Around BGK models: numerical methods for conservation laws and more

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Goal of the talk

Use the kinetic viewpoint to construct numerical approximations of systems of PDEs arising in fluid dynamics, plasmas, and more.

The fundamental example Finite volumes for Euler equations:

- Godunov method
  Riemann solver with resolution of algebraic equations at each cell interface
- Kinetic method

$$\begin{aligned} &\textit{Euler} &&= \textit{limit of BGK} \\ &\textit{discretized Euler} &&= \textit{limit of discretized BGK} \end{aligned}$$

Resolution of transport equations

First paper in that direction: R.H. Sanders and K.H. Prendergast, On the origin of the 3 kiloparsec arm, Astrophys. J. (1974)

BGK (Bhatnagar-Gross-Krook) model for Euler equations - 1954

It is a simplification of the Boltzmann equation governing the distribution function $f(x, t, \xi)$ of an homogeneous gas:

$$\partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \left(M(Pf^\varepsilon, \xi) - f^\varepsilon \right), \; Pf^\varepsilon = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ \frac{1}{2}|\xi|^2 \end{pmatrix} f^\varepsilon d\xi.$$

linear transport equation with source-term.

ε: proportionnal to the Knudsen number.

$M(U, \xi)$: maxwellian function satisfying

$$\forall U = (\rho, \rho u, \mathcal{E}), \quad PM(U, \xi) = U, \quad P(\xi M(U, \xi)) = F(U)$$

and entropy properties. F: flux of Euler equations.

Macroscopic limit: $\varepsilon \to 0$

$$f^{\varepsilon} \to f = M(U), \quad U = Pf$$

and U is an entropy solution of Euler equations.

The idea is to discretize the BGK equation in such a way that when $\varepsilon \to 0$ we obtain a consistent and stable discretization of Euler equations, ie to construct an Asymptotic Preserving (AP) scheme

Finite Volume method for hyperbolic systems of conservation laws

$$\partial_t U + \partial_x F(U) = 0, \quad (x,t) \in \mathbb{R} \times [0,T]$$

with $U(x,t) \in \mathcal{V} \subset \mathbb{R}^K$.

$C_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[,\ x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = \Delta x,\ t_0 = 0,\ t_{n+1} = t_n + \Delta t.$

$$\int_{t_n}^{t_{n+1}} \int_{C_j} \partial_t U + \partial_x F(U) dx\, dt = 0$$

$$\int_{C_j} U(x, t_{n+1}) dx = \int_{C_j} U(x, t_n) dx - \int_{t_n}^{t_{n+1}} F(U(x_{j+\frac{1}{2}}, t)) dt + \int_{t_n}^{t_{n+1}} F(U(x_{j-\frac{1}{2}}, t)) dt$$

Finite Volume method for hyperbolic systems of conservation laws

$$U_j^n \sim \frac{1}{\Delta x}\int_{C_j} U(x,t_n)dx, \quad F_{j+\frac{1}{2}}^n \sim \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(U(x_{j+\frac{1}{2}},t))dt$$

Approximate formula:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right).$$

The numerical approximation is determined by the choice of $F_{j+\frac{1}{2}}^n$.

$$F_{j+\frac{1}{2}}^n = \mathcal{F}(U_j^n, U_{j+1}^n), \quad \mathcal{F}(U,U) = F(U) \quad \text{(conservativity)}.$$

Simple choices as centred formula $\mathcal{F}(U,V) = \frac{1}{2}(F(U)+F(V))$ are not stable even in the linear case.

Finite Volume method for conservation laws

In order to compute physically relevant solutions, the scheme must satisfy some discrete discrete entropy inequality: if η is an entropy for the system:

$$\frac{\eta(U_j^{n+1}) - \eta(U_j^n)}{\Delta t} + \frac{\mathcal{G}_{j+\frac{1}{2}}^n - \mathcal{G}_{j-\frac{1}{2}}^n}{\Delta x} \leq 0.$$

$\mathcal{G}_{j+\frac{1}{2}}^n$: numerical entropy flux.

Finite Volume method for conservation laws

Godunov method: solve the Riemann problem at interface exactly, *i.e.* : $\forall x_{j+\frac{1}{2}}$, find the exact solution $U_{j+\frac{1}{2}}$ of the system with

$$U_{j+\frac{1}{2}}(x,t_n) = U_j^n \quad \text{if} \quad x < x_{j+\frac{1}{2}}, \quad U_{j+\frac{1}{2}}(x,t_n) = U_{j+1}^n \quad \text{if} \quad x > x_{j+\frac{1}{2}}.$$

Then set as numerical flux

$$F_{j+\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}},t))dt.$$

May be difficult and expensive.

Under CFL condition, this is equivalent to set U_j^{n+1} as the average of the exact solution at $t = t_{n+1}$ when $U(x,t_n) = \sum_j U_j^n \mathbb{1}_{C_j}(x)$.

Alternatives to Godunov method

In order to avoid the exact resolution of the Riemann problem at each interface: approximate solvers and Lax-Friedrichs type schemes. Among them:

- HLL solver
- kinetic solvers
- relaxation solvers

These methods are closely linked.

In what follows we give a brief overview of the history of kinetic approach of conservation laws.

HLL solver

A. Harten, P.D. Lax, B. Van Leer, On upstream differencing and Godunov-type schemes for hyperbolic conservation laws, SIAM Rev. (1983)

Exact solution of the Riemann problem

$$U_{j+\frac{1}{2}}(x,t) = W\left(\frac{x - x_{j+\frac{1}{2}}}{t - t_n}; U_j^n, U_{j+1}^n\right)$$

and W is a superposition of simple waves (shocks, rarefactions, contact discontinuities).

Denoting $a_i(U)$ the eigenvalues of $F'(U)$:

$$\lambda^- = \min_i\{a_i(U), a_i(V)\}, \quad \lambda^+ = \max_i\{a_i(U), a_i(V)\}.$$

For $y < \lambda^-$: $W(y, U, V) = U$.

For $y > \lambda^+$: $W(y, U, V) = V$.

HLL solver

HLL scheme : $W(y, U, V)$ is replaced by the superposition of three constant states U, Z, V:

$$w(y, U, V) = \begin{cases} U & \text{if} \quad y < \lambda^-, \\ Z & \text{if} \quad \lambda^- < y < \lambda^+, \\ V & \text{else.} \end{cases}$$

HLL solver: the numerical flux

Some more considerations lead to

$$\mathcal{F}(U,V) = \begin{cases} F(U) & \text{if} \quad \lambda^- > 0, \\ F(V) & \text{if} \quad \lambda^+ < 0, \\ \dfrac{-\lambda^- F(V) + \lambda^+ F(U)}{\lambda^+ - \lambda^-} + \dfrac{\lambda^+ \lambda^-}{\lambda^+ - \lambda^-}(V - U) & \text{else.} \end{cases}$$

Explicit, easy to implement.

Kinetic interpretation of flux splitting methods

Euler equations:

$$F'(U) = A(U) = R(U)\Lambda(U)R(U)^{-1}, \quad \Lambda(U) = \mathrm{diag}(a_i(U)).$$

By homogeneity of F the Euler identity holds:

$$F(U) = A(U)U.$$

Denoting $\Lambda^{\pm}$ =diag($a_i^{\pm}$) and $A^{\pm} = R\Lambda^{\pm}R^{-1}$:

$$F(U) = F^{+}(U) - F^{-}(U), \quad F^{\pm}(U) = A^{\pm}(U)U.$$

Kinetic interpretation of flux splitting methods

Steger-Warming scheme:

$$F^n_{j+\frac{1}{2}} = F^+(U^n_j) - F^-(U^n_{j+1})$$

Harten, Lax and Van Leer give a kinetic interpretation of this scheme with a Maxwellian function supported by the characteristic velocities $a_i(U)$.

A. Harten, P.D. Lax, B. Van Leer, On upstream differencing and Godunov-type schemes for hyperbolic conservation laws, SIAM Rev. (1983)

Kinetic numerical methods

Y. Brenier, Résolution déquations d'évolution quasilinéaires en dimension N d'espace à l'aide d'équations linéaires en dimension N+1. JDE (1983).

Scalar conservation law

$$\partial_t U + \sum_{n=1}^{N} \partial_{x_n} F_n(U) = 0, \quad (x,t) \in \mathbb{R}^N \times [0,T]$$

with $U(x,t) \in \mathcal{V} \subset \mathbb{R}$.

Kinetic equation (linear related equation):

$$\partial_t s + \sum_{n=1}^{N} F_n'(\xi) \partial_{x_n} s = 0.$$

Maxwellian function:

$$\chi(u,\xi) = \begin{cases} 1 & \text{if} \quad 0 < \xi < u \\ -1 & \text{if} \quad u < \xi < 0 \\ 0 & \text{else.} \end{cases}$$

Moment operator:

$$\forall g \in L^1(\mathbb{R}), \quad Pg = \int_{\mathbb{R}} g(\xi) d\xi.$$

The following time-splitting procedure converges in L^1 when $\Delta t \to 0$:

- Projection on the Maxwellian state:

$$s^n(x,\xi) = \chi(u^n(x),\xi).$$

- Transport:

$$\forall (x,\xi) \in \mathbb{R}^N \times \mathbb{R}, \quad s^{n+1}(x,\xi) = s^n(x - tF'(\xi)).$$

- Moment operator:

$$\forall x \in \mathbb{R}^N, \quad u^{n+1}(x) = P(s^{n+1}(x,.)).$$

Related BGK model

$$\partial_t f + \sum_{n=1}^{N} F_n'(\xi)\partial_{x_n} f = \frac{1}{\varepsilon}\left(\chi(Pf,\xi) - f\right). \qquad (1)$$

One has

$$\int_{\mathbb{R}} \chi(U,\xi)d\xi = U, \quad \int_{\mathbb{R}} F'(\xi)\chi(U,\xi)d\xi = F(U) - F(0).$$

The same model appears in Y. Giga and T. Miyakawa, A kinetic construction of global solutions of first order quasilinear equations, Duke Math. J. (1983).

Convergence of (1) to weak entropy solutions of the scalar conservation law:
B. Perthame and E. Tadmor, A kinetic equation with kinetic entropy functions for scalar conservation laws, Commun. Math. Phys. (1991).

Kinetic interpretation of Engquist-Osher scheme

Consider the 1D scalar conservation law with F strictly convex, $UF'(U) \geq 0$, $F(0) = 0$:

$$\partial_t U + \partial_x F(U) = 0.$$

$$F^-(U) = -F(U) \quad \text{if } U < 0, \qquad 0 \quad \text{else}$$
$$F^+(U) = F(U) \quad \text{if } U > 0, \qquad 0 \quad \text{else.}$$

F^+ and F^- are increasing functions. Numerical flux of Engquist-Osher:

$$F^n_{j+\frac{1}{2}} = F^+(U^n_j) - F^-(U^n_{j+1})$$

This scheme can be obtained as follows: for all $n \geq 0$

- Maxwellian projection: $f_j^n(\xi) = \chi(U_j^n, \xi)$,
- Upwind scheme on the transport equation:

$$f_j^{n+\frac{1}{2}}(\xi) = f_j^n(\xi) - F^{+\prime}(\xi)\frac{\Delta t}{\Delta x}\left(f_j^n(\xi) - f_{j-1}^n(\xi)\right) + F^{-\prime}(\xi)\frac{\Delta t}{\Delta x}\left(f_{j+1}^n(\xi) - f_j^n(\xi)\right)$$

- Moment operator:

$$U_j^{n+1} = \int_{\mathbb{R}} f_j^{n+\frac{1}{2}}(\xi) d\xi.$$

Kinetic schemes for hyperbolic systems

Models for gas dynamics:
B. Perthame, Boltzmann type schemes for gas dynamics and the entropy property, SINUM (1990).

See the book B. Perthame, Kinetic formulation of conservation laws. Oxford lecture series in mathematics and its applications, 21 (2002) and ref. therein.

Jin and Xin's relaxation scheme

S. Jin and Z. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions. CPAM (1995). 1D simplified version : consider a hyperbolic system

$$(1) \qquad \partial_t U + \partial_x F(U) = 0, \quad U(x,t) \in \mathcal{V} \subset \mathbb{R}^K.$$

Relaxation approximation by a semilinear hyperbolic system of $2K$ equations

$$\begin{cases} \partial_t U^\varepsilon + \partial_x V^\varepsilon = 0 \\ \partial_t V^\varepsilon + a^2 \partial_x U^\varepsilon = \dfrac{1}{\varepsilon}\left(F(U^\varepsilon) - V^\varepsilon\right), \quad a > 0. \end{cases}$$

Formally when $\varepsilon \to 0$, $(U^\varepsilon, V^\varepsilon) \to (U, F(U))$ and U is a solution of (1).

Stability condition

Chapman-Enskog expansion:

$$V^\varepsilon = F(U^\varepsilon) - \varepsilon V^{(1)}.$$

$$\begin{aligned} V^{(1)} &= \partial_t V^\varepsilon + a^2 \partial_x U^\varepsilon \\ &= \partial_t F(U^\varepsilon) + a^2 \partial_x U^\varepsilon + O(\varepsilon) \\ &= F'(U^\varepsilon)\partial_t U^\varepsilon + a^2 \partial_x U^\varepsilon + O(\varepsilon) \\ &= -F'(U^\varepsilon)\partial_x V^\varepsilon + a^2 \partial_x U^\varepsilon + O(\varepsilon) \end{aligned}$$

Hence:

$$\partial_t U^\varepsilon + \partial_x F(U^\varepsilon) = \varepsilon \partial_x \left((a^2 I - F'(U^\varepsilon)^2) \partial_x U^\varepsilon \right).$$

A subcharacteristic condition (TP Liu) is necessary: a has to be large enough w.r.t. F'.

Relaxing and relaxed schemes

Relaxing scheme: for $\varepsilon > 0$ fixed, fractional step method:

1) Upwind scheme on the homogeneous part.
2) Exact resolution of the source-term on $[t_n, t_{n+1}]$:

$$\begin{cases} \partial_t U^\varepsilon = 0 \\ \partial_t V^\varepsilon = \dfrac{1}{\varepsilon}\left(F(U^\varepsilon) - V^\varepsilon\right). \end{cases}$$

Obtention of the relaxed scheme by making $\varepsilon \longrightarrow 0$ in the relaxed scheme. A Lax-Friedrichs type scheme is obtained:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{2\Delta x}(F(U_{j+1}^n) - F(U_{j-1}^n)) + a\frac{\Delta t}{2\Delta x}(U_{j+1}^n - 2U_j^n + U_{j-1}^n).$$

Higher order in space and time is possible and easy.

Convergence in the 1D scalar case for relaxing and relaxed schemes: D. A.-D. and R. Natalini, Convergence of relaxation schemes for conservation laws, App Anal. (1996).
Convergence of the model: R. Natalini, CPAM (1996).

BGK interpretation of Jin and Xin's model

$$\begin{cases} \partial_t U^\varepsilon + \partial_x V^\varepsilon = 0 \\ \partial_t V^\varepsilon + a^2 \partial_x U^\varepsilon = \dfrac{1}{\varepsilon}\left(F(U^\varepsilon) - V^\varepsilon\right), \quad a > 0. \end{cases}$$

Diagonalization:

$$f^- = \frac{1}{2}\left(U - \frac{V}{a}\right), \quad f^+ = \frac{1}{2}\left(U + \frac{V}{a}\right).$$

Denoting $M^\pm(U) = \dfrac{\pm F(U) + aU}{2a}$:

$$\begin{cases} \partial_t f^- - a\partial_x f^- = \dfrac{1}{\varepsilon}\left(M^-(f^- + f^+) - f^-\right) \\ \partial_t f^+ + a\partial_x f^+ = \dfrac{1}{\varepsilon}\left(M^+(f^- + f^+) - f^+\right) \end{cases}$$

This is formally a discrete vectorial BGK model:

$$f = \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \mathbb{R}^{2K}, \quad Pf = f^- + f^+, \quad M = \begin{pmatrix} M^- \\ M^+ \end{pmatrix}$$

The moment operator P is an integration with the counting measure satisfying

$$PM(U) = U, \quad P\Lambda M(U) = F(U)$$

with $\Lambda = diag(-aI_K, aI_K)$. Discrete analogue of

$$\int_{\mathbb{R}} M(U,\xi)d\xi = U, \quad \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ \frac{1}{2}|\xi|^2 \end{pmatrix} \xi M(U,\xi)d\xi = F(U)$$

for Euler equations.

Generalization

Multidimensional hyperbolic system

$$\partial_t U + \sum_{d=1}^{D} \partial_{x_d} F_d(U) = 0, \quad U(x,t) \in \mathcal{V} \subset \mathbb{R}^K.$$

BGK model:

$$\partial_t f^\varepsilon + \sum_{d=1}^{D} \Lambda_d \partial_{x_d} f^\varepsilon = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right), \quad f^\varepsilon(x,t) \in (\mathbb{R}^K)^L.$$

$\Lambda_d = \mathrm{diag}\left(v_{d,1}\mathrm{I}_K, \dots, v_{d,L}\mathrm{I}_K\right)$, $v_{d,l} \in \mathbb{R}$,

$P \in \mathcal{L}\left((\mathbb{R}^K)^L, \mathbb{R}^K\right)$, and $M = (M_1, \dots, M_L)$: $\mathcal{V} \to (\mathbb{R}^K)^L$.

Compatibility conditions

$$\forall U \in \mathcal{V}, \quad PM(U) = U, \quad P\Lambda_d M(U) = F_d(U),\ d = 1, \ldots, D.$$

Often $Pf = \sum_l f_l$.

Chapman-Enskog expansion can be generalized:

$$U^\varepsilon = Pf^\varepsilon$$

$$\partial_t U^\varepsilon + \sum_d \partial_{x_d} F_d(U^\varepsilon) = \varepsilon \sum_d \left(\partial_{x_d} \sum_j B_{dj}(U^\varepsilon) \partial_{x_d} U^\varepsilon \right)$$

with

$$B_{dj}(U) = P\Lambda_d \Lambda_j M'(U) - F_d'(U) F_j'(U).$$

Stability

Convergence of $U^\varepsilon = Pf^\varepsilon$ towards an entropy solution of the system of conservation laws?

Scalar case: U_0: initial condition. We suppose that the Maxwellian functions are monotone nondecreasing functions:

$$\forall U \in [-\|U_0\|_\infty, \|U_0\|_\infty], \quad M_l'(U) \geq 0, \; l = 1, \dots, L.$$

Example: Jin and Xin's model: $L = 2$,

$$M^-(U) = \frac{-F(U) + aU}{2a}, \quad M^+(U) = \frac{F(U) + aU}{2a}.$$

M^- and M^+ are increasing functions if and only if the subcharacteristic condition is satisfied:

$$\forall U \in [-\|U_0\|_\infty, \|U_0\|_\infty], \quad -a \leq F'(U) \leq a.$$

Scalar case

Theorem 1 R. Natalini, JDE 1998. $U_0 \in L^\infty(\mathbb{R}^D)$ and $f_0 = M(U_0)$. For all l, M_l is nondecresing on $[-\|U_0 l_\infty, \|U_0 l_\infty]$. For $\varepsilon > 0$ fixed the BGK system has a unique solution $f^\varepsilon \in C([0, \infty[, L^1_{loc} \cap L^\infty)$. Moreover

$$M_l(-\|U_0\|_\infty) \leq f_l^\varepsilon \leq M_l(\|U_0\|_\infty), \quad l = 1, \ldots, L.$$

Main argument: The BGK system is quasimonotone (B. Hanouzet and R. Natalini, Diff. Int. Eq. 1996).

Theorem 2 R. Natalini, JDE 1998. Same assumptions. $U^\varepsilon = Pf^\varepsilon$ converges to the unique entropy solution of the Cauchy problem for the conservation law.

Boundary conditions: V. Milisic, Proc. Amer. Math. Soc. 2003.

Systems

F. Bouchut, J. Stat. Phys. 1999: let E be a set of entropies for the system of conservation laws. For $\eta \in E$ a related kinetic entropy H_η is a convex function s.t. $G_{l,\eta} = H_\eta \circ M_l$ satisfies:

- $PG_\eta(U) = \eta(U)$,
- $\forall U_f$ s.t. $f = M(U_f)$, $PG_\eta(U_f) \leq PH_\eta(f)$.

The existence of H_η for all $\eta \in E$ is equivalent to the fact that $\eta \in E$ $(M_l')^t \eta''$ is symmetric and

$$\forall U \in \mathcal{V}, \quad \sigma(M_l'(U)) \subset [0, +\infty[; \quad l = 1, \ldots, L.$$

In this case the Chapman-Enskog expansion is η-dissipative.

see also D. Serre, Ann. Inst. H. Poincaré 2000 for 2x2 1D systems.
S. Bianchini, CPAM 2006 for 1D strictly hyperbolic systems (data small in BV).

Examples of BGK vectorial models

1D, L=2. $Pf = \sum_l f_l$.

$$\begin{cases} \partial_t f_1 + \lambda^- \partial_x f_1 = \frac{1}{\varepsilon}\left(M_1(f_1+f_2) - f_1\right) \\ \partial_t f_2 + \lambda^+ \partial_x f_2 = \frac{1}{\varepsilon}\left(M_2(f_1+f_2) - f_2\right) \end{cases}$$

with

$$M_1(U) = \frac{\lambda^+ U - F(U)}{\lambda^+ - \lambda^-}, \quad M_2(U) = \frac{-\lambda^- U + F(U)}{\lambda^+ - \lambda^-},$$

Stability conditions: $(M_l')^t \eta''$ is symmetric for all entropy η.

$$\lambda^- \leq F'(U) \leq \lambda^+.$$

2D version

L=4. $Pf = \sum_l f_l$.

$$\Lambda_1 = \begin{pmatrix} \lambda_1^- I_K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^+ I_K & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2^- I_K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^+ I_K \end{pmatrix}.$$

Stability conditions: $(M_l')^t \eta''$ is symmetric for all entropy η.

$$\frac{1}{2}\lambda_d^- \le F_d'(U) \le \frac{1}{2}\lambda_d^+.$$

see D. A-D. and R. Natalini, SINUM 2000 for more examples with linear combinations of U and $F_d(U)$.

Flux decomposition in 1D

As for Engquist-Osher of Steger-Warming, suppose that $F(U) = F^+(U) - F^-(U)$ with $\sigma(F^{\pm\prime}(U)) \subset [0, \infty[$. BGK system: $\lambda > 0$, $U = \sum_l f_l$.

$$\partial_t f_1 - \lambda \partial_x f_1 = \frac{1}{\varepsilon}\left(\frac{F^-(U)}{\lambda} - f_1\right)$$

$$\partial_t f_2 = \frac{1}{\varepsilon}\left(U - \frac{F^+(U) + F^-(U)}{\lambda} - f_2\right)$$

$$\partial_t f_3 + \lambda \partial_x f_3 = \frac{1}{\varepsilon}\left(\frac{F^+(U)}{\lambda} - f_3\right)$$

see also F. Bouchut, Entropy satisfying flux vector splitting and kinetic BGK models, Numer. Math. 2003

Numerical methods

D. A-D. and R. Natalini, SINUM 2000

Each BGK model provides a numerical method by the same procedure as above:

1) Numerical approximation of the BGK model by fractional step method:

- Upwind method on the free transport equations: obtention of $f^{\varepsilon,n+1/2}$
- Exact resolution of the source-term:

$$f_j^{\varepsilon\prime} = \frac{1}{\varepsilon}\left(M(Pf_j^{\varepsilon}) - f_j^{\varepsilon}\right)$$

  We have $Pf^{\varepsilon\prime} = 0$ so

$$U_j^{\varepsilon,n+1} = Pf_j^{\varepsilon,n+1/2},$$

$$f_j^{\varepsilon,n+1} = M(U_j^{\varepsilon,n+1}) + \mathrm{e}^{-\Delta t/\varepsilon}\left(f_j^{\varepsilon,n+1/2} - M(U_j^{\varepsilon,n+1})\right).$$

2)Relaxed limit of the scheme:$\varepsilon \to 0$

- Projection onto equilibrium:

$$f_j^n = M(U_j^n)$$

- Transport (upwind). In 1D

$$f_j^{n+\frac{1}{2}} = f_j^n - \frac{\Delta t}{\Delta x}\Lambda(f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n)$$

$$U_j^{n+1} = Pf_j^{n+\frac{1}{2}}.$$

Same procedure as Brenier (1983): transport-collapse.

First order relaxed numerical flux

If $f^n_{j+\frac{1}{2}} = \Phi(f^n_j, f^n_{j+1})$ then

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}})$$

with

$$F^n_{j+\frac{1}{2}} = \mathcal{F}(U_j^n, U_{j+1}^n) = P\Lambda\Phi(M(U_j^n), M(U_{j+1}^n))$$

Consistence:

$$\mathcal{F}(U,U) = P\Lambda\Phi((MU), M(U)) = F(U).$$

Scalar case: convergence to the unique entropy solution of Cauchy Problem (monotony).
Boundary conditions: D. A-D. and V. Milisic, Numer. Math. 2004.

Application to the presented BGK models

Application to model 2: HLL method.

Application to FDM: flux vector splitting.

Higher order schemes

In time: RK2 Heun. In space, there are 2 viewpoints:

I. Replace the upwind scheme by a second order MUSCL type scheme: $f_j^{n+\frac{1}{2}}$ is obtained by affine reconstruction, exact transport and exact average on C_j.

Works well, good stability properties, at least in the scalar case (TVD, L^∞).

Problems:
Computation of $K \times L$ slopes $\forall j, n$.
For systems: positivity is not necessarily preserved.

II. Forget BGK and apply the second order procedure on the relaxed numerical flux :

1) Affine reconstruction of U_j^n:

$$\overline{U_j^n}(x) = U_j^n + \sigma_j^n (x - x_j).$$

2) Modified numerical flux:

$$\overline{F^n_{j+\frac{1}{2}}} = \mathcal{F}\left(\overline{U_j^n}(x + \Delta x/2), \overline{U_{j+1}^n}(x - \Delta x/2)\right).$$

Interest and drawbacks of such methods

- BGK models provide flexible, easy to implement finite volume schemes.
- Great stability
- Drawback: contact discontinuities are not accurate

Other numerical techniques

- Finite elements T. Katsaounis and C. Makridakis, Math of Comp. 2001).
- Discontinuous Galerkin method, lattice Boltzmann method: B. Graille, JCP 1994,
  Coulette et al, Computers and Fluids 2019
- Residual distribution schemes: D. Torlo, PhD thesis with R. Abgrall, 2020.

Other problems

- Parabolic problems: kinetic models with diffusive scaling: F. Bouchut, F. Guarguaglini, R. Natalini, Indiana Univ. Math. J. 2000.
  D. A-D., R. Natalini, S. Tang, Math. of Comp. 2003.
- Incompressible Navier-Stokes equations: M.F. Carfora, R. Natalini, M2ANB 2008,
  Bouchut et al, SIAM J. Comput. Math. 2018.
- A non conservative hyperbolic system : the Euler bitemperature model for plasma out of thermic equilibrium: will be the topic of the second talk.