Cayley octads plane quartic curves Del Pezzo surfaces of degree 2 and double Veronese cones

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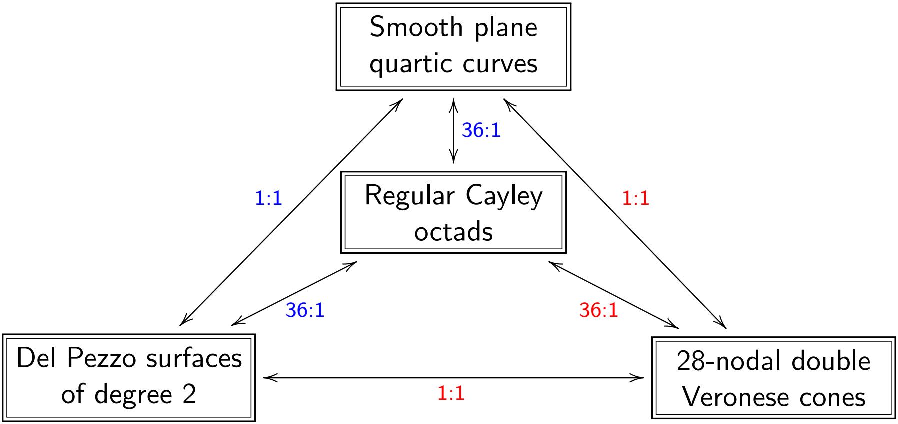
Kye Words

- 0-dim — Regular Cayley octads
- 1-dim — Smooth plane quartic curves
- 2-dim — Smooth Del Pezzo surfaces of degree 2
- 3-dim — 28-nodal double Veronese cones

Their Relations



Regular Cayley Octads

Definition

A Cayley octad is a unordered set of eight points in $\mathbb{P}^3$ that consist of an intersection of three quadrics $\mathbb{P}^3$.

For a given Cayley octad Φ, let $\mathcal{L}(\Phi)$ be the linear system of quadrics passing through all the eight points of Φ. This linear system is known to be a net, i.e., 2-dimensional.

Regular Cayley Octads

Therefore, there are three linearly independent quadric homogenous polynomials F_0, F_1, F_2 over $\mathbb{P}^3$ that generate the linear system $\mathcal{L}(\Phi)$, i.e., an element in $\mathcal{L}(\Phi)$ is defined by the quadric homogenous equation

$$xF_0 + yF_1 + zF_2 = 0 \tag{1}$$

for some $[x : y : z] \in \mathbb{P}^2$. We can express (1) as a 4×4 symmetric matrix $M(\Phi)$ with entries of linear forms in x, y, z. Then $\det(M(\Phi)) = 0$ defines a plane quartic curve $H(\Phi)$ in $\mathbb{P}^2$. This plane quartic curve is called the *Hessian quartic* of the net $\mathcal{L}(\Phi)$. It parametrizes the singular members in $\mathcal{L}(\Phi)$.

Definition

A Cayley octad is called regular if its Hessian quartic is smooth.

Smooth Plane Quartic Curves

Definition

A smooth plane quartic curve C is a smooth curve in the projective plane $\mathbb{P}^2$ defined by a homogenous polynomial $F(x, y, z)$ of degree 4.

Smooth Plane Quartic Curves

- There are 28 bitangent lines to a smooth plane quartic curves.

$$t_o + t_h = 28$$

- There are two kinds of inflection points

$$i_o + 2i_h = 24$$

  where $i_h = t_h$.
- Its dual curve is a plane curve of degree 12 with t_o nodes, i_o simple cusps, t_h singular points of type E_6.
- There are 36 inequivalent symmetric linear determinantal expressions.

Smooth Plane Quartic Curves

Theorem (Hesse (1855), Dixon (1902))

There are 36 inequivalent symmetric linear determinantal expressions.

For a given smooth plane quartic curve C, we can obtain 36 distinct nets of quadrics in $\mathbb{P}^3$ up to projective transformations.

$$C : \det\left(a_{ij}x + b_{ij}y + c_{ij}z\right) = 0 \text{ in } \mathbb{P}^2$$

$$\left(a_{ij}x + b_{ij}y + c_{ij}z\right) = x\left(a_{ij}\right) + y\left(b_{ij}\right) + z\left(c_{ij}\right) \text{ net of quadrics in } \mathbb{P}^3$$

Definition

A theta characteristic on a smooth curve C is a divisor class θ such that

$$2\theta = K_C.$$

A theta characteristic θ is said to be even (resp. odd) if $\mathrm{h}^0(C, \mathcal{O}_C(\theta))$ is even (resp. odd).

- The number of theta characteristics of a smooth plane quartic is 64.
- The number of odd theta characteristics of a smooth plane quartic is 28.
- The number of even theta characteristics of a smooth plane quartic is 36.
- An even theta characteristic θ of a smooth plane quartic C defines an embedding of C into $\mathbb{P}^3$ via the linear system $|K_C + \theta|$:

  $$\varphi_{|K_C+\theta|} : C \to S \subset \mathbb{P}^3.$$

  The space curve S has degree 6.

Regular Cayley Octads vs Smooth Plane Quartic Curves

A given regular Cayley octad Φ defines a net $\mathcal{L}(\Phi)$ of quadrics in $\mathbb{P}^3$. The net $\mathcal{L}$ yields its Hessian quartic curve $H(\Phi)$ in $\mathbb{P}^2$, which is smooth. Meanwhile, the singular points of quadrics in the net $\mathcal{L}(\Phi)$ sweep out a smooth curve of degree 6 in $\mathbb{P}^3$, which is called the Steinerian curve of the net. There is an even theta characteristic $\theta(\Phi)$ such that the linear system $|K_{H(\Phi)} + \theta(\Phi)|$ defines an isomorphism of $H(\Phi)$ with $S(\Phi)$.

Regular Cayley Octads vs Smooth Plane Quartic Curves

Let $\mathcal{N}$ be the set that consists of isomorphism classes of regular Cayley octads modulo projective transformations. Let $\mathcal{T}$ be the set that consists of the pairs (C, θ), where C is a smooth plane quartic considered up to isomorphism, and θ is an even theta characteristic on C.
Define the map

$$\Theta\colon \mathcal{N} \to \mathcal{T}$$

by assigning $\Theta(\mathcal{L}) = (H(\mathcal{L}), \theta(\mathcal{L}))$.

Theorem (Beauville (1977))

The map Θ is bijective.

Smooth del Pezzo surfaces of degree 2

Definition

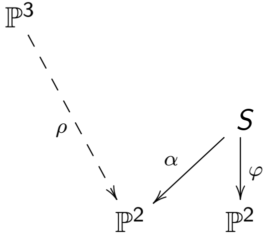
A smooth del Pezzo surface of degree 2 is a smooth surface whose anticanonical divisor

- *is ample;*
- *has anticanonical degree 2.*

Smooth del Pezzo surfaces of degree 2

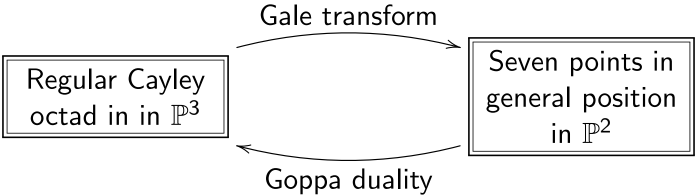
- Every smooth del Pezzo surface of degree 2 can be obtained by blowing up $\mathbb{P}^2$ at 7 points in general position, i.e., no three of them lie on a single line, no six of them lie on a single conic. The anticanonical linear system defines a double covering of $\mathbb{P}^2$ branched along a smooth quartic curve.
- Every smooth del Pezzo surface of degree 2 can be obtained by taking the double cover of $\mathbb{P}^2$ branched along a smooth quartic curve. The pull-backs of 28 bitangent lines to the branch quartic curve defines 56 (-1)-curves on the double cover. We may choose 7 disjoint (-1)-curves out of the 56 (-1)-curves.

Regular Cayley Octads vs Smooth del Pezzo surfaces of degree 2



For a given regular Cayley octad Φ, choose point P in Φ. Project the other 7 points to $\mathbb{P}^2$ from P. These seven points are in gernal position. Blow up $\mathbb{P}^2$ along these 7 points to obtain a smooth del Pezzo surface of degree 2. It is a double cover of $\mathbb{P}^2$ branched along the Hessian quartic curve of the net $\mathcal{L}(\Phi)$.

Regular Cayley Octads vs Smooth del Pezzo surfaces of degree 2



28-nodal Double Veronese Cones

Definition

A double Veronese cone is a 3-fold with Gorenstein terminal singularities whose anticanonical divisor

- *is ample;*
- *is divisible by 2 in the Picard group;*
- *has anticanonical degree 8.*

28-nodal Double Veronese Cones from regular Cayley octads

Lemma

Seven distinct points $P_1, \ldots, P_7$ in $\mathbb{P}^3$ are from a regular Cayley octad if and only if the two conditions

(A) *every element of $\mathcal{L}(P_1, \ldots, P_7)$ is irreducible;*

(B) *the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ consists of eight distinct points,*

are satisfied.

28-nodal Double Veronese Cones from regular Cayley octads

Lemma

For seven distinct points $P_1, \dots, P_7$ in $\mathbb{P}^3$, the two conditions

(A) *every element of $\mathcal{L}(P_1, \dots, P_7)$ is irreducible;*

(B) *the base locus of $\mathcal{L}(P_1, \dots, P_7)$ consists of eight distinct points,*

are satisfied if and only if the following three conditions hold:

(A′) *no four points of $P_1, \dots, P_7$ are coplanar (and in particular no three are collinear);*

(B′) *all points $P_1, \dots, P_7$ are not contained in a single twisted cubic;*

(C′) *for each i, the twisted cubic passing through the points of $\{P_1, \dots, P_7\} \setminus \{P_i\}$ and the line passing through the point P_i and one point in $\{P_1, \dots, P_7\} \setminus \{P_i\}$ meet neither twice nor tangentially.*

28-nodal Double Veronese Cones from regular Cayley octads

- Let Φ be a regular Cayley octad. Choose one point P from Φ. Denote by $P_1, \ldots, P_7$ the remaining 7 points.
- For $i < j$, let L_{ij} be the line in $\mathbb{P}^3$ determined by P_i and P_j. (21 such lines).
- For i, let C_i be the twisted cubic determined by $\Phi \setminus \{P, P_i\}$. (7 such twisted cubics).

28-nodal Double Veronese Cones from regular Cayley octads

- Let $\pi\colon \widehat{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ at the points $P_1, \ldots, P_7$.
- $-K_{\widehat{\mathbb{P}}^3}$ is nef and big with $(-K_{\widehat{\mathbb{P}}^3})^3 = 8$.
- Let $\widetilde{L}_{ij}$ be the proper transform of L_{ij}.
- Let $\widetilde{C}_i$ be the proper transform of C_i.
- We have $\widetilde{L}_{ij} \cdot (-K_{\widehat{\mathbb{P}}^3}) = \widetilde{C}_i \cdot (-K_{\widehat{\mathbb{P}}^3}) = 0$ and

$$\mathcal{N} = \mathcal{O}(-1) \bigoplus \mathcal{O}(-1).$$

28-nodal Double Veronese Cones from regular Cayley octads

Proposition (ACPS, Prokhorov)

Let $\pi\colon \widehat{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ at the points $P_1, \ldots, P_7$ and let $\phi\colon \widehat{\mathbb{P}}^3 \dashrightarrow V$ be the map given by the linear system $|-2K_{\widehat{\mathbb{P}}^3}|$. Then

- *the map ϕ is a birational morphism;*
- *the exceptional locus of ϕ is a disjoint union of the proper transforms of the lines passing through pairs of the points P_i and the twisted cubics passing through six-tuples of the points P_i;*
- *the variety V is a 28-nodal double Veronese cone.*

Theorem (Prokhorov)

Every double Veronese cone with 28 singular points can be obtained in this way.

28-nodal Double Veronese Cones from smooth plane quartic curves

We start with a smooth quartic curve C in the projective plane $\mathbb{P}^2$ given by an equation

$$H(x, y, z) = \sum_{i+j+k=4} a_{ijk} x^i y^j z^k = 0. \qquad (2)$$

We regard $[s : t : u]$ as a general point in the dual projective plane $\check{\mathbb{P}}^2$. Then the corresponding line on $\mathbb{P}^2$ is a general line $L_{s,t,u}$ given by

$$sx + ty + uz = 0.$$

The line $L_{s,t,u}$ hits the quartic C at four distinct points x_1, x_2, x_3, x_4 lying on $L_{s,t,u} \setminus \{z = 0\}$. We may regard these four points as points on the affine line, so that we could define their cross-ratio as follows:

$$\lambda(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_3)(x_4 - x_2)}{(x_1 - x_2)(x_4 - x_3)}.$$

This has six different values according to the order of the four points.

28-nodal Double Veronese Cones from smooth plane quartic curves

However, the following j-function is invariant with respect to the reordering x_1, x_2, x_3, x_4.

$$
\begin{aligned}
j(x_1,x_2,x_3,x_4) &= 256\frac{\left(1-\lambda(x_1,x_2,x_3,x_4)\left(1-\lambda(x_1,x_2,x_3,x_4)\right)\right)^3}{\lambda(x_1,x_2,x_3,x_4)^2\left(1-\lambda(x_1,x_2,x_3,x_4)\right)^2} \\
&= 2^8\frac{\left((x_1-x_2)^2(x_4-x_3)^2-(x_1-x_3)(x_4-x_2)(x_4-x_1)(x_3-x_2)\right)^3}{(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2}.
\end{aligned}
\tag{3}
$$

28-nodal Double Veronese Cones from smooth plane quartic curves

By plugging $z = -\frac{sx+ty}{u}$ into (2), we obtain

$$u^4 H\left(x, y, -\frac{sx+ty}{u}\right) = \sum_{i+j+k=4} a_{ijk}(-u)^{4-k}\left(sx+ty\right)^k x^i y^j$$
$$= \sum_{r=0}^{4} b_{4-r} x^r y^{4-r},$$

where

$$b_r = \sum_{j=0}^{r} \sum_{i+k=4-j} (-1)^k a_{ijk} \binom{k}{k+j-r} s^{k+j-r} t^{r-j} u^{4-k}.$$

28-nodal Double Veronese Cones from smooth plane quartic curves

Then we have the following identities for elementary symmetric functions of x_1, x_2, x_3, x_4:

$$x_1 + x_2 + x_3 + x_4 = -\frac{b_1}{b_0};$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{b_2}{b_0};$$

$$x_2x_3x_4 + x_1x_2x_4 + x_1x_2x_4 + x_1x_2x_3 = -\frac{b_3}{b_0};$$

$$x_1x_2x_3x_4 = \frac{b_4}{b_0}.$$

Since the denominator and the numerator of the j-function in (3) are symmetric polynomials in x_1, x_2, x_3, x_4, the j-function in (3) may be regarded as a rational function in b_0, b_1, b_2, b_3, b_4.

28-nodal Double Veronese Cones from smooth plane quartic curves

Indeed, one has

$$j(b_0, b_1, b_2, b_3, b_4) = 1728\frac{4h_2(b_0, b_1, b_2, b_3, b_4)^3}{4h_2(b_0, b_1, b_2, b_3, b_4)^3 - 27h_3(b_0, b_1, b_2, b_3, b_4)^2},$$

where

$$h_2(b_0, b_1, b_2, b_3, b_4) = \frac{1}{3}\Big(-3b_1b_3 + 12b_0b_4 + b_2^2\Big);$$

$$h_3(b_0, b_1, b_2, b_3, b_4) = \frac{1}{27}\Big(72b_0b_2b_4 - 27b_0b_3^2 - 27b_1^2b_4 + 9b_1b_2b_3 - 2b_2^3\Big).$$

28-nodal Double Veronese Cones from smooth plane quartic curves

Regarding $h_2(b_0, b_1, b_2, b_3, b_4)$ and $h_3(b_0, b_1, b_2, b_3, b_4)$ as polynomials in s, t, u, one can see that

$$h_2(b_0, b_1, b_2, b_3, b_4) = u^4 g_4(s, t, u),$$

$$h_3(b_0, b_1, b_2, b_3, b_4) = u^6 g_6(s, t, u),$$

where $g_4(s, t, u)$ and $g_6(s, t, u)$ are homogenous polynomials of degrees 4 and 6, respectively, in s, t, u. Consequently, the rational function $j(b_0, b_1, b_2, b_3, b_4)$ may be regarded as a rational function j_C in s, t, u, so that it is a rational function on $\check{\mathbb{P}}^2$. More precisely, one has

$$j_C(s, t, u) = 1728 \frac{4g_4(s, t, u)^3}{4g_4(s, t, u)^3 - 27g_6(s, t, u)^2}.$$

28-nodal Double Veronese Cones from smooth plane quartic curves

$$j_C(s,t,u) = 1728\frac{4g_4(s,t,u)^3}{4g_4(s,t,u)^3 - 27g_6(s,t,u)^2}.$$

- The equation $g_4(s,t,u) = 0$ of degree 4 describes the points of $\check{\mathbb{P}}^2$ corresponding to lines that intersect C by equianharmonic quadruples of points. In other words, these lines with the quadruples of points define elliptic curves of j-invariant 0. Elliptic curves of j-invariant 0 are isomorphic to the Fermat plane cubic curve.
- The equation $g_6(s,t,u) = 0$ of degree 6 describes the points of $\check{\mathbb{P}}^2$ corresponding to lines that intersect C by harmonic quadruples of points. In this case, lines with the quadruples of points define elliptic curves of j-invariant 1728.

28-nodal Double Veronese Cones from smooth plane quartic curves

Eventually, with $g_4(s,t,u)$ and $g_6(s,t,u)$, we obtain a hypersurface V of degree 6 in $\mathbb{P}(1,1,1,2,3)$ given by an equation

$$w^2 = v^3 - g_4(s,t,u)v + g_6(s,t,u),$$

where $\mathrm{wt}(w) = 3, \mathrm{wt}(v) = 2$.

This is a double Veronese Cone.

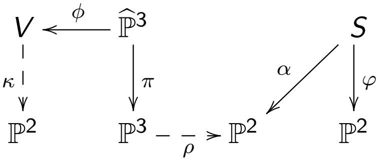
28-nodal Double Veronese Cones from smooth plane quartic curves

$$w^2 = v^3 - g_4(s,t,u)v + g_6(s,t,u).$$

- It has 28 nodes.
- These points come from 28 bitangents of the given smooth plane quartic curve C.
- V has a rational elliptic fibration structure over $\check{\mathbb{P}}^2$.
- These points lie over the ordinary double points and E_6-type singular points of the dual curve of C.

28-nodal Double Veronese Cones from smooth plane quartic curves

Smooth plane quartic curves $\overset{1:1}{\longleftrightarrow}$ 28-nodal double Veronese cones

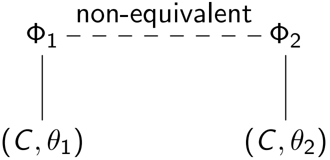


Here ϕ is a small resolution of all singular points of the 3-fold V, the morphism π is the blow up of $\mathbb{P}^3$ at the seven distinct points $P_1, \ldots, P_7$, and the rational map κ is given by the half-anticanonical linear system of V.

28-nodal Double Veronese Cones

$$
\begin{array}{ccc}
\widehat{\mathbb{P}^3} & \xrightarrow{\ \phi\ } & V \\
{\scriptstyle \pi}\downarrow & & \vdots\ {\scriptstyle \kappa} \\
\mathbb{P}^3 & \dashrightarrow & \mathbb{P}^2
\end{array}
\tag{4}
$$

Where are the theta characteristics?



- Φ_1 can be transformed onto Φ_2 by a Cremona transform.
- $\widehat{\mathbb{P}}^3_1$ is transformed onto $\widehat{\mathbb{P}}^3_2$ by flops.

Where are the theta characteristics?

- A (unordered) set of seven distinct odd theta characteristics $\theta_1, \ldots, \theta_7$ on a smooth plane quartic curve C is called an *Aronhold system* if they satisfy the condition that $\theta_i + \theta_j + \theta_k - K_C$ is an even theta characteristic for each choice of three distinct indices i, j, k.

Where are the theta characteristics?

- A choice of a point in a given Cayley octad Φ is a choice of an Aronhold system.
- The diagram (4) is not unique.
- The diagram (4) is given by a choice of an Aronhold system up to automorphisms preserving the sums of the odd theta characteristics in Aronhold systems.
- In general, there are $288 = 8 \times 36$ diagrams.