Minimal rational curves on the moduli spaces of symplectic and orthogonal bundles over a curve

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[Joint work with Kiryong Chung and Sanghyeon Lee]

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- From the induced isomorphism $V \cong V^\vee$, we have $\deg(V) = 0$.
  (In the symplectic case, $\det(V) \cong \mathcal{O}_C$.)
- Rank 2 case:
  –Every vector bundle V with $\det(V) \cong \mathcal{O}_C$ is a symplectic bundle.
  –Every orthogonal bundle V with $\det(V) \cong \mathcal{O}_C$ is isom. to $L \oplus L^\vee$.

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- There is a (injective) forgetful morphism

$$\mathcal{M}\mathrm{S}_C(2n),\ \mathcal{M}\mathrm{O}_C(2n) \longrightarrow SU_C(2n).$$

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- (Xiaotao Sun) Hecke curves have minimal degree among the rational curves passing through a general point of $\mathcal{M}$.
- Many applications of Hecke curves to study the geometry of $\mathcal{M}$: non-abelian Torelli theorem for $\mathcal{M}$, structure of $\mathrm{Aut}(\mathcal{M})$, deformation rigidity of $\mathcal{M}$, stability of the associated bundles, ...

Hecke curves on $\mathcal{M} = SU_C(m)$ [J.-M. Hwang]

- Given a point $[V_0] \in \mathcal{M}$, choose $\mu \in \mathbb{P}(V|_x)^\vee \cong \mathbb{P}^{m-1}$ and define V^μ by

$$0 \to V^\mu \to V_0 \xrightarrow{\mu} \mathbb{C}_x \to 0$$

whose restriction to the fiber at x gives the sequence:

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- Hecke curves $\mathcal{C}^{\mu\nu}$ through $[V_0]$ are parameterized by ν lying over μ.

Overview on the results (with K. Chung and S. Lee)

- Construction of "Hecke curves" on $\mathcal{M}\mathrm{S}_C(2n)$ and $\mathcal{M}\mathrm{O}_C(2n)$.

  –The Hecke curves on $\mathcal{M}\mathrm{S}_C(2n)$ are special type of $\mathcal{C}^{\mu\nu}$'s on $\mathcal{M}$.
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- Applications:
  –non-abelian Torelli theorem for $\mathcal{M}\mathrm{S}_C(2n)$ and $\mathcal{M}\mathrm{O}_C(2n)$
  –description of $\mathrm{Aut}(\mathcal{M}\mathrm{S}_C(2n))$ and $\mathrm{Aut}(\mathcal{M}\mathrm{O}_C(2n))$.

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- Given a point $[(V_0, \omega_0)]$, choose $\Lambda \in IG(2, V_0|_x)$ and define V^Λ by

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- For each 2-dim'l subspace $H \subset V^\Lambda|_x$, we get a vector bundle V_H:
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- Lemma V_H has an orthogonal form $\Leftrightarrow H \in IG(2, \ker(\omega^\Lambda|_x))$.

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- Orthogonal Hecke curves $\mathcal{C}^\Lambda$ through $[(V_0, \omega_0)] \in \mathcal{MO}_C(2n)$ are parameterized by $\Lambda \in IG(2, V_0|_x) \cong IG(2, 2n)$.

Degree of Hecke curves

For the computation of the degree of $\mathbb{P}^1 \subset \mathcal{M}\mathrm{S}_C(2n)$ and $\mathcal{M}\mathrm{O}_C(2n)$, we basically follow X. Sun's approach for $\mathbb{P}^1 \subset \mathcal{M}$.

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A rational curve $\mathbb{P}^1 \subset \mathcal{M}$ gives a vector bundle $\mathcal{V} \to \mathbb{P}^1 \times C$.
Its degree is computed by using the relative Harder–Narasimhan filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{V}$$

which restricts to the H–N filtration of $\mathcal{V}|_{\mathbb{P}^1 \times \{x\}}$ for a general $x \in C$.

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Lemma The above filtration for $\mathbb{P}^1 \subset \mathcal{M}\mathrm{S}_C(2n)$ and $\mathcal{M}\mathrm{O}_C(2n)$ is symmetric about the middle:

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell \subset \mathcal{E}_\ell^\perp \subset \mathcal{E}_{\ell-1}^\perp \subset \cdots \mathcal{E}_1^\perp \subset \mathcal{E}_0^\perp = \mathcal{V},$$

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where $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_\ell$ are isotropic.

Therefore, we may compute the degree of $\mathbb{P}^1$ by the degrees of isotropic subbundles, which are "controllable" in the symplectic/orthogonal setting.

Non-abelian Torelli theorem

For two applications below, we need further to show:

Lemma The symplectic / orthogonal Hecke curves passing through a general point $[V]$ of $\mathcal{M}\mathrm{S}_C(2n)$ / $\mathcal{M}\mathrm{O}_C(2n)$ are effectively parameterized by $\mathbb{P}(V)$ / $IG(2, V)$ (under a suitable assumption on $g(C)$).

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Theorem Under the above assumption on $g(C)$,

$$\mathcal{M}\mathrm{O}_C(2n) \cong \mathcal{M}\mathrm{O}_{C'}(2n) \implies C \cong C'.$$

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Proof An isom. ϕ of moduli spaces induces an isom. of parameter spaces of minimal rational curves: $\mathbb{P}(V) \cong \mathbb{P}(\phi(V))$ or $IG(2, V) \cong IG(2, \phi(V))$. Since these are rational fibrations over C and C' respectively, $C \cong C'$.

Automorphisms of the moduli spaces

There are two sources of the automorphisms of $\mathcal{M}\mathrm{S}_C(2n)$ and $\mathcal{M}\mathrm{O}_C(2n)$:

- automorphism $\sigma \in \mathrm{Aut}(C)$
- 2-torsion point $L \in Pic^0(C)$: if $V \in \mathcal{M}\mathrm{S}_C(2n)$, then so is $V \otimes L$, since

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Theorem Under the above assumption on $g(C)$, the automorphism groups of $\mathcal{M}\mathrm{S}_C(2n)$ and $\mathcal{M}\mathrm{O}_C(2n)$ are generated by these two sources.

Proof An automorphism $\tau \in \mathrm{Aut}(\mathcal{M}\mathrm{S}_C(2n))$ induces an isomorphism $\widetilde{\tau} : \mathbb{P}(V) \cong \mathbb{P}(\tau(V))$. By composing with an $\sigma \in \mathrm{Aut}(C)$, we may assume that $\widetilde{\tau}$ preserves the fibers. Then $\tau(V) \cong V \otimes L$ for some $L \in Pic^0(C)$. Specializing V to the trivial symplectic bundle $\mathcal{O}_C^{\oplus 2n}$, we get $L^2 \cong \mathcal{O}_C$.

More general context

- For a line bundle L over C, an L-valued symplectic/orthogonal bundle is a vector bundle V of rank $2n$ equipped with a symplectic/orthogonal form $\omega : V \otimes V \to L$.
  From $V \cong V^\vee \otimes L$, we get $\det(V) = n \deg(L)$.
- An L-valued orthogonal bundle V satisfies $(\det V)^2 \cong L^{2n}$, but $\det V \ncong L^n$ in general. Together with the 2nd Stiefel–Whitney class $w_2(V)$, the class $c_1(V)$ produces several components of $\mathcal{M}\mathrm{O}_C(2n, L)$.
  (Ex: $\mathcal{M}\mathrm{O}_C(2n)$ is the component of $\mathcal{M}\mathrm{O}_C(2n, \mathcal{O}_C)$ with $w_2 = 0$.)
- We may also consider the orthogonal bundles of odd rank.

The discussions for minimal rational curves on $\mathcal{M}\mathrm{S}_C(2n)$ and $\mathcal{M}\mathrm{O}_C(2n)$ also works for $\mathcal{M}\mathrm{S}_C(2n, L)$ and $\mathcal{M}\mathrm{O}_C(n, L)$ in general.