# Recent progress on regularity problem due to Castelnuovo-Mumford-Eisenbud-Goto

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# Outline

- Introduction
- Regularity conjecture and known results
- $\mathcal{O}_X$-regularity conjecture: smooth cases and singular cases
  - Double point divisors for smooth cases
  - Counterexamples due to J. McCullough-I. Peeva (2018)
- Boundary cases of $\mathcal{O}_X$-regularity for smooth varieties

# Introduction

- $X$ : a projective (not necessary smooth) variety defined over an algebraically closed field $k$ with $\mathrm{char}(k) = 0$.
- $\mathcal{L}$ : a very ample line bundle on $X$.
- For a polarized pair $(X, \mathcal{L})$, Serre vanishing theorem implies that

$$H^i(X, \mathcal{L}^{\otimes m}) = 0, \forall i \geq 1, m >> 0.$$

**Question:** What is the effective lower bound $m_0(X, \mathcal{L})$ such that $H^i(X, \mathcal{L}^{\otimes m}) = 0, \forall i \geq 1, m \geq m_0(X, \mathcal{L})$ ?

- **(Forklore conjecture)** $m_0(X, \mathcal{L})$ is (the delta genus of $\mathcal{L}$) $+ 1$, i.e.

$$m_0(X, \mathcal{L}) = \triangle(X, \mathcal{L}) + 1 := \mathcal{L}^{\dim(X)} + \dim(X) - h^0(X, \mathcal{L}) + 1.$$

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- Let $\mathcal{L}$ be an ample and globally generated line bundle on $X$. A coherent sheaf $\mathcal{F}$ on $X$ is $m$-regular with respect to $\mathcal{L}$ if $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(m-i)}) = 0$ for $i \geq 1$.
- $\mathrm{reg}_{\mathcal{L}}(\mathcal{F})$ is the minimum of $m$ such that $\mathcal{F}$ is $m$-regular with respect to $\mathcal{L}$. For example, $\mathrm{reg}(\mathcal{O}_X)$ is the minimum $m$ such that

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The $m$-regularity of $\mathcal{F}$ with respect to $\mathcal{L}$ has nice properties as follows:

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# General setting

$X^n \subset \mathbb{P}^{n+e}$: a non-degenerate projective variety of dim $n$, codim $e$, and degree $d$ defined over $k = \overline{k}$ with $\mathrm{char}(k) = 0$.

**Definition**

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  1. (Castelnuovo normality) $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m-1)) \twoheadrightarrow H^0(\mathcal{O}_X(m-1))$ is surjective, i.e. $X$ is $(m-1)$-normal;
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Give a bound for $m_0$ in terms of $\deg(X), \operatorname{codim}(X)$ such that for all $m \geq m_0$, $H^1(\mathbb{P}^{n+e}, \mathcal{I}_{X|\mathbb{P}^{n+e}}(m)) = 0$, i.e.

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$X^n \subset \mathbb{P}^{n+e}$ : irreducible and reduced of codim $e$ and degree $d$.

**Regularity Conjecture(1984)**

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### Theorem (Castelnuovo 1893)

*Let $C \subset \mathbb{P}^3$ be a non-degenerate smooth projective curve of degree $d$. Then $\mathrm{reg}(C) \leq d - 1$.*

### Theorem (Gruson-Lazarsfeld-Peskine 1983)

*Let $C \subset \mathbb{P}^r$ be a projective curve (not necessarily smooth) of degree $d$ and codimension $e$.*

- $\mathrm{reg}(C) \leq d - e + 1$.
- *the equality holds $\Leftrightarrow$ $C \subset \mathbb{P}^r$ is a plane curve, an elliptic normal curve, a rational normal curve, a rational curve with $d = e + 2$, or a smooth rational curve having a $(d - e + 1)$-secant line.*

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In higher dimensional cases, we only have partial results. Assume that $X$ is smooth.

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3. (Mumford, Bertram-Ein-Lazarsfeld) In general, we only have $\mathrm{reg}(X) \leq \min\{e, n+1\}(d-1) - n + 1$.

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# Remark

**Lemma**

*Let $X^n \subset \mathbb{P}^{n+e}$ be a projective variety of dimension $n \geq 2$, and let $Y \subseteq \mathbb{P}^{n+e-1}$ be a general hyperplane section.*

- *If $Y \subseteq \mathbb{P}^{n+e-1}$ is $k$-normal for $k \geq k_0$, then $H^1(X, \mathcal{O}_X(k)) = 0$ for $k \geq k_0 - 1$;*
- *For $i \geq 2$, $H^{i-1}(Y, \mathcal{O}_Y(k)) = 0$ for $k \geq k_0$, then $H^i(X, \mathcal{O}_X(k)) = 0$ for $k \geq k_0 - 1$.*
- *In particular, $\mathrm{reg}(Y) \leq k_0$ implies $\mathrm{reg}(\mathcal{O}_X) \leq k_0 - 1$.*

- Therefore, for a singular surface $X$, $\mathrm{reg}(\mathcal{O}_X) \leq d - e$.
- For any threefold $X$ with at worst finite singular points, $\mathrm{reg}(\mathcal{O}_X) \leq d - e$.

Mysterious dichotomy between smooth varieties and singular varieties.
Positive results for smooth cases

- variants of Kodaira vanishing theorem.
- projection methods with the locus of multisecant lines.
- The fact that the base locus of the double point divisor is empty or at worst finite plays a crucial role to guarantee the semi-ampleness of the double point divisors (Zariski-Fujita theorem).

Negative results for singular cases

- McCullough-Peeva constructed counterexamples to regularity conjecture. Starting from a projective subscheme with bad regularity, they could construct the prime ideal(via step-by step homogenization process with Rees-like algebra) whose regularity is almost same.
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# Threefolds in $\mathbb{P}^5$

This is the first nontrivial case on regularity for smooth threefolds and also the nontrivial case for $\mathcal{O}_X$-regularity for singular threefolds.

- (K-, 1998) Let $X$ be a smooth threefold in $\mathbb{P}^5$. Then
  - $X$ is $m$-normal for all $m \geq d - 4$;
  - $\mathrm{reg}(X) \leq d - 1$ because of Lazarsfeld method with the following facts: Zak's linearly normality theorem, $h^1(\mathcal{O}_X) = 0$ (Barth Theorem) and the locus of 5-secant lines is 4-dimensional due to Z. Ran's (dimension +2)-secant lemma.
- (MP, 2018) constructed a singular threefold $X \subset \mathbb{P}^5$ with $\dim Sing(X) = 1$ such that $I_X = (f_1, f_2, \ldots, f_{19})$, $7 \leq \deg(f_i) \leq 105$, $\deg(X) = 94 < \mathrm{reg}(X) = 105$, $\mathrm{reg}(\mathcal{O}_X) = 39$. More precisely, $h^1(\mathcal{I}_X(104)) = 0$ but, $h^1(\mathcal{I}_X(103)) \neq 0$. Note that $X$ is a linear section of $Y^6 \subset \mathbb{P}^8$ whose depth is 4 and so $\mathrm{reg}(X) = \mathrm{reg}(Y)$.

# Threefolds in $\mathbb{P}^5$

This is the first nontrivial case on regularity for smooth threefolds and also the nontrivial case for $\mathcal{O}_X$-regularity for singular threefolds.

- (K-, 1998) Let $X$ be a smooth threefold in $\mathbb{P}^5$. Then
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# Positive results

**Proposition** (Birational double point formula)
Let $\varphi\colon V^n \to M^{n+1}$ be a morphism of smooth projective varieties such that $\varphi\colon V \twoheadrightarrow W := \varphi(V) \subset M$ is birational.
Then, $\varphi^*(K_M + W) - K_V \sim D - E$ where $D$ and $E$ are effective divisors on $V$ such that $E$ is $\varphi$-exceptional. Moreover, if $\varphi$ is isomorphic at $x \in V$, then $x \notin \operatorname{Supp}(D - E)$.

*Proof.* see Lemma 10.2.8(Positivity in Algebraic Geometry II).

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## Double point divisors from inner projections

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at $\Lambda$ and the blow-up $\widetilde{X}$ of $X$ at $x_1, \ldots, x_{e-1}$ with the following diagram:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\sigma} & X & \hookrightarrow & \mathbb{P}^{n+e} \\ & \searrow^{\widetilde{\pi}} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_\Lambda} \\ & & \overline{X}_\Lambda & \hookrightarrow & \mathbb{P}^{n+1} \end{array}$$

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From the morphism $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda \subset \mathbb{P}^{n+1}$ and $\deg(\overline{X}_\Lambda) = d - (e-1)$, the birational double point formula implies that

$$\widetilde{\pi}^*(K_{\mathbb{P}^{n+1}} + \overline{X}_\Lambda) - K_{\widetilde{X}} = (d - n - e - 1)\widetilde{H} - K_{\widetilde{X}} \sim D(\widetilde{\pi}) - \widetilde{E}.$$

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If we assume $\widetilde{E} = \emptyset$, then, the non-isomorphic double point locus $D(\widetilde{\pi})$ of $\widetilde{\pi}$ is equivalent to $(d-n-e-1)\widetilde{H} - K_{\widetilde{X}}$. Define $D(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X} \setminus E_1 \cup \cdots \cup E_{e-1}})}$ which is called the double point divisor from inner projection $\pi_\Lambda$ and linearly equivalent to $B_{inn} := (d-n-e-1)H - K_X$.

**Proposition (Noma)**

- *Suppose that $X$ is not a scroll over a smooth projective curve, the Veronese surface in $\mathbb{P}^5$, or a Roth variety. Then, $B_{inn}$ is semiample.*
- $\mathrm{reg}_H(\mathcal{O}_X) \leq d-e$ *unless $X$ is a scroll over a curve.*

Remark that b.p.f. implies "semiample" which also implies nefness. The base locus of $B_{inn}$ is contained in the non-birational locus $\mathcal{C}(X) := \{x \in X \mid \pi_x : X \to \mathbb{P}^{n+e-1} \text{ is non birational}\}$ which is finite. So, Fujita-Zariski Theorem guarantee the semiampleness.

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So, the double point dvisor $B_{inn} = (d - n - e - 1)H - K_X$ is nef.
On the other hand,

$$(d - e - i)H = K_X + (n + 1 - i)H + B_{inn}.$$

Thus, Kodaira vanishing give a proof of $\operatorname{reg}(\mathcal{O}_X) \leq d - e$.
We have the following (jointly with J. Park, to appear):

## Proposition

*Let $X \subseteq \mathbb{P}^r$ be a non-degenerate scroll of degree $d$ and codimension $e$ over a smooth projective curve of genus $g$. Suppose that $n = \dim(X) \geq 2$. Then we have the following:*

1. *If $g = 0$, then $\operatorname{reg}(\mathcal{O}_X) = 1$.*
2. *If $g = 1$, then $\operatorname{reg}(\mathcal{O}_X) = 2$.*
3. *If $g \geq 2$, then $\operatorname{reg}(\mathcal{O}_X) \leq d - e - 2$.*

## Theorem

*Let $X \subseteq \mathbb{P}^r$ be a non-degenerate smooth projective variety of degree $d$ and codimension $e$. Then we have the upper bound and classification of boundary cases(jointly with J. Park, to appear):*

1. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$.
2. $\mathrm{reg}(\mathcal{O}_X) = d - e$ *if and only if* $X \subseteq \mathbb{P}^r$ *is a hypersurface or a linearly normal variety with* $d = e + 1$ *or* $e + 2$.
3. $\mathrm{reg}(\mathcal{O}_X) = d - e - 1$ *if and only if* $X \subseteq \mathbb{P}^r$ *is an isomorphic projection of a projective variety in* $(a)$ *at one point, a linearly normal variety with* $d = e + 3$ *and* $e \geq 2$*, or a complete intersection of type* $(2, 3)$.

- Thank you very much for your concern!