maps, and moduli of stable sheaves on CY3).

The notion of intrinsic normal cones and perfect obstruction theories can be generalized to DM stacks.

Virtual Fundamental Classes (Construction)

Definition (Virtual Fundamental Class)

Let $\mathcal{X}$ be a Deligne-Mumford stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$. The *virtual fundamental class* of $\mathcal{X}$ is defined to be

$$[\mathcal{X}]^{\mathrm{vir}} := 0^!_{\mathfrak{E}}[\mathfrak{C}_{\mathcal{X}}] \in CH_*(\mathcal{X}).$$

Definition (Virtual Fundamental Class)

Let $\mathcal{X}$ be a Deligne-Mumford stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$. The *virtual fundamental class* of $\mathcal{X}$ is defined to be

$$[\mathcal{X}]^{\mathrm{vir}} := 0^!_{\mathfrak{E}}[\mathfrak{C}_{\mathcal{X}}] \in CH_*(\mathcal{X}).$$

Virtual Enumerative Invariants If $\mathcal{X}$ is proper, then we can define virtual invariants by

$$\int_{[\mathcal{X}]^{\mathrm{vir}}} c_{i_1}(E_1) \cdots c_{i_r}(E_r) \in \mathbb{Q}$$

for vector bundles $E_1, \cdots, E_r$ on X.

Virtual Fundamental Classes (Construction)

Definition (Virtual Fundamental Class)

Let $\mathcal{X}$ be a Deligne-Mumford stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$. The *virtual fundamental class* of $\mathcal{X}$ is defined to be

$$[\mathcal{X}]^{\mathrm{vir}} := 0^!_{\mathfrak{E}}[\mathfrak{C}_{\mathcal{X}}] \in CH_*(\mathcal{X}).$$

Virtual Enumerative Invariants If $\mathcal{X}$ is proper, then we can define virtual invariants by

$$\int_{[\mathcal{X}]^{\mathrm{vir}}} c_{i_1}(E_1)\cdots c_{i_r}(E_r) \in \mathbb{Q}$$

for vector bundles $E_1, \cdots, E_r$ on X.

Example Gromov-Witten invariants, Donaldson-Thomas invariants and Pandharipande-Thomas invariants are defined in this way.

Virtual Pullback (Manolache, J. Algebraic Geom. 2012)
Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM-type morphism of algebraic stacks with a relative perfect obstruction theory $\imath : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$. Then there is a *virtual pullback*

$$f^! : CH_*(\mathcal{Y}) \to CH_{*+d}(\mathcal{X}).$$

If $\mathcal{X}$ and $\mathcal{Y}$ are equipped with perfect obstruction theories which are compatible with the obstruction theory of f, then we have

$$[\mathcal{X}]^{\mathrm{vir}} = f^![\mathcal{Y}]^{\mathrm{vir}} \in CH_*(\mathcal{X}).$$

Torus Localization (Graber-Pandharipande, Invent. Math. 1999)
If $\mathcal{X}$ is a DM stack equipped with a $T = \mathbb{G}_m$-action and a T-equivariant perfect obstruction theory $\imath : \mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{E}$, then the fixed point locus $\mathcal{X}^T$ also has a natural perfect obstruction theory and

$$[\mathcal{X}]^{\mathrm{vir}} = \jmath_*\left(\frac{[\mathcal{X}^T]^{\mathrm{vir}}}{e(N^{\mathrm{vir}}_{\mathcal{X}^T/\mathcal{X}})}\right) \in CH^T_*(\mathcal{X}) \otimes_{\mathbb{Z}[t]} \mathbb{Q}[t, t^{-1}].$$

where $\jmath : \mathcal{X}^T \hookrightarrow \mathcal{X}$ is the inclusion.

Cosection Localization (Kiem-Li, J. Amer. Math. Soc. 2013)
If $\mathcal{X}$ is a DM stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{E}$ and a cosection $\sigma : \mathfrak{E} \to \mathbb{A}^1_X$, then there is a cosection localized virtual fundamental class $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{loc}} \in CH_*(\mathcal{X}(\sigma))$ in the zero locus $\mathcal{X}(\sigma)$ such that

$$[\mathcal{X}]^{\mathrm{vir}} = \jmath_*[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{loc}} \in CH(\mathcal{X})$$

where $\jmath : \mathcal{X}(\sigma) \hookrightarrow \mathcal{X}$ is the inclusion.

Recall that the *Chow groups*

$$CH_*(X) = \frac{\text{algebraic cycles on } X}{\text{rational equivalences}}$$

has following additional structures :

1. Projective Pushforward) $f_* : CH_*(X) \to CH_*(Y) : [\xi] \mapsto \deg(f|_\xi)[f(\xi)]$;
2. Smooth Pullback) $f^* : CH_*(Y) \to CH_{*+e}(X) : [\eta] \mapsto [f^{-1}\eta]$;
3. Exterior Product) $CH_*(X) \otimes CH_*(Y) \to CH(X \times Y) : [\xi] \otimes [\eta] \mapsto [\xi] \times [\eta]$;
4. Gysin Pullback) $\imath^! : CH_*(Y) \to CH_{*-c}(X) : \eta \mapsto [Y] \cap [\eta]$.

Recall that the *Chow groups*

$$CH_*(X) = \frac{\text{algebraic cycles on } X}{\text{rational equivalences}}$$

has following additional structures :

1. Projective Pushforward) $f_* : CH_*(X) \to CH_*(Y) : [\xi] \mapsto \deg(f|_\xi)[f(\xi)]$;
2. Smooth Pullback) $f^* : CH_*(Y) \to CH_{*+e}(X) : [\eta] \mapsto [f^{-1}\eta]$;
3. Exterior Product) $CH_*(X) \otimes CH_*(Y) \to CH(X \times Y) : [\xi] \otimes [\eta] \mapsto [\xi] \times [\eta]$;
4. Gysin Pullback) $\imath^! : CH_*(Y) \to CH_{*-c}(X) : \eta \mapsto [Y] \cap [\eta]$.

Also, Chow groups satisfy following properties :

1. Excision) $CH_*(Z) \to CH_*(X) \to CH_*(U) \to 0$ is exact for a closed immersion $Z \hookrightarrow X$;
2. Homotopy) $CH_*(X) \xrightarrow{\cong} CH_{*+r}(E)$ is an isomorphism for a vector bundle torsor $E \to X$.

Definition (Intersection Theory)

An *intersection theory* H_* for schemes is a collection of graded abelian groups

$$H_*(X)$$

for each quasi-projective scheme X equipped with projective pushforwards, smooth pullbacks, Gysin pullbacks, and exterior products satisfying natural functorial properties and homotopy property, excision property, and projective bundle formula.

Definition (Intersection Theory)

An *intersection theory* H_* for schemes is a collection of graded abelian groups

$$H_*(X)$$

for each quasi-projective scheme X equipped with projective pushforwards, smooth pullbacks, Gysin pullbacks, and exterior products satisfying natural functorial properties and homotopy property, excision property, and projective bundle formula.

Example (Chow Theory) The Chow groups $CH_*(X)$ of algebraic cycles modulo rational equivalences on X form an intersection theory.

Definition (Intersection Theory)

An *intersection theory* H_* for schemes is a collection of graded abelian groups

$$H_*(X)$$

for each quasi-projective scheme X equipped with projective pushforwards, smooth pullbacks, Gysin pullbacks, and exterior products satisfying natural functorial properties and homotopy property, excision property, and projective bundle formula.

Example (Chow Theory) The Chow groups $CH_*(X)$ of algebraic cycles modulo rational equivalences on X form an intersection theory.

Example (Algebraic K-Theory) The Grothendieck groups $K_0(X)$ of coherent sheaves on X form an intersection theory.

Definition (Intersection Theory)

An *intersection theory* H_* for schemes is a collection of graded abelian groups

$$H_*(X)$$

for each quasi-projective scheme X equipped with projective pushforwards, smooth pullbacks, Gysin pullbacks, and exterior products satisfying natural functorial properties and homotopy property, excision property, and projective bundle formula.

Example (Chow Theory) The Chow groups $CH_*(X)$ of algebraic cycles modulo rational equivalences on X form an intersection theory.

Example (Algebraic K-Theory) The Grothendieck groups $K_0(X)$ of coherent sheaves on X form an intersection theory.

There are infinitely many intersection theories because we can always construct a new theory by twisting a given theory with Todd classes.

Example (Algebraic Cobordism)

The *algebraic cobordism groups*

$$\Omega_*(X) = \frac{\text{cobordism cycles on } X}{\text{double point relations}}$$

for schemes X form an intersection theory.

- A *cobordism cycle* is a $\mathbb{Z}$-linear combination of projective morphisms

$$[f : Z \to X]$$

  from smooth quasi-projective schemes Z.
- Let $W \to X \times \mathbb{P}^1$ be a projective morphism from a smooth scheme W such that the fiber of $W \to \mathbb{P}^1$ over $\infty \in \mathbb{P}^1(\mathbf{k})$ is smooth and the fiber over $0 \in \mathbb{P}^1(\mathbf{k})$ is the union $W_0 = A \cup B$ of two smooth divisors intersecting transversely. The associated *double point relation* is

$$[W_\infty \to X] = [A \to X] + [B \to X] - [P \to X]$$

  where $D = A \cap B$, $P = \mathbb{P}_D(N_{D/A} \oplus \mathcal{O}_D)$.

Theorem (Levine-Pandharipande, Levine-Morel)

Algebraic cobordism Ω *is a universal intersection theory.*

Intersection Theories

Theorem (Levine-Pandharipande, Levine-Morel)

Algebraic cobordism Ω is a universal intersection theory.

By the universality, we have natural maps

$$\begin{array}{ccc} & \Omega_*(X) & \\ \swarrow & & \searrow \\ K_0(X)[\beta, \beta^{-1}] & & CH_*(X) \end{array}$$

for any scheme X.

Intersection Theories

Theorem (Levine-Pandharipande, Levine-Morel)

Algebraic cobordism Ω is a universal intersection theory.

By the universality, we have natural maps

$$K_0(X)[\beta, \beta^{-1}] \longleftarrow \Omega_*(X) \longrightarrow CH_*(X)$$

for any scheme X.

Under the above natural maps, algebraic cobordism recovers both the algebraic K-theory and the Chow theory:

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \cong K_0(X)[\beta, \beta^{-1}],$$
$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z} \cong CH_*(X).$$

Virtual fundamental classes have also been studied in other intersection theories.

Virtual Fundamental Classes in Other Intersection Theories

Virtual fundamental classes have also been studied in other intersection theories.

Virtual Structure Sheaves (in Algebraic K-Theory)

1. construction of virtual fundamental classes [Lee, Duke Math. J. 2004]
2. virtual pullback [Qu, Ann. Inst. Fourier 2018]
3. virtual torus localization [Qu, Ann. Inst. Fourier 2018]
4. cosection localization [Kiem-Li, Int. Math. Res. Not. 2018]
5. virtual Riemann-Roch [Fantechi-Gottsche, Geom. Topol 2010]

Virtual Fundamental Classes in Other Intersection Theories

Virtual fundamental classes have also been studied in other intersection theories.

Virtual Structure Sheaves (in Algebraic K-Theory)

1. construction of virtual fundamental classes [Lee, Duke Math. J. 2004]
2. virtual pullback [Qu, Ann. Inst. Fourier 2018]
3. virtual torus localization [Qu, Ann. Inst. Fourier 2018]
4. cosection localization [Kiem-Li, Int. Math. Res. Not. 2018]
5. virtual Riemann-Roch [Fantechi-Gottsche, Geom. Topol 2010]

Virtual Cobordism Classes (in Algebraic Cobordism)

1. construction of virtual fundamental classes (for quasi-projective schemes) [Shen, J. Lond. Math. Soc. 2016].

Question : Can we develop a theory of virtual fundamental classes in all intersection theories?

Question : Can we develop a theory of virtual fundamental classes in all intersection theories?

Mostly, it is sufficient to consider algebraic cobordism because it is universal.

One of the main problem is that it is hard to extend an intersection theory for schemes to stacks.

Intersection Theories for Stacks

One of the main problem is that it is hard to extend an intersection theory for schemes to stacks.

For Chow theory, it took a long time and many groundbreaking papers to extend the theory for schemes to stacks

(schemes) [Fulton, 1984]

↓

(DM stacks) [Vistoli, Invent. Math. 1989]

↓

(Artin stacks) [Kresch, Invent. Math. 1999]

Intersection Theories for Stacks

One of the main problem is that it is hard to extend an intersection theory for schemes to stacks.

For Chow theory, it took a long time and many groundbreaking papers to extend the theory for schemes to stacks

(schemes) [Fulton, 1984]

$\downarrow$

(DM stacks) [Vistoli, Invent. Math. 1989]

$\downarrow$

(Artin stacks) [Kresch, Invent. Math. 1999]

Since Kresch's theory uses the structure of the Chow theory, it seems difficult to apply it directly to other intersection theories (even to algebraic cobordism).

Definition (Kiem-P.)

Let H_* be an intersection theory for schemes. For any algebraic stack $\mathcal{X}$, the limit intersection theory is defined to be the inverse limit

$$\mathcal{H}_d(\mathcal{X}) := \varprojlim_{t:T\to\mathcal{X}} H_{d+d(t)}(T)$$

where the limit is taken over all smooth morphisms $t : T \to \mathcal{X}$ from quasi-projective schemes T, and the transition maps are given by the lci pullbacks $s^* : H_{*+d(t_2)}(T_2) \to H_{*+d(t_1)}(T_1)$ for commutative diagrams

$$\begin{array}{ccc} T_1 & \xrightarrow{\;s\;} & T_2 \\ & {\scriptstyle t_1}\searrow \quad \swarrow{\scriptstyle t_2} & \\ & \mathcal{X} & \end{array}$$

with t_1 and t_2 being smooth morphisms from quasi-projective schemes.

Limit Intersection Theories

Theorem (Kiem-P.)

The limit intersection theory $\mathcal{H}_*$ is a "weak" intersection theory for algebraic stacks which admit "good approximations by schemes".

Theorem (Kiem-P.)

The limit intersection theory $\mathcal{H}_*$ is a "weak" intersection theory for algebraic stacks which admit "good approximations by schemes".

A *weak* intersection theory is a theory which has all the structures and properties of intersection theories except that the excision property is replaced by a weaker version of it.

Theorem (Kiem-P.)

The limit intersection theory $\mathcal{H}_*$ is a "weak" intersection theory for algebraic stacks which admit "good approximations by schemes".

A *weak* intersection theory is a theory which has all the structures and properties of intersection theories except that the excision property is replaced by a weaker version of it.

Examples of good approximations

1. All quotient stacks have good approximations, using Totaro's algebraic approximations of classifying spaces.
2. All vector bundle stacks and cone stacks over quotient stacks have good approximations.

The (2-)category of algebraic stacks which have good approximations is closed under basic operations.

The limit intersection theories recover *equivariant intersection theories*.

Limit Intersection Theories

The limit intersection theories recover *equivariant intersection theories*.

Let $\mathcal{X} = [X/G]$ be a global quotient stack (precisely X is a quasi-projective scheme and G is a linear algebraic group acting on X linearly).

The limit intersection theories recover *equivariant intersection theories*.

Let $\mathcal{X} = [X/G]$ be a global quotient stack (precisely X is a quasi-projective scheme and G is a linear algebraic group acting on X linearly).

- If $H_* = CH_*$ is Fulton's Chow theory, then we have an isomorphism

$$CH_*^{limit}(\mathcal{X}) \cong CH_*^G(X)$$

to Edidin-Graham's equivariant Chow theory.

The limit intersection theories recover *equivariant intersection theories*.

Let $\mathcal{X} = [X/G]$ be a global quotient stack (precisely X is a quasi-projective scheme and G is a linear algebraic group acting on X linearly).

- If $H_* = CH_*$ is Fulton's Chow theory, then we have an isomorphism

$$CH_*^{limit}(\mathcal{X}) \cong CH_*^G(X)$$

  to Edidin-Graham's equivariant Chow theory.
- If $H_* = \Omega_*$ is Levine-Morel's algebraic cobordism, then we have an isomorphism

$$\Omega_*^{limit}(\mathcal{X}) \cong \Omega_*^G(X)$$

  to Heller-Malagon-Lopez and Krishna's equivariant algebraic cobordism.

Main Theorem

Main Theorem

Theorem (Kiem-P.)

Let H_* be an intersection theory for schemes. For a quasi-projective DM stack X equipped with a perfect obstruction theory, there is a virtual fundamental class

$$[X]^{\mathrm{vir}} \in H_*(X)$$

satisfying

1. *virtual pullback formula,*
2. *torus localization formula, and*
3. *cosection localization principle.*

Virtual Cobordism Classes

Corollary

If X is a quasi-projective scheme, then the virtual cobordism class maps to the virtual structure sheaf and the virtual fundamental class (in Chow) under the canonical maps:

$$\begin{array}{ccc} & \Omega_*(X) & \\ \swarrow & & \searrow \\ K_0(X)[\beta^{\pm 1}] & & CH_*(X) \end{array} \qquad \begin{array}{ccc} & [X]^{\mathrm{vir}}_{\Omega} & \\ \swarrow & & \searrow \\ \mathcal{O}_X^{\mathrm{vir}} & & [X]^{\mathrm{vir}}_{\mathrm{CH}} \end{array}$$

In addition, the three key techniques on virtual structure sheaves and virtual fundametal classes comes from those techniques on virtual cobordism classes.

Virtual Cobordism Classes

Corollary

If X is a quasi-projective scheme, then the virtual cobordism class maps to the virtual structure sheaf and the virtual fundamental class (in Chow) under the canonical maps:

$$\begin{array}{ccc} & \Omega_*(X) & \\ \swarrow & & \searrow \\ K_0(X)[\beta^{\pm 1}] & & CH_*(X) \end{array} \qquad \begin{array}{ccc} & [X]^{\mathrm{vir}}_{\Omega} & \\ \swarrow & & \searrow \\ \mathcal{O}^{\mathrm{vir}}_X & & [X]^{\mathrm{vir}}_{\mathrm{CH}} \end{array}$$

In addition, the three key techniques on virtual structure sheaves and virtual fundametal classes comes from those techniques on virtual cobordism classes.

$\Longrightarrow$ This unifies the theory of virtual structure sheaves and the theory of virtual fundamental classes (in Chow).

Construction of the Virtual Fundamental Class

Construction of the Virtual Fundamental Class

Recall that $[X]^{\mathrm{vir}}_{\mathrm{CH}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in CH_*(X)$.

Construction of the Virtual Fundamental Class

Recall that $[X]^{\mathrm{vir}}_{\mathrm{CH}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in CH_*(X)$.

In general intersection theories, the *fundamental classes* may not exist for singular schemes [Levine, Fundamental classes in algebraic cobordism, 2003].

Construction of the Virtual Fundamental Class

Recall that $[X]^{\mathrm{vir}}_{\mathrm{CH}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in CH_*(X)$.

In general intersection theories, the *fundamental classes* may not exist for singular schemes [Levine, Fundamental classes in algebraic cobordism, 2003].

We can still define the fundamental class of the intrinsic normal cone using Fulton-MacPherson's deformation to the normal cone.

Construction of the Virtual Fundamental Class

Recall that $[X]^{\mathrm{vir}}_{\mathrm{CH}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in CH_*(X)$.

In general intersection theories, the *fundamental classes* may not exist for singular schemes [Levine, Fundamental classes in algebraic cobordism, 2003].

We can still define the fundamental class of the intrinsic normal cone using Fulton-MacPherson's deformation to the normal cone. Embed X into a smooth quasi-projective DM stack Y. Let

$$[\mathfrak{C}_X] := (k^*)^{-1} \circ \mathrm{sp}_{X/Y}[Y] \in \mathcal{H}_0(\mathfrak{C}_X).$$

$\mathrm{sp}_{X/Y} : \mathcal{H}_*(Y) \to \mathcal{H}_*(C_{X/Y})$ is the specialization map given by $M^\circ_{X/Y}$;
$k^* : \mathcal{H}_*(\mathfrak{C}_X) \xrightarrow{\cong} \mathcal{H}_{*+e}(C_{X/Y})$ is given by the homotopy property.

Construction of the Virtual Fundamental Class

Recall that $[X]^{\mathrm{vir}}_{\mathrm{CH}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in CH_*(X)$.

In general intersection theories, the *fundamental classes* may not exist for singular schemes [Levine, Fundamental classes in algebraic cobordism, 2003].

We can still define the fundamental class of the intrinsic normal cone using Fulton-MacPherson's deformation to the normal cone. Embed X into a smooth quasi-projective DM stack Y. Let

$$[\mathfrak{C}_X] := (k^*)^{-1} \circ \mathrm{sp}_{X/Y}[Y] \in \mathcal{H}_0(\mathfrak{C}_X).$$

$\mathrm{sp}_{X/Y} : \mathcal{H}_*(Y) \to \mathcal{H}_*(C_{X/Y})$ is the specialization map given by $M^\circ_{X/Y}$;
$k^* : \mathcal{H}_*(\mathfrak{C}_X) \xrightarrow{\cong} \mathcal{H}_{*+e}(C_{X/Y})$ is given by the homotopy property.

Then as in the original construction, we can define the virtual fundamental class by $[X]^{\mathrm{vir}} := 0^!_{\mathfrak{E}} \circ \imath_*[\mathfrak{C}_X] \in \mathcal{H}_*(X)$.

Construction of the Virtual Fundamental Class

Recall that $[X]^{\mathrm{vir}}_{\mathrm{CH}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in CH_*(X)$.

In general intersection theories, the *fundamental classes* may not exist for singular schemes [Levine, Fundamental classes in algebraic cobordism, 2003].

We can still define the fundamental class of the intrinsic normal cone using Fulton-MacPherson's deformation to the normal cone. Embed X into a smooth quasi-projective DM stack Y. Let

$$[\mathfrak{C}_X] := (k^*)^{-1} \circ \mathrm{sp}_{X/Y}[Y] \in \mathcal{H}_0(\mathfrak{C}_X).$$

$\mathrm{sp}_{X/Y} : \mathcal{H}_*(Y) \to \mathcal{H}_*(C_{X/Y})$ is the specialization map given by $M^\circ_{X/Y}$;
$k^* : \mathcal{H}_*(\mathfrak{C}_X) \xrightarrow{\cong} \mathcal{H}_{*+e}(C_{X/Y})$ is given by the homotopy property.

Then as in the original construction, we can define the virtual fundamental class by $[X]^{\mathrm{vir}} := 0^!_{\mathfrak{E}} \circ \imath_*[\mathfrak{C}_X] \in \mathcal{H}_*(X)$.

(In progress) This can be generalized to *any* DM stack, without assuming the existence of a global embedding into a smooth DM stack.

Virtual Pullback

Let $f : X \to Y$ be a quasi-projective morphism of algebraic stacks equipped with a relative perfect obstruction theory $\imath : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$.

Virtual Pullback

Let $f : X \to Y$ be a quasi-projective morphism of algebraic stacks equipped with a relative perfect obstruction theory $\imath : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$.

1) Assume that Y is a quasi-projective scheme.

Sketch of the proof of the main theorem

Virtual Pullback

Let $f : X \to Y$ be a quasi-projective morphism of algebraic stacks equipped with a relative perfect obstruction theory $\imath : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$.

1) Assume that Y is a quasi-projective scheme. Factor f by

$$\begin{array}{ccc} & & Z \\ & \overset{g}{\nearrow} & \big\downarrow h \\ X & \xrightarrow{\ f\ } & Y, \end{array}$$

a closed immersion $g : X \hookrightarrow Z$ and a smooth morphism $h : Z \to Y$.

Sketch of the proof of the main theorem

Virtual Pullback

Let $f : X \to Y$ be a quasi-projective morphism of algebraic stacks equipped with a relative perfect obstruction theory $\imath : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$.

1) Assume that Y is a quasi-projective scheme. Factor f by

$$\begin{array}{ccc} & & Z \\ & \overset{g}{\nearrow} & \downarrow h \\ X & \xrightarrow{f} & Y, \end{array}$$

a closed immersion $g : X \hookrightarrow Z$ and a smooth morphism $h : Z \to Y$. We define the specialization map $\mathrm{sp}_{X/Y}$ by the unique map that fits into the diagram

$$\begin{array}{ccc} H_{*+e}(Z) & \xrightarrow{\mathrm{sp}_{X/Z}} & H_{*+e}(C_{X/Z}) \\ \uparrow h^* & & \uparrow \cong \\ H_*(Y) & \overset{\mathrm{sp}_{X/Y}}{\dashrightarrow} & \mathcal{H}_*(\mathfrak{C}_{X/Y}). \end{array}$$

Sketch of the proof of the main theorem

Virtual Pullback

Let $f : X \to Y$ be a quasi-projective morphism of algebraic stacks equipped with a relative perfect obstruction theory $\imath : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$.

1) Assume that Y is a quasi-projective scheme. Factor f by

$$\begin{array}{ccc} & & Z \\ & \overset{g}{\nearrow} & \downarrow h \\ X & \xrightarrow{f} & Y, \end{array}$$

a closed immersion $g : X \hookrightarrow Z$ and a smooth morphism $h : Z \to Y$. We define the specialization map $\mathrm{sp}_{X/Y}$ by the unique map that fits into the diagram

$$\begin{array}{ccc} H_{*+e}(Z) & \xrightarrow{\mathrm{sp}_{X/Z}} & H_{*+e}(C_{X/Z}) \\ \uparrow h^* & & \uparrow \cong \\ H_*(Y) & \overset{\mathrm{sp}_{X/Y}}{\dashrightarrow} & \mathcal{H}_*(\mathfrak{C}_{X/Y}). \end{array}$$

Define the virtual pullback by the composition

$$f^! : H_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} \mathcal{H}_*(\mathfrak{C}_{X/Y}) \xrightarrow{\imath_*} \mathcal{H}_*(\mathfrak{E}) \xrightarrow{0^!_{\mathfrak{E}}} H_{*+d}(X).$$

2) Assume that Y is an algebraic stack which has a good system of approximations $\{y_i : Y_i \to Y\}_i$.

2) Assume that Y is an algebraic stack which has a good system of approximations $\{y_i : Y_i \to Y\}_i$. Let

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow & & \downarrow{\scriptstyle y_i} \\ X & \xrightarrow{f} & Y \end{array}$$

be a Cartesian square.

Sketch of the proof of the main theorem

2) Assume that Y is an algebraic stack which has a good system of approximations $\{y_i : Y_i \to Y\}_i$. Let

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow & & \downarrow{\scriptstyle y_i} \\ X & \xrightarrow{f} & Y \end{array}$$

be a Cartesian square. Then we can define the virtual pullbacks

$$f_i^! : H_*(Y_i) \to H_{*+d}(X_i)$$

for the induced maps $f_i : X_i \to Y_i$ and the induced perfect obstruction theories since they are morphisms of schemes.

Sketch of the proof of the main theorem

2) Assume that Y is an algebraic stack which has a good system of approximations $\{y_i : Y_i \to Y\}_i$. Let

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow & & \downarrow{\scriptstyle y_i} \\ X & \xrightarrow{f} & Y \end{array}$$

be a Cartesian square. Then we can define the virtual pullbacks

$$f_i^! : H_*(Y_i) \to H_{*+d}(X_i)$$

for the induced maps $f_i : X_i \to Y_i$ and the induced perfect obstruction theories since they are morphisms of schemes. Then we can define the virtual pullback

$$f^! := \varprojlim f_i^! : \mathcal{H}_*(Y) \to \mathcal{H}_{*+d}(X)$$

of f by taking the inverse limit of the virtual pullbacks $f_i^!$.

Sketch of the proof of the main theorem

2) Assume that Y is an algebraic stack which has a good system of approximations $\{y_i : Y_i \to Y\}_i$. Let

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow & & \downarrow{\scriptstyle y_i} \\ X & \xrightarrow{f} & Y \end{array}$$

be a Cartesian square. Then we can define the virtual pullbacks

$$f_i^! : H_*(Y_i) \to H_{*+d}(X_i)$$

for the induced maps $f_i : X_i \to Y_i$ and the induced perfect obstruction theories since they are morphisms of schemes. Then we can define the virtual pullback

$$f^! := \varprojlim f_i^! : \mathcal{H}_*(Y) \to \mathcal{H}_{*+d}(X)$$

of f by taking the inverse limit of the virtual pullbacks $f_i^!$.

3) (In progress) This can be generalized to any DM-type morphism $f : X \to Y$ if Y has good approximations.

Sketch of the proof of the main theorem

Cosection-Localized Gysin Map

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\sigma} & \mathbb{A}^1_X \\ \downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

Cosection-Localized Gysin Map

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\sigma} & \mathbb{A}^1_X \\ \downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

In this setting, we will define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to H_{*-r}(X(\sigma)).$$

Sketch of the proof of the main theorem

Cosection-Localized Gysin Map

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\ \sigma\ } & \mathbb{A}^1_X \\ \downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

In this setting, we will define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to H_{*-r}(X(\sigma)).$$

1) Assume that $E = \mathfrak{E}$ is a vector bundle and $D = X(\sigma)$ is a divisor of X.

Sketch of the proof of the main theorem

Cosection-Localized Gysin Map

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\sigma} & \mathbb{A}^1_X \\ \downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

In this setting, we will define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to H_{*-r}(X(\sigma)).$$

1) Assume that $E = \mathfrak{E}$ is a vector bundle and $D = X(\sigma)$ is a divisor of X. Then $K = \ker(E \xrightarrow{\sigma} \mathcal{O}_X(-D))$ is a vector bundle and

$$E(\sigma) = K \cup E|_D.$$

Cosection-Localized Gysin Map

$$\begin{array}{ccc}\mathfrak{E} & \xrightarrow{\sigma} & \mathbb{A}^1_X \\ \big\downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

In this setting, we will define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to H_{*-r}(X(\sigma)).$$

1) Assume that $E = \mathfrak{E}$ is a vector bundle and $D = X(\sigma)$ is a divisor of X. Then $K = \ker(E \xrightarrow{\sigma} \mathcal{O}_X(-D))$ is a vector bundle and

$$E(\sigma) = K \cup E|_D.$$

Then $(a_*, b_*) : H_*(E|_D) \oplus H_*(K) \to H_*(E(\sigma))$ is surjective by the excision property.

Cosection-Localized Gysin Map

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\sigma} & \mathbb{A}^1_X \\ \downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

In this setting, we will define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to H_{*-r}(X(\sigma)).$$

1) Assume that $E = \mathfrak{E}$ is a vector bundle and $D = X(\sigma)$ is a divisor of X. Then $K = \ker(E \xrightarrow{\sigma} \mathcal{O}_X(-D))$ is a vector bundle and

$$E(\sigma) = K \cup E|_D.$$

Then $(a_*, b_*) : H_*(E|_D) \oplus H_*(K) \to H_*(E(\sigma))$ is surjective by the excision property. Let

$$0^!_{E,\sigma}(a_*\alpha + b_*\beta) = 0^!_{E|_D}\alpha + (-D) \cdot (0^!_K \beta).$$

Cosection-Localized Gysin Map

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\sigma} & \mathbb{A}^1_X \\ \downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

In this setting, we will define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to H_{*-r}(X(\sigma)).$$

1) Assume that $E = \mathfrak{E}$ is a vector bundle and $D = X(\sigma)$ is a divisor of X. Then $K = \ker(E \xrightarrow{\sigma} \mathcal{O}_X(-D))$ is a vector bundle and

$$E(\sigma) = K \cup E|_D.$$

Then $(a_*, b_*) : H_*(E|_D) \oplus H_*(K) \to H_*(E(\sigma))$ is surjective by the excision property.
Let

$$0^!_{E,\sigma}(a_*\alpha + b_*\beta) = 0^!_{E|_D}\alpha + (-D) \cdot (0^!_K \beta).$$

Here $(-D) \cdot \gamma \neq -(D \cdot \gamma)$. It depends on the *formal group law* of H_*.

Sketch of the proof of the main theorem

Cosection-Localized Gysin Map

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\sigma} & \mathbb{A}^1_X \\ \downarrow & & \\ X & & \end{array}$$

X : quasi-projective scheme
$\mathfrak{E}$: vector bundle stack of rank r
σ : cosection
$X(\sigma)$: zero locus of σ in X
$\mathfrak{E}(\sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X$: *kernel cone stack*

In this setting, we will define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to H_{*-r}(X(\sigma)).$$

1) Assume that $E = \mathfrak{E}$ is a vector bundle and $D = X(\sigma)$ is a divisor of X. Then $K = \ker(E \xrightarrow{\sigma} \mathcal{O}_X(-D))$ is a vector bundle and

$$E(\sigma) = K \cup E|_D.$$

Then $(a_*, b_*) : H_*(E|_D) \oplus H_*(K) \to H_*(E(\sigma))$ is surjective by the excision property. Let

$$0^!_{E,\sigma}(a_*\alpha + b_*\beta) = 0^!_{E|_D}\alpha + (-D) \cdot (0^!_K \beta).$$

Here $(-D) \cdot \gamma \neq -(D \cdot \gamma)$. It depends on the *formal group law* of H_*.

2) If $X(\sigma)$ is not a divisor, then blowup X along $X(\sigma)$. Then we can define the localized Gysin map by a similar manner.

Sketch of the proof of the main theorem

3) If $\mathfrak{E}$ is a vector bundle stack, then $\mathfrak{E} = [E_1/E_0]$ for some vector bundles E_1, E_0 and σ extends to a cosection $\tau : E_1 \to \mathbb{A}^1_X$. Also $X(\sigma) = X(\tau)$, $\mathfrak{E}(\sigma) = [E_1(\tau)/E_0]$. Then we can define the localized Gysin map by the composition

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*\mathfrak{E}(\sigma) \xleftarrow{\cong} H_{*+r_0}(E_1(\tau)) \xrightarrow{0^!_{E_1,\tau}} H_{*-r}X(\sigma).$$

3) If $\mathfrak{E}$ is a vector bundle stack, then $\mathfrak{E} = [E_1/E_0]$ for some vector bundles E_1, E_0 and σ extends to a cosection $\tau : E_1 \to \mathbb{A}^1_X$. Also $X(\sigma) = X(\tau)$, $\mathfrak{E}(\sigma) = [E_1(\tau)/E_0]$. Then we can define the localized Gysin map by the composition

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*\mathfrak{E}(\sigma) \xleftarrow{\cong} H_{*+r_0}(E_1(\tau)) \xrightarrow{0^!_{E_1,\tau}} H_{*-r}X(\sigma).$$

4) If X is an algebraic stack with a good system of approximations $\{x_i : X_i \to X\}_i$, then we can define the localized Gysin map by the inverse limit

$$0^!_{\mathfrak{E},\sigma} := \varprojlim_i 0^!_{\mathfrak{E}_i,\sigma_i} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to \mathcal{H}_{*-r}(X(\sigma))$$

where $\mathfrak{E}_i = \mathfrak{E} \times_X X_i$ and $\sigma_i = x_i^*\sigma$.

3) If $\mathfrak{E}$ is a vector bundle stack, then $\mathfrak{E} = [E_1/E_0]$ for some vector bundles E_1, E_0 and σ extends to a cosection $\tau : E_1 \to \mathbb{A}^1_X$. Also $X(\sigma) = X(\tau)$, $\mathfrak{E}(\sigma) = [E_1(\tau)/E_0]$. Then we can define the localized Gysin map by the composition

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*\mathfrak{E}(\sigma) \xleftarrow{\cong} H_{*+r_0}(E_1(\tau)) \xrightarrow{0^!_{E_1,\tau}} H_{*-r}X(\sigma).$$

4) If X is an algebraic stack with a good system of approximations $\{x_i : X_i \to X\}_i$, then we can define the localized Gysin map by the inverse limit

$$0^!_{\mathfrak{E},\sigma} := \varprojlim_i 0^!_{\mathfrak{E}_i,\sigma_i} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to \mathcal{H}_{*-r}(X(\sigma))$$

where $\mathfrak{E}_i = \mathfrak{E} \times_X X_i$ and $\sigma_i = x_i^*\sigma$.

Cosection-Localized Virtual Fundamental Class

Let X be a quasi-projective DM stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$ and a cosection $\sigma : \mathfrak{E} \to \mathbb{A}^1_X$.

3) If $\mathfrak{E}$ is a vector bundle stack, then $\mathfrak{E} = [E_1/E_0]$ for some vector bundles E_1, E_0 and σ extends to a cosection $\tau : E_1 \to \mathbb{A}^1_X$. Also $X(\sigma) = X(\tau)$, $\mathfrak{E}(\sigma) = [E_1(\tau)/E_0]$. Then we can define the localized Gysin map by the composition

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*\mathfrak{E}(\sigma) \xleftarrow{\cong} H_{*+r_0}(E_1(\tau)) \xrightarrow{0^!_{E_1,\tau}} H_{*-r}X(\sigma).$$

4) If X is an algebraic stack with a good system of approximations $\{x_i : X_i \to X\}_i$, then we can define the localized Gysin map by the inverse limit

$$0^!_{\mathfrak{E},\sigma} := \varprojlim_i 0^!_{\mathfrak{E}_i,\sigma_i} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to \mathcal{H}_{*-r}(X(\sigma))$$

where $\mathfrak{E}_i = \mathfrak{E} \times_X X_i$ and $\sigma_i = x_i^*\sigma$.

Cosection-Localized Virtual Fundamental Class

Let X be a quasi-projective DM stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$ and a cosection $\sigma : \mathfrak{E} \to \mathbb{A}^1_X$.

Cone Reduction Theorem (Kiem-Li) $(\mathfrak{C}_X)_{\mathrm{red}} \subseteq \mathfrak{E}(\sigma)$

3) If $\mathfrak{E}$ is a vector bundle stack, then $\mathfrak{E} = [E_1/E_0]$ for some vector bundles E_1, E_0 and σ extends to a cosection $\tau : E_1 \to \mathbb{A}^1_X$. Also $X(\sigma) = X(\tau)$, $\mathfrak{E}(\sigma) = [E_1(\tau)/E_0]$. Then we can define the localized Gysin map by the composition

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*\mathfrak{E}(\sigma) \xleftarrow{\cong} H_{*+r_0}(E_1(\tau)) \xrightarrow{0^!_{E_1,\tau}} H_{*-r}X(\sigma).$$

4) If X is an algebraic stack with a good system of approximations $\{x_i : X_i \to X\}_i$, then we can define the localized Gysin map by the inverse limit

$$0^!_{\mathfrak{E},\sigma} := \varprojlim_i 0^!_{\mathfrak{E}_i,\sigma_i} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to \mathcal{H}_{*-r}(X(\sigma))$$

where $\mathfrak{E}_i = \mathfrak{E} \times_X X_i$ and $\sigma_i = x_i^*\sigma$.

Cosection-Localized Virtual Fundamental Class

Let X be a quasi-projective DM stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$ and a cosection $\sigma : \mathfrak{E} \to \mathbb{A}^1_X$.

Cone Reduction Theorem (Kiem-Li) $(\mathfrak{C}_X)_{\mathrm{red}} \subseteq \mathfrak{E}(\sigma)$

Since $\mathcal{H}_*((\mathfrak{C}_X)_{\mathrm{red}}) = \mathcal{H}_*(\mathfrak{C}_X)$, we can define the *cosection-localized virtual fundamental class* by

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} := 0^!_{\mathfrak{E},\sigma}[\mathfrak{C}_X] \in \mathcal{H}_*(X(\sigma)).$$

3) If $\mathfrak{E}$ is a vector bundle stack, then $\mathfrak{E} = [E_1/E_0]$ for some vector bundles E_1, E_0 and σ extends to a cosection $\tau : E_1 \to \mathbb{A}^1_X$. Also $X(\sigma) = X(\tau)$, $\mathfrak{E}(\sigma) = [E_1(\tau)/E_0]$. Then we can define the localized Gysin map by the composition

$$0^!_{\mathfrak{E},\sigma} : \mathcal{H}_*\mathfrak{E}(\sigma) \xleftarrow{\cong} H_{*+r_0}(E_1(\tau)) \xrightarrow{0^!_{E_1,\tau}} H_{*-r}X(\sigma).$$

4) If X is an algebraic stack with a good system of approximations $\{x_i : X_i \to X\}_i$, then we can define the localized Gysin map by the inverse limit

$$0^!_{\mathfrak{E},\sigma} := \varprojlim_i 0^!_{\mathfrak{E}_i,\sigma_i} : \mathcal{H}_*(\mathfrak{E}(\sigma)) \to \mathcal{H}_{*-r}(X(\sigma))$$

where $\mathfrak{E}_i = \mathfrak{E} \times_X X_i$ and $\sigma_i = x_i^*\sigma$.

Cosection-Localized Virtual Fundamental Class

Let X be a quasi-projective DM stack equipped with a perfect obstruction theory $\imath : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$ and a cosection $\sigma : \mathfrak{E} \to \mathbb{A}^1_X$.

Cone Reduction Theorem (Kiem-Li) $(\mathfrak{C}_X)_{\mathrm{red}} \subseteq \mathfrak{E}(\sigma)$

Since $\mathcal{H}_*((\mathfrak{C}_X)_{\mathrm{red}}) = \mathcal{H}_*(\mathfrak{C}_X)$, we can define the *cosection-localized virtual fundamental class* by

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} := 0^!_{\mathfrak{E},\sigma}[\mathfrak{C}_X] \in \mathcal{H}_*(X(\sigma)).$$

Then $\jmath_*[X]^{\mathrm{vir}}_{\mathrm{loc}} = [X]^{\mathrm{vir}} \in \mathcal{H}_*(X)$ where $\jmath : X(\sigma) \hookrightarrow X$ is the inclusion.

Torus Localization Theorem

Torus Localization Theorem

Let X be a quasi-projective scheme with a linear $\mathbb{T} = \mathbb{G}_m^{\times r}$-action. Then we have an isomorphism

$$\jmath : H_*^{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{H_*^{\mathbb{T}}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}] \xrightarrow{\cong} H^{\mathbb{T}}(X) \otimes_{H_*^{T}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}]$$

where $Q \subseteq H_*^{\mathbb{T}}(\mathbf{k})$ is the multiplicative subset generated by the first Chern classes of one-dimensional $\mathbb{T}$-representations $\mathbf{k}(\lambda)$ of weight $\lambda \neq 0 \in \widehat{G}$.

Torus Localization Theorem

Let X be a quasi-projective scheme with a linear $\mathbb{T} = \mathbb{G}_m^{\times r}$-action. Then we have an isomorphism

$$\jmath : H_*^{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{H_*^{\mathbb{T}}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}] \xrightarrow{\cong} H^{\mathbb{T}}(X) \otimes_{H_*^{T}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}]$$

where $Q \subseteq H_*^{\mathbb{T}}(\mathbf{k})$ is the multiplicative subset generated by the first Chern classes of one-dimensional $\mathbb{T}$-representations $\mathbf{k}(\lambda)$ of weight $\lambda \neq 0 \in \widehat{G}$.

- For Chow, it was proved in [Edidin-Graham, Amer. J. Math. 1998].

Torus Localization Theorem

Let X be a quasi-projective scheme with a linear $\mathbb{T} = \mathbb{G}_m^{\times r}$-action. Then we have an isomorphism

$$\jmath : H_*^{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{H_*^{\mathbb{T}}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}] \xrightarrow{\cong} H^{\mathbb{T}}(X) \otimes_{H_*^{T}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}]$$

where $Q \subseteq H_*^{\mathbb{T}}(\mathbf{k})$ is the multiplicative subset generated by the first Chern classes of one-dimensional $\mathbb{T}$-representations $\mathbf{k}(\lambda)$ of weight $\lambda \neq 0 \in \widehat{G}$.

- For Chow, it was proved in [Edidin-Graham, Amer. J. Math. 1998].

Virtual Torus Localization Formula

In [Chang-Kiem-Li, Adv. Math. 2017], it was discovered that the virtual torus localization formula follows from the virtual pullback formula and the torus localization theorem (in Chow).

Torus Localization Theorem

Let X be a quasi-projective scheme with a linear $\mathbb{T} = \mathbb{G}_m^{\times r}$-action. Then we have an isomorphism

$$\jmath : H_*^{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{H_*^{\mathbb{T}}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}] \xrightarrow{\cong} H^{\mathbb{T}}(X) \otimes_{H_*^{T}(\mathbf{k})} H_*^{\mathbb{T}}(\mathbf{k})[Q^{-1}]$$

where $Q \subseteq H_*^{\mathbb{T}}(\mathbf{k})$ is the multiplicative subset generated by the first Chern classes of one-dimensional $\mathbb{T}$-representations $\mathbf{k}(\lambda)$ of weight $\lambda \neq 0 \in \widehat{G}$.

- For Chow, it was proved in [Edidin-Graham, Amer. J. Math. 1998].

Virtual Torus Localization Formula

In [Chang-Kiem-Li, Adv. Math. 2017], it was discovered that the virtual torus localization formula follows from the virtual pullback formula and the torus localization theorem (in Chow).

This also works for general intersection theories.

Definition

Let X be a projective DM stack equipped with a perfect obstruction theory. Then we have a virtual cobordism class $[X]^{\mathrm{vir}}_{\Omega} \in \Omega^{\mathrm{lim}}_*(X)$ in the limit algebraic cobordism. The *cobordism-valued virtual invariant* of X can be defined by

$$q_*[X]^{\mathrm{vir}} \in \Omega_*(\mathrm{Spec}(\mathbf{k}))_{\mathbb{Q}} = \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \cdots]$$

where $q : \mathcal{X} \to \mathrm{Spec}(\mathbf{k})$ is the structural map. Here the proper pushforward $q_* : \Omega^{\mathrm{lim}}_*(X)_{\mathbb{Q}} \to \Omega_*(\mathrm{Spec}(\mathbf{k}))_{\mathbb{Q}}$ can be defined using a finite surjective map $F \to X$ from a projective scheme F.

Definition

Let X be a projective DM stack equipped with a perfect obstruction theory. Then we have a virtual cobordism class $[X]^{\mathrm{vir}}_{\Omega} \in \Omega^{\lim}_*(X)$ in the limit algebraic cobordism. The *cobordism-valued virtual invariant* of X can be defined by

$$q_*[X]^{\mathrm{vir}} \in \Omega_*(\mathrm{Spec}(\mathbf{k}))_{\mathbb{Q}} = \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \cdots]$$

where $q : \mathcal{X} \to \mathrm{Spec}(\mathbf{k})$ is the structural map. Here the proper pushforward $q_* : \Omega^{\lim}_*(X)_{\mathbb{Q}} \to \Omega_*(\mathrm{Spec}(\mathbf{k}))_{\mathbb{Q}}$ can be defined using a finite surjective map $F \to X$ from a projective scheme F.

Example We can define cobordism-valued GW-invariants, DT-invariants, and PT-invariants.

The End

Definition (Kiem-P.)

Let H_* be an intersection theory for schemes. A *good system of approximations for an algebraic stack* $\mathcal{X}$ consists of morphisms

$$\{x_i : X_i \to \mathcal{X}\}_{i \geq 0}, \quad \{x_{i,i+1} : X_i \to X_{i+1}\}_{i \geq 0}$$

such that

1. $x_{i+1} \circ x_{i,i+1}$ and x_i are 2-isomorphic,
2. x_i is smooth morphism from a quasi-projective scheme X_i,
3. for any quasi-projective morphism $S \to \mathcal{X}$ from a quasi-projective scheme S, we have a natural isomorphism
$$H_d(S) \cong \varprojlim_i H_{d+d(x_i)}(S \times_{\mathcal{X}} X_i),$$
4. for any quasi-projective morphism $\mathcal{Y} \to \mathcal{X}$ of algebraic stacks, $H_{*+d(x_{i+1})}(\mathcal{Y} \times_{\mathcal{X}} X_{i+1}) \to H_{*+d(x_i)}(\mathcal{Y} \times_{\mathcal{X}} X_i)$ are surjective.

Definition

We say that H_* has the *excision property* if the sequence

$$H_*(\mathcal{Z}) \to H_*(\mathcal{X}) \to H_*(\mathcal{X} - \mathcal{Z}) \to 0$$

is exact for any closed immersion $\mathcal{Z} \to \mathcal{X}$.

Definition

We say that H_* has the *weak excision property* if

1. $H_*(\mathcal{X}) \to H_*(U)$ is surjective for any open immersion $U \hookrightarrow \mathcal{X}$;
2. for a regular immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ with a trivial normal bundle $N_{\mathcal{Z}/\mathcal{X}}$, there is a map
$$\lambda_{\mathcal{Z}/\mathcal{X}} : H_*(\mathcal{X} - \mathcal{Z}) \to H_{*-c}(\mathcal{Z})$$
which factors the Gysin pullback $H_*(\mathcal{X}) \to H_{*-c}(\mathcal{Z})$.

The weak excision property is enough to define the *specialization map*

$$\mathrm{sp}_{\mathcal{X}/\mathcal{Y}} : H_*(\mathcal{Y}) \to H_*(C_{\mathcal{X}/\mathcal{Y}})$$

for a closed immersion $\mathcal{X} \hookrightarrow \mathcal{Y}$. (Apply it to the deformation space $M^\circ_{\mathcal{X}/\mathcal{Y}}$.)

1. The first Chern class of a line bundle L over a scheme X is defined by $c_1(L) := 0^* \circ 0_* : H_*(X) \to H_{*-1}(X)$ where $0 : X \to L$ is the zero section.

2. For any intersection theory H_* for schemes, there is a formal group law $F_H(u, v) \in H_*(\mathrm{Spec}(\mathbf{k}))[[u, v]]$ such that

$$c_1(L \otimes N) = F_H(c_1(L), c_1(N)).$$

   Hence we have a formal inverse $u \cdot g(u) \in H_*(\mathrm{Spec}(\mathbf{k}))[[u]]$ such that $c_1(L^\vee) = c_1(L) \circ g(c_1(L))$.

3. Let D be an effective Cartier divisor of a scheme X. We define the refined intersection map by

$$-D\cdot := g(c_1(N_{D/X})) \circ \imath^* : H_*(X) \to H_{*-1}(D)$$

   where $\imath : D \hookrightarrow X$ is the inclusion. Then we have $\imath_* \circ (-D\cdot) = c_1(\mathcal{O}_X(D))$.

It seems plausible to define the higher Chow groups of an algebraic stack $\mathcal{X}$ by the inverse limit

$$CH_*(\mathcal{X}, \cdot) := \varprojlim_{t:T \to \mathcal{X}} CH_*(T, \cdot)$$

as the zeroth Chow group.

- This makes sense for smooth stacks $\mathcal{X}$.
- For singular stacks there is a problem. We need lci pullbacks to define the limit but pullbacks for higher Chow groups are only defined for smooth schemes in Bloch's original paper.

There is a natural map

$$\alpha(\mathcal{X}) : CH_*^{\text{Kresch}}(\mathcal{X}) \to CH_*^{\text{limit}}(\mathcal{X})$$

from Kresch's Chow theory to the limit Chow theory for any algebraic stack $\mathcal{X}$.

- The map $\alpha(\mathcal{X})$ is an isomorphism for global quotient stacks (because Kresch's Chow and the limit Chow both coincide with Edidin-Graham's equivariant Chow groups).

Question Is $\alpha(\mathcal{X})$ an isomorphism for stacks which is not a global quotient stack?

There are other definitions of virtual cobordism classes.

1. Recently, Levine also constructed virtual cobordism classes using motivic stable homotopy theory in [Levine, Intrinsic stable normal cone, Arxiv, 2017].
2. Lowrey and Schrug also constructed virtual cobordism classes for quasi-smooth derived schemes in [Lowrey-Schrug, Derived algebraic cobordism, J. Inst. Jussieu, 2016].
3. Khan constructed another version using the motivic stable homotopy theories of derived stacks [Khan, Virtual fundamental classes of derived stacks I, Arxiv, 2019].

Question Is the above definitions equivalent to ours in a reasonable setting?