Ulrich bundles on intersection of two 4-dimensional quadrics

Cho, Yonghwa

KIAS

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Introduction

(joint work w/ Yeongrak Kim & Kyoung-Seog Lee)

$S = k[x_0, \ldots, x_{n+1}] \ni F$ homogeneous of degree d

$X = (F = 0) \subset \mathbb{P}^{n+1}$ smooth hypersurface

Question. $\exists\, M \in \mathrm{Mat}_{d\times d}(S_1)$ such that $F = \det M$?

Theorem(Segre 1951). *$X_3 \subset \mathbb{P}^3_k$ smooth cubic defined by $(F = 0)$. TFAE*

(1) $F = \det M$ *for some* $M \in \mathrm{Mat}_{3\times 3}(S_1)$

(2) X_3 *contains a twisted cubic curve*

(3) $X_3 \sim_{bir} \mathbb{P}^2_k$

(4) X_3 *contains a k-rational point and 6 disjoint lines*

Introduction

We work over $k = \mathbb{C}$

Set-theoretic analogy:

Proposition. $X^n \subset \mathbb{P}^{n+1}$ *smooth hypersurface. TFAE*

(1) $F^r = \det M$ *for some* $M \in \mathrm{Mat}_{dr \times dr}(S_1)$

(2) $\exists\, \mathcal{E}$ *vector bundle of rank* r *s.t.*

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{dr} \to \mathcal{O}_{\mathbb{P}^{n+1}}^{dr} \to \mathcal{E} \to 0$$

Ulrich bundles

Proposition-Definition(Eisenbud-Schreyer)**.**

$i\colon X^n \hookrightarrow \mathbb{P}^{n+c}$ smooth projective, $\mathcal{E}$ locally free sheaf over X. TFAE

(1) *$\mathcal{E}$ admits a linear resolution*

$$0 \to \mathcal{O}(-c)^{b_c} \to \mathcal{O}(-c+1)^{b_{c-1}} \to \ldots \to \mathcal{O}(-1)^{b_1} \to \mathcal{O}^{b_0} \to i_*\mathcal{E} \to 0$$

(2) $H^\bullet(\mathcal{E}(-j)) = 0$ *for* $j = 1, \ldots, n$

(3) *For any finite linear projection $\pi\colon X \to \mathbb{P}^n$, $\pi_*\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}^t$ for some t.*

$\mathcal{E}$ is called an **Ulrich sheaf(bundle)** if it satisfies above.

Remark. Definition depends on $\mathcal{O}_X(1)$

Question(Eisenbud-Schreyer)**.** Does any X admit Ulrich bundles?

Examples

1. $X = \mathbb{P}^n \qquad \mathcal{E}$ Ulrich $\Leftrightarrow \mathcal{E} = \mathcal{O}^t$ for some t.

   $H^\bullet(\mathcal{E}(-j)) = 0,\ j = 1, \ldots, n \ \Leftrightarrow\ \mathcal{E} \in {}^\perp\langle\, \mathcal{O}(1),\ \ldots,\ \mathcal{O}(n)\,\rangle$

   $\mathrm{D^b}(\mathbb{P}^n) = \langle\, \mathcal{O},\ \mathcal{O}(1),\ \ldots,\ \mathcal{O}(n)\,\rangle$ full exceptional collection

   $\langle \mathcal{O} \rangle \cap \mathbf{Coh}(X) = \{\mathcal{O}^t : t \in \mathbb{Z}_{\geq 1}\} \cup \{0\}$

2. $Q^n \subset \mathbb{P}^{n+1}$ quadric

   $\mathcal{E}$ is Ulrich & indecomposable $\Leftrightarrow$ $\mathcal{E}$ is a spinor bundle

$$\mathrm{D^b}(Q^n) = \begin{cases} \langle\, \mathcal{S},\ \mathcal{O}(1),\ \ldots, \mathcal{O}(n)\,\rangle & n \text{ odd} \\ \langle\, \mathcal{S}^+,\ \mathcal{S}^-,\ \mathcal{O}(1),\ \ldots, \mathcal{O}(n)\,\rangle & n \text{ even} \end{cases}$$

Corollary. $(x_0^2 + \ldots + x_{n+1}^2)^r = \det M$ *for some* $M \in \mathrm{Mat}_{2r\times 2r}(S_1)$ *if and only if* $2^{[\frac{n-1}{2}]}$ *divides* r.

Fano threefolds of index 2

[Beauville] Fano threefold of index 2 admits an Ulrich bundle of rank 2

[Casanellas-Hartshorne-Geiß-Schreyer] X a general cubic threefold, $r \geq 2$
$\Rightarrow$ there is an $r^2 + 1$ dimensional family of stable Ulrich bundles

Remark. Both use Serre construction: $0 \to \mathcal{O}_X^{r-1} \to \mathcal{E} \to \mathcal{I}_C \otimes \mathcal{L} \to 0$

[Lahoz-Macrì-Stellari] removes genericity assumption from [CHGS]

Derived categories

X a Fano threefold of index 2 $(\omega_X = \mathcal{O}_X(-2))$

$\Rightarrow \mathrm{D}^{\mathrm{b}}(X) = \langle\, O(-1),\, \mathcal{O},\, \mathcal{A}_X \,\rangle$ semiorthogonal decomposition

Ulrich condition: $H^\bullet(\mathcal{E}(-j)) = 0$, $j = 1, 2, 3$, equivalently

$$\mathrm{Ext}^\bullet(\mathcal{E}^\vee(1), \mathcal{O}(1-j)) = 0, \quad 1 - j = 0, -1, -2$$

$\Rightarrow \mathcal{E}^\vee(1) \in \mathcal{A}_X$ & $\mathrm{Ext}^\bullet(\mathcal{E}^\vee(1), \mathcal{O}(-2)) = 0.$

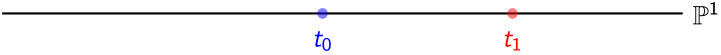
Strategy. Study Ulrich bundles as objects in $\mathcal{A}_X$

From now on, $X := Q_0 \cap Q_\infty$ (smooth), $Q_0, Q_\infty \in |\mathcal{O}_{\mathbb{P}^5}(2)|$

In this case, $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(C)$ for some curve of $g(C) = 2$ [Bondal-Orlov]

Fourier-Mukai kernel: restricted spinor bundles

Quadric pencil $|Q_t|_{t \in \mathbb{C} \cup \{\infty\}}$, $Q_t = (q_0 + t q_\infty = 0) \subset \mathbb{P}^5$



If Q_{t_0} is smooth $\rightarrow$ $\exists\, 2$ spinor bundles $\mathcal{S}^{\pm}_{Q_{t_0}}$ $\left(\mathcal{S}^{\pm}_{t_0} := \mathcal{S}^{\pm}_{Q_{t_0}}\big|_X\right)$

If Q_{t_1} is singular $\rightarrow$ $Q_{t_1} = \mathrm{Cone}(v, Q^3)$, unique spinor bundle $\mathcal{S}_{Q_{t_1}}$ over Q^3

$\mathcal{S}_{t_1} :=$ pullback along $X \rightarrow Q_{t_1} \setminus \{v\} \rightarrow Q^3$

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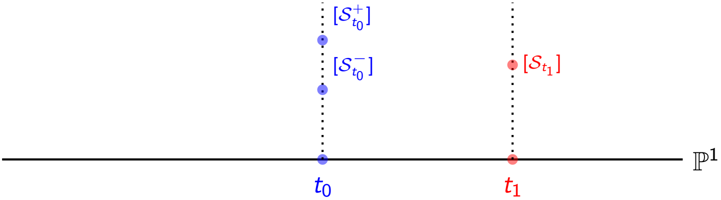

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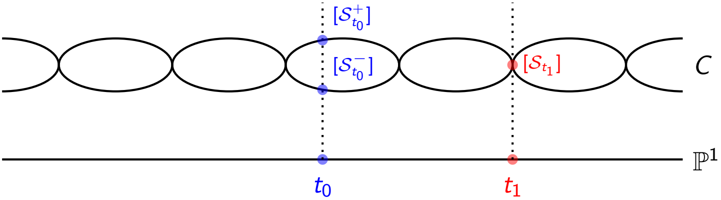

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$\mathcal{S}_{t_1}$:= pullback along $X \rightarrow Q_{t_1} \setminus \{v\} \rightarrow Q^3$

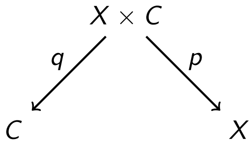
$C \xrightarrow{2:1} \mathbb{P}^1$, branched over 6 points $(g(C) = 2)$

$\mathcal{S}$:= a universal family, *i.e.* vector bundle over $X \times C$ s.t.

$$\mathcal{S}\big|_{X\times\{c\}} = \mathcal{S}_c, \quad c = [\mathcal{S}_c]$$

Fourier-Mukai Transform

$\mathrm{D^b}(X) = \langle\, \mathcal{O}(-1),\, \mathcal{O},\, \mathcal{A}_X \,\rangle$



[Bondal-Orlov]: $\Phi_{\mathcal{S}}$ induces $\mathrm{D^b}(C) \simeq \mathcal{A}_X$

Note. The right adjoint

$$\Phi^!_{\mathcal{S}} \colon \mathrm{D^b}(X) \to \mathrm{D^b}(C), \qquad \mathcal{E}^\bullet \mapsto q_*(p^*\mathcal{E}^\bullet \otimes \mathcal{S}^\vee) \otimes \omega_C[1]$$

is the left inverse to $\Phi_{\mathcal{S}}$. ($\Phi^!_{\mathcal{S}}\Phi_{\mathcal{S}} = Id$)

Fourier-Mukai Transform

$$\mathrm{D^b}(X) = \langle\, \mathcal{O}(-1),\, \mathcal{O},\, \mathcal{A}_X \,\rangle$$

$$\begin{array}{ccc} & \mathrm{D^b}(X \times C) & \\ {}^{q^*}\nearrow & & \searrow^{p_*} \\ \mathrm{D^b}(C) & & \mathrm{D^b}(X) \\ F^\bullet & \xmapsto{\ \Phi_{\mathcal{S}}\ } & p_*(q^* F^\bullet \otimes \mathcal{S}) \end{array}$$

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Main Theorem

X, C, $\Phi_{\mathcal{S}}$, ... as before

$\mathcal{M}^{\mathrm{s}}_X(r) :=$ moduli space of stable Ulrich bundles of rank r over X

$\mathcal{U}^{\mathrm{s}}_C(r, 2r) :=$ moduli space of stable bundles of (rk,deg) $= (r, 2r)$ over C

Theorem(C.-Kim-Lee). *$\Phi^!_{\mathcal{S}}$ induces an open embedding*

$$\varphi \colon \mathcal{M}^{\mathrm{s}}_X(r) \hookrightarrow \mathcal{U}^{\mathrm{s}}_C(r, 2r), \qquad \mathcal{E} \mapsto \Phi^!_{\mathcal{S}}(\mathcal{E}^\vee(1))$$

Moreover, $\mathcal{M}^{\mathrm{s}}_X(r)$ is nonempty for $r \geq 2$.

Remark.

(1) $\mathcal{M}^{\mathrm{s}}_X(r)$ is nonsingular, quasi-projective of dimension $r^2 + 1$

(2) φ is not surjective in general

Sketch of the proof

$$\mathcal{E} \in \mathcal{M}^{\mathrm{s}}_X(r) \iff H^\bullet(\mathcal{E}(-j)) = 0,\ j = 1,2,3$$

$$\iff \mathrm{Ext}^p_X(\mathcal{E}^\vee(1), \mathcal{O}(-j')) = 0,\ j' = 0,1,2$$

$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{O}(-1), \mathcal{O}, \mathcal{A}_X \rangle$

$$\iff \mathcal{E}^\vee(1) \in \mathcal{A}_X \ \&\ \mathrm{Ext}^p_X(\mathcal{E}^\vee(1), \omega_X) = 0$$

Use $\Phi^!_{\mathcal{S}} \colon \mathcal{A}_X \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(C)$: $\Phi^!_{\mathcal{S}}(\mathcal{E}^\vee(1)),\ \Phi^!_{\mathcal{S}}\omega_X \in \mathrm{D}^{\mathrm{b}}(C)$

$\Phi^!_{\mathcal{S}}\omega_X$ is related to the *second Raynaud bundle*, Indeed,

Proposition. $\Phi^!_{\mathcal{S}}\omega_X = \mathcal{R}^\vee \otimes \omega_C^2[-2]$, *where* $\mathcal{R} \in \mathcal{U}^{\mathrm{s}}_C(4,4)$ *with the property*

$$H^0(\mathcal{R} \otimes L) = H^1(\mathcal{R} \otimes L) = \mathbb{C},\ \forall L \in \mathrm{Pic}^0\, C.$$

Sketch of the proof

Proposition. *Let* $\mathcal{E}$ *be Ulrich bundle of rank* r. *Then* $F := \Phi^!_{\mathcal{S}}(\mathcal{E}^\vee(1))$ *is a locally free sheaf concentrated at degree* 0 *(as complex). Moreover,* $\operatorname{rk}\mathcal{F} = r$, $\deg \mathcal{F} = 2r$, *and*

$$\mathcal{E} \text{ is (semi)stable } \Leftrightarrow \ F \text{ is (semi)stable.}$$

Proof(outline). The pth cohomology sheaf of $\Phi^!_{\mathcal{S}}(\mathcal{E}^\vee(1))$ is

$$\mathcal{H}^p := R^{p+1}p_*(q^*(\mathcal{E}^\vee(1)) \otimes \mathcal{S}^\vee) \otimes \omega_C$$

Recall: $\Phi^!_{\mathcal{S}}(\mathcal{G}) = p_*(q^*\mathcal{G} \otimes \mathcal{S}^\vee) \otimes \omega_C[1]$

Base change: $\mathcal{H}^p \otimes \kappa(c) \to H^{p+1}(\mathcal{E}^\vee(1) \otimes \mathcal{S}^\vee_c)$

Sketch of the proof

From the properties of spinor bundles on quadrics:

$$0 \to \mathcal{S}_{\tau c}^\vee \to \mathcal{O}_X^4 \to \mathcal{S}_c^\vee(1) \to 0.$$

Since $H^\bullet(\mathcal{E}^\vee(j)) = 0$ for $j = 1, 0, -1$, get

$$H^{p+1}(\mathcal{E}^\vee(1) \otimes \mathcal{S}_c^\vee) \simeq H^{p+2}(\mathcal{E}^\vee(1) \otimes \mathcal{S}_c^\vee(-1)) = \dots,$$

and conclude $H^{p+1}(\dots) = 0$ for $p > 0$.

If $\operatorname{ch} F = s + d[P_C]$, $[P_C] \in H^2(C, \mathbb{Z})$, then GRR formula reads

$$\begin{aligned}\operatorname{ch} \Phi_{\mathcal{S}} F &= p_*(\operatorname{ch}(q^* F) \cdot \operatorname{ch} S \cdot \operatorname{td} \mathcal{T}_q) \\ &= (2d - 3s) + (d - 2s)[H_X] + s[L_X] + \tfrac{1}{3}(2s - d)[P_X]\end{aligned}$$

For $\mathcal{E}$ an Ulrich bundle of rank r, $\operatorname{ch} \mathcal{E} = r - r[L_X] \implies s = r,\ d = 2r$.

Sketch of the proof

If $\mathcal{E} \in \mathcal{M}^{s}_{X}(r)$, then $F := \Phi^{!}_{\mathcal{S}}(\mathcal{E}^{\vee}(1))$ stable bundle with $\mathrm{rk} = r$, $\deg = 2r$, and

$$\begin{aligned} 0 = \mathrm{Ext}^{p}_{X}(\mathcal{E}^{\vee}(1), \omega_X) &= \mathrm{Hom}_{\mathrm{D^b}(X)}(\Phi_{\mathcal{S}} F, \omega_X[p]) \\ &= \mathrm{Hom}_{\mathrm{D^b}(C)}(F, \mathcal{R}^{\vee} \otimes \omega_C^2[p-2]) \\ &= \mathrm{Ext}^{p-2}_{C}(F, \mathcal{R}^{\vee} \otimes \omega_C^2) \end{aligned}$$

Converse:

$\Phi_{\mathcal{S}} F$ is a vector bundle

Proposition. $\mathcal{M}^{s}_{X}(r) \simeq \left\{ F \in \mathcal{U}^{s}_{C}(r, 2r) : \begin{array}{l} H^1(F \otimes \mathcal{S}_x) = 0,\ \forall x \in X \\ \mathrm{Ext}^{p}(F, \mathcal{R}^{\vee} \otimes \omega_C^2) = 0,\ \forall p \end{array} \right\}$

Claim. General F satisfies the above properties.

(remark: it suffices to study the cases $r = 2, 3$)

Sketch of the proof

Key: $X = \mathcal{SU}_C(2, \xi^*)$ for some $\xi \in \mathrm{Pic}^1 C$ with $\mathcal{S}$ the universal family

$H^1(F \otimes \mathcal{S}_x) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{S}_x^\vee \otimes \omega_C)^*$. Assume $\exists f \colon F \to \mathcal{S}_x^\vee \otimes \omega_C$ nonzero

$$0 \to F' \to F \xrightarrow{f} \mathcal{S}_x^\vee \otimes \omega_C \to 0$$

F' semistable with $\mu = \frac{2r-5}{r-2} = 2 - \frac{1}{r-2}$

$$\begin{aligned}\dim\{F\} &\leq \quad \dim\{F'\} \quad + \dim\{x\} + \dim \mathbb{P}\,\mathrm{Ext}^1(\mathcal{S}_x^\vee \otimes \omega_C, F') \\ &= (r^2 - 4r + 5) + \quad 3 \quad + \qquad (3r-5) \\ &< r^2 + 1 = \dim \mathcal{U}_C^s(r, 2r) \text{ (if } r \geq 3)\end{aligned}$$

Similar argument for proving $\mathrm{Ext}^p(F, \mathcal{R}^\vee \otimes \omega_C^2) = 0$:

$(r = 3)$ take nonzero $f \colon \mathcal{R} \to F^\vee \otimes \omega_C^2$, then $\mu(\mathrm{im}\, f) \in \{\frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}\}$.

$\to$ Do case-by-case dimension counting...

Further questions

In general, $\mathcal{M}^{s}_{X}(r) \neq \mathcal{U}^{s}_{C}(r, 2r)$

Example. Let $P \in C$ be a point, and $F := \mathcal{R}^{\vee} \otimes \omega_C^2 \otimes \mathcal{O}_C(-P) \in \mathcal{U}^{s}_{C}(4, 8)$. Clearly, there is a nonzero map $F \to \mathcal{R}^{\vee} \otimes \omega_C^2$.

Question. $\overline{\mathcal{M}^{s}_{X}}(r) =?$

Theorem[Qin]**.** *Let $\mathcal{I}_X(2)$ be the moduli space of instanton sheaves of charge 2. Then, $\mathcal{I}_X(2)$ is projective and contains $\mathcal{M}^{s}_{X}(2)$.*