Around BGK models: numerical methods for conservation laws and more

D. Aregba-Driollet

Institut de Mathématiques de Bordeaux

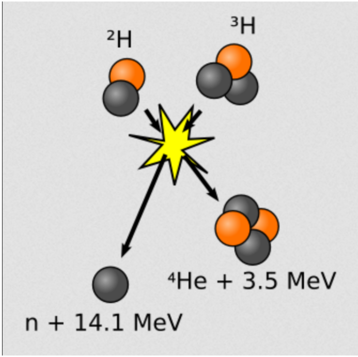
2. BGK approximations of a nonconservative system: Euler bitemperature model

Collaborations:

- Stéphane Brull (IMB, Bordeaux), Bruno Dubroca
- Corentin Prigent : PhD thesis, IMB and CELIA

The physical context

Inertial Confinment Fusion experiment



In that goal: powerful laser beams transform a microcapsule of deuterieum-tritium into plasma.

The physical context

- Shock waves in a very small volume
- High temperatures : 10^7 K
- Time scale : 10^{-9} seconds
- During a small time interval the temperature of ions differ from the one of electrons

The bitemperature Euler model

Constant ionization $Z = \frac{n_e}{n_i}$ (quasi-neutrality)

Notations: e: electrons, i: ions.

- c_e, c_i: massic fractions

$$\rho_e = \rho c_e = m_e n_e, \quad \rho_i = \rho c_i = m_i n_i, \quad c_e + c_i = 1.$$

  Consequence: c_e et $c_i$are constant.

- $u = u_e = u_i$ (the velocities are in equilibrium)

- The Ionic and electronic temperatures are out of equilibrium: two distinct energies

$$\mathcal{E}_\beta = \rho_\beta \varepsilon_\beta + \frac{1}{2}\rho_\beta u^2, \quad \beta = e, i.$$

Two species Euler nonconservative system

2 pressure laws and 2 temperatures:

$$p_\alpha = (\gamma_\alpha - 1)\rho_\alpha \varepsilon_\alpha = n_\alpha k_B T_\alpha, \quad \alpha = e, i.$$

Equations:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p_e + \nabla_x p_i = 0, \\ \partial_t \mathcal{E}_e + \nabla_x \cdot (u(\mathcal{E}_e + p_e)) - u \cdot (c_i \nabla_x p_e - c_e \nabla_x p_i) = \nu_{ei}(T_i - T_e), \\ \partial_t \mathcal{E}_i + \nabla_x \cdot (u(\mathcal{E}_i + p_i)) + u \cdot (c_i \nabla_x p_e - c_e \nabla_x p_i) = -\nu_{ei}(T_i - T_e), \end{cases}$$

[Coquel, Marmignon, 1998], [Coquel, Chalons, 2005]

Nonconservativity

- Source terms
- System of form

$$\partial_t \mathcal{U} + \sum_{d=1}^{D} A_d(\mathcal{U}) \partial_{x_d}(\mathcal{U}) = S(\mathcal{U})$$

  A flux function F such that $F'_d(\mathcal{U}) = A_d(\mathcal{U})$ does not exist. No divergential form.

- If $\gamma_e = \gamma_i$ then $(\rho, \rho u, \mathcal{E}_e + \mathcal{E}_i)$ satisfies Euler system. But even in this case, the problem of determining T_e and T_i separately remains.

Properties

- Hyperbolicity: diagonalisable with 3 eigenvalues $u \cdot \omega$, $u \cdot \omega \pm a$

$$a = \sqrt{\frac{\gamma_e p_e + \gamma_i p_i}{\rho}}$$

- [Aregba-Driollet, Breil, Brull, Estibals, Dubroca, 2018] Existence of a dissipative strictly convex entropy

$$\eta = \sum_{\alpha=e,i} \left(-\frac{\rho_\alpha}{m_\alpha(\gamma_\alpha - 1)} \ln \frac{p_\alpha}{\rho_\alpha^{\gamma_\alpha}} \right), \quad Q = u\eta.$$

For smooth solutions

$$\partial_t \eta + \mathrm{div} Q = -\frac{\nu_{ei}}{k_B T_i T_e}(T_i - T_e)^2$$

Strong solutions: the entropy does not provide a symmetrizer

Hyperbolic system in divergential form

$$\partial_t U + \mathrm{div} F(U) = Q(U)$$

η is a strictly convex entropy if and only if $\eta''(U)F_d'(U)$ is symmetric for all $d = 1, \dots, D$.

Here : $\partial_t U + \sum_{d=1}^{D} A_d(U)\partial_d U = Q(U)$ with $A_d(U) \neq F_d'(U)$. Even for $D = 1$

$\eta''(U)A(U)$ is symmetric if and only if $T_i = T_e$.

[Aregba-Driollet, Brull, Peng 2021]:

- Existence of a symmetrizer (hence local existence for smooth solutions)
- 1D for $\gamma_i \neq \gamma_e$: global existence of smooth solutions of the Cauchy problem for small data.

Weak solutions

Weak solutions are physical, our goal is to approximate them numerically

Terms $u \cdot \nabla(c_i p_e - c_e p_i)$:

- u continuous (contact discontinuities or rarefactions): OK.
- u discontinuous (shocks) ? Theoritical definition : [Dal Maso, Le Floch, Murat, 1995], [Berthon, Coquel, Le Floch, 2012]. In order to select admissible shocks one needs for information from elsewhere: viscosity, underlying system from physics...

Here: kinetic viewpoint

In this talk

1. Construction of the bitemperature Euler system as the hydrodynamic limit of a BGK model
2. Entropy properties
3. Discrete 2D BGK scheme
   - First order
   - Extension to order 2

Construction of the bitemperature model

One species

$f(t,x,v)$: distribution function

$$\underbrace{\frac{\partial f}{\partial t} + v.\nabla_x f}_{transport} + \underbrace{F.\nabla_v f}_{force\ \ term} = \underbrace{\mathcal{C}(f,f)}_{collision\ \ term}$$

ρ, u, T: mass, velocity and temperature

$$\rho = \int_{\mathbb{R}^3} m\, f\, dv, \qquad u = \frac{1}{\rho}\int_{\mathbb{R}^3} vf\, dv, \qquad T = \frac{1}{3\rho}\int_{\mathbb{R}^3} |v-u|^2 f\, dv.$$

Fluid models:

- Equilibrium states: $\mathcal{C}(f,f) = 0 \ \Leftrightarrow f = \mathcal{M}_f$
- $f = \mathcal{M}_f$ and take the moments on the kinetic equation with respect to $(1, v, v^2)$
  $\Rightarrow$ Euler system
- $f = \mathcal{M}_f + \varepsilon f_1$ and take the moments on the kinetic equation with respect to $(1, v, v^2)$
  $\Rightarrow$ système de Navier-Stokes

Macroscopic mixing quantities

$\alpha = e$ or i. Density: n_α, velocity: u_α, temperature: T_α

$f_\alpha(t, x, v)$: distribution function of α species

$$\rho_\alpha = n_\alpha m_\alpha = m_\alpha \int_{\mathbb{R}^3} f_\alpha dv, \quad u_\alpha = \frac{1}{n_\alpha} \int_{\mathbb{R}^3} v f_\alpha dv,$$
$$\mathcal{E}_\alpha = \frac{3}{2}\rho_\alpha \frac{k_B}{m_\alpha} T_\alpha + \frac{1}{2}\rho_\alpha u_\alpha^2 = \int_{\mathbb{R}^3} m_\alpha \frac{v^2}{2} f_\alpha dv.$$

Mixing

$$u = \frac{\rho_e u_e + \rho_i u_i}{\rho_e + \rho_i}, \quad n k_B T = \sum_\alpha \frac{1}{2}\rho_\alpha (u_\alpha^2 - u^2) + \sum_\alpha (n_\alpha k_B T_\alpha).$$

Total charge: $Q = \int_{\mathbb{R}^3} (q_e f_e + q_i f_i) dv = n_e q_e + n_i q_i,$

Current: $j = \int_{\mathbb{R}^3} v (q_e f_e + q_i f_i) dv = n_e q_e u_e + n_i q_i u_i$

Kinetic model

Two species BGK model: $\alpha = e, i$

$$\partial_t f_\alpha + v \cdot \nabla_x f_\alpha + \frac{q_\alpha}{m_\alpha}(E + v \wedge B).\nabla_v f_\alpha = \frac{1}{\tau_\alpha}(\mathcal{M}_\alpha(f_\alpha) - f_\alpha) + \frac{1}{\tau_{ei}}(\overline{\mathcal{M}_\alpha}(f_e, f_i) - f_\alpha),$$

$\tau_\alpha > 0$, $\tau_{ei} > 0$, $\tau_{ie} > 0$, $\tau_{ei} = \tau_{ie}$. $\tau_{ei} \neq \tau_{ie}$ possible

$E(x,t)$: electric field, $B(x,t)$: magnetic field.

$$\begin{aligned}
\mathcal{M}_\alpha(f_\alpha) &= \frac{n_\alpha}{(2\pi k_B T_\alpha/m_\alpha)^{3/2}} exp(-\frac{|v-u_\alpha|^2}{2k_B T_\alpha/m_\alpha}) \\
\overline{\mathcal{M}_\alpha}(f_e, f_i) &= \frac{n_\alpha}{(2\pi k_B T/m_\alpha)^{3/2}} exp(-\frac{|v-u|^2}{2k_B T/m_\alpha})
\end{aligned}$$

Coupling with Maxwell's equations

$$\begin{cases} \cancel{-c^{-2}\partial_t E} + \mathrm{rot} B = \mu_0 j, \\ \partial_t B + \mathrm{rot} E = 0, \\ \cancel{\epsilon_0 \mathrm{div} E} = Q, \\ \mathrm{div} B = 0. \end{cases}$$

Quasi-neutrality: $Q = 0$: $Z = n_e / n_i = Cte$. If $\rho = \rho_e + \rho_i$ then

$$\rho_\alpha = c_\alpha \rho, \quad \alpha = e, i$$

with c_e and c_i constant.

The velocities depend on the current:

$$\begin{cases} u_e = u - \dfrac{m_i}{\rho e Z} j = u - \dfrac{m_i}{\rho q_i} j \\ u_i = u + \dfrac{m_e}{\rho e} j = u - \dfrac{m_e}{\rho q_e} j \end{cases}$$

Kinetic model

$$\begin{cases}\partial_t f_e + v\cdot\nabla_x f_e + \dfrac{q_e}{m_e}(E + v\wedge B)\cdot\nabla_v f_e = \dfrac{1}{\varepsilon}(\mathcal{M}_e - f_e) + \dfrac{1}{\tau_{ei}}(\overline{\mathcal{M}_e} - f_e),\\ \partial_t f_i + v\cdot\nabla_x f_i + \dfrac{q_i}{m_i}(E + v\wedge B)\cdot\nabla_v f_i = \dfrac{1}{\varepsilon}(\mathcal{M}_i - f_i) + \dfrac{1}{\tau_{ie}}(\overline{\mathcal{M}_i} - f_i)\end{cases}$$

with

$$\begin{cases}Q = 0, \quad \mathrm{rot}B = \mu_0 j\\ \partial_t B + \mathrm{rot}E = 0\\ \mathrm{div}B = 0\end{cases}$$

$\varepsilon \to 0$: $f_\alpha = \mathcal{M}_\alpha$.

Hydrodynamic limit

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \mathrm{div}\left(\rho u \otimes u + \frac{m_e m_i}{\rho e^2 Z} j \otimes j\right) + \nabla\left(p_e + p_i\right) + B \wedge j = 0$$

$$\partial_t \mathcal{E}_\alpha + \mathrm{div}(u_\alpha(\mathcal{E}_\alpha + p_\alpha)) - \frac{q_\alpha c_\alpha}{m_\alpha} \rho E \cdot u_\alpha = \nu_{1,\alpha\beta}(T_\beta - T_\alpha) + \nu_{2,\alpha\beta} j \cdot j + \nu_{3,\alpha\beta} j \cdot u,$$

$$\alpha = e, i.$$

Generalized Ohm's law: $\mu = Z\frac{m_e}{m_i}$, e: charge of the electron

$$\partial_t j + \mathrm{div}\left(u \otimes j + j \otimes u - \frac{m_i}{\rho Z e}(1-\mu) j \otimes j + \sum_\alpha \left(\frac{q_\alpha}{m_\alpha} p_\alpha\right) I\right)$$
$$+ \frac{1-\mu}{\mu}\frac{Ze}{m_i} j \wedge B + \rho \frac{q_e q_i}{m_e m_i}(E + u \wedge B)$$
$$= \frac{j}{\frac{\tau_{ie}}{c_i} + \frac{\tau_{ei}}{c_e}}(\mu - \frac{1}{\mu}).$$

Simplifications of Ohm's law

1. $B = 0 \Longrightarrow j = 0$ and

$$\rho E = \nabla \left(\frac{p_e m_i}{q_i} + \frac{p_i m_e}{q_e} \right)$$

   One obtains the bitemperature Euler model.

2. One keeps B and sets

$$\sum_\alpha \frac{q_\alpha}{m_\alpha} \nabla p_\alpha + \rho \frac{q_e q_i}{m_e m_i} (E + u \wedge B) = 0.$$

   1D Tranverse Magnetic field: same as [Brull, Dubroca, Lhébrard, 2021]

Here we continue with $B = 0$.

Entropy

Entropy

Macroscopic entropy function

$$\eta(\mathcal{U}) = \eta_e(\mathbf{U}_e(\mathcal{U})) + \eta_i(\mathbf{U}_i(\mathcal{U})), \quad Q(\mathcal{U}) = u\eta(\mathcal{U}).$$

$$\eta_\alpha(\rho_\alpha, \rho_\alpha u, \mathcal{E}_\alpha) = -\frac{\rho_\alpha}{m_\alpha(\gamma_\alpha - 1)}\left(\ln\left(\frac{(\gamma_\alpha - 1)\rho_\alpha\varepsilon_\alpha}{\rho_\beta^{\gamma_\alpha}}\right) + C\right), \quad C \geq 0$$

$$\mathbf{U}_\beta(\mathcal{U}) = (c_\alpha\rho, c_\alpha\rho u, \mathcal{E}_\alpha)$$

Boltzmann entropy function

$$\mathcal{H}(f_e, f_i) = \mathcal{H}_s(f_e) + \mathcal{H}_s(f_i), \quad \mathcal{H}_s(f) = \int_{\mathbb{R}^3} (f\, ln(f) - f) dv.$$

$$\mathcal{H}_s(\mathcal{M}_\alpha(f_\alpha)) = \eta_\alpha(\rho_\alpha, \rho_\alpha u, \mathcal{E}_\alpha)$$

Entropy dissipation

The entropy η is compatible with the Boltzmann entropy :

Theorem If $\mathcal{U}$ is a solution of the bitemperature Euler model which is the hydrodynamic limit of the kinetic model then

$$\partial_t \eta(\mathcal{U}) + \mathrm{div}_x \cdot Q(\mathcal{U}) \leq -\frac{\nu_{ei}}{k_B T_i T_e}(T_i - T_e)^2.$$

A solution is said to be admissible if it satisfies this inequality.
[Aregba-Driollet, Breil, Brull, Estibals, Dubroca, 2018]

Numerical scheme : discrete vectorial BGK model

1D Schemes

1. At the fluid level [Aregba-Driollet, Breil, Brull, Estibals,Dubroca, 2018]
    - Discretisation of the kinetic model and moment operator (K)
    - Pressure relaxation (Suliciu)
    - Lagrange-projection (LP)
    - Discrete BGK (BGKD)
2. C. Prigent's PhD Thesis: DVM from the "physical" kinetic model [Brull, Dubroca, Prigent, 2020] (DVM)

If there is no shock, all those schemes behave similarly

In the presence of shocks

All those schemes produce the same correct shock and contact discontinuity propagation speeds. Some incomplete Rankine-Hugoniot relations hold

$\Longrightarrow$ No problem with propagation speeds

(K), (BGKD), (DVM) product the same shocks

(BGKD) has a discrete entropy inequality

Vectorial BGK model

System of conservation laws

$$\partial_t U + \sum_{d=1}^{D} \partial_{x_d} F_d(U) = 0,$$

where $U(x,t) \in \Omega$, $\Omega \subset \mathbb{R}^K$, convex.

$$\partial_t f^\varepsilon + \sum_{d=1}^{D} \Lambda_d \partial_{x_d} f^\varepsilon = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right),$$

$f^\varepsilon = (f_1^\varepsilon, \ldots, f_L^\varepsilon)$, $\quad f^\varepsilon(x,t) \in (\mathbb{R}^K)^L$, $\Lambda_d = \mathrm{diag}\left(v_{d,1} \mathrm{I}_K, \ldots, v_{d,L} \mathrm{I}_K\right)$, $v_{d,l} \in \mathbb{R}$,
$P \in \mathcal{L}\left((\mathbb{R}^K)^L, \mathbb{R}^K\right)$, et $M = (M_1, \ldots, M_L) \colon \Omega \to (\mathbb{R}^K)^L$.

also reads as

$$\partial_t f_l^\varepsilon + \sum_{d=1}^{D} v_{d,l} \partial_{x_d} f_l^\varepsilon = \frac{1}{\varepsilon}\left(M_l(Pf^\varepsilon) - f_l^\varepsilon\right), \quad 1 \le l \le L.$$

Compatibility conditions

$$\forall U \in \Omega, \qquad P(M(U)) = U, \qquad P(\Lambda_d M(U)) = F_d(U), \qquad d = 1, \ldots, D.$$

Moment operator P

$$\partial_t(Pf^\varepsilon) + \sum_{d=1}^{D} \partial_{x_d} P(\Lambda_d f^\varepsilon) = 0.$$

If $f^\varepsilon \to f$ then $f = M(Pf)$

$U = Pf$ is a solution of the fluid model

How can we use this framework here?

We need an equivalent of the force term $E \cdot \nabla_v f$.
1D first order: [Aregba-Driollet, Breil, Brull, Dubroca,Estibals, 2018]

Choice of a vectorial BGK model for conservative Euler model

Dimension 2 and 4 velocities. $D = 2$, $L = 4$.
Definition of P

$$\forall f \in (\mathbb{R}^4)^4, \quad Pf = U = \sum_{l=1}^{4} f_l \, .$$

$$\begin{cases} \partial_t f_1 + \lambda_1^- \partial_{x_1} f_1 = \dfrac{1}{\varepsilon}(M_1(U) - f_1) \\ \partial_t f_2 + \lambda_2^- \partial_{x_2} f_2 = \dfrac{1}{\varepsilon}(M_2(U) - f_2) \\ \partial_t f_3 + \lambda_1^+ \partial_{x_1} f_3 = \dfrac{1}{\varepsilon}(M_3(U) - f_3) \\ \partial_t f_4 + \lambda_2^+ \partial_{x_2} f_4 = \dfrac{1}{\varepsilon}(M_4(U) - f_4) \end{cases}$$

Maxwellian functions

$$M(U) = \begin{pmatrix} \dfrac{1}{\lambda_1^+ - \lambda_1^-}\left(\dfrac{\lambda_1^+}{2}U - F_1(U)\right) \\ \dfrac{1}{\lambda_2^+ - \lambda_2^-}\left(\dfrac{\lambda_2^+}{2}U - F_2(U)\right) \\ \dfrac{1}{\lambda_1^+ - \lambda_1^-}\left(\dfrac{-\lambda_1^-}{2}U + F_1(U)\right) \\ \dfrac{1}{\lambda_2^+ - \lambda_2^-}\left(-\dfrac{\lambda_2^-}{2}U + F_2(U)\right) \end{pmatrix}.$$

Subcharacteristic condition:

$$\forall U \in \Omega,\ \sigma(F_d'(U)) \subset \left]\frac{\lambda_d^-}{2}, \frac{\lambda_d^+}{2}\right[,\ d = 1,2 \iff \forall U \in \Omega,\ \forall l\ \sigma(M_l'(U)) \subset]0,+\infty[.$$

BGK model for the bitemperature equations

For $f^\alpha \in (\mathbb{R}^4)^4$: $Pf^\alpha = U^\alpha = (\rho^\alpha, \rho^\alpha u^\alpha, \mathcal{E}^\alpha)$

$$\begin{cases} \partial_t f_l^{e,\varepsilon} + \sum_{d=1}^{2} v_{d,l} \partial_{x_d} f_l^{e,\varepsilon} + \frac{q^e}{m^e} N(E^\varepsilon) f_l^{e,\varepsilon} = \frac{1}{\varepsilon}\left(M_l^e(U^{e,\varepsilon}) - f_l^{e,\varepsilon}\right) + B_l^{ei}(f^{e,\varepsilon}, f^{i,\varepsilon}), \\ \partial_t f_l^{i,\varepsilon} + \sum_{d=1}^{2} v_{d,l} \partial_{x_d} f_l^{i,\varepsilon} + \frac{q^i}{m^i} N(E^\varepsilon) f_l^{i,\varepsilon} = \frac{1}{\varepsilon}\left(M_l^i(U^{i,\varepsilon}) - f_l^{i,\varepsilon}\right) + B_l^{ie}(f^{e,\varepsilon}, f^{i,\varepsilon}), \\ \cancel{-c^{-2}\partial_t E^\varepsilon} = \mu_0 \left(\frac{q^e}{m^e}\rho^{e,\varepsilon} u^{e,\varepsilon} + \frac{q^i}{m^i}\rho^{i,\varepsilon} u^{i,\varepsilon}\right), \\ \cancel{\epsilon_0 \mathrm{div} E^\varepsilon} = \frac{q^e}{m^e}\rho^{e,\varepsilon} + \frac{q^i}{m^i}\rho^{i,\varepsilon} \\ \mathrm{rot} E^\varepsilon = 0 \end{cases}$$

$B^{\alpha\beta}$: source term $\Rightarrow$ interactions ions-electrons

$$PB^{\alpha\beta} \Rightarrow (0, 0, 0, \nu^{\alpha\beta}(T^\beta - T^\alpha)).$$

Quasi-neutral limit

$$\rho^e = c^e \rho, \quad \rho^i = c^i \rho, \quad u = u^e = u^i$$

Hydrodynamic limit

Force term:

$$\forall g \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \quad N(E)g = -(0, g_1 E, g_2 \cdot E)$$

hence if $U^\alpha = (\rho^\alpha, \rho^\alpha u, \mathcal{E}^\alpha)$ then

$$N(E)U^\alpha = -(0, \rho^\alpha E, \rho^\alpha u \cdot E)$$

$$\begin{aligned}
\partial_t \rho^\alpha + \nabla \cdot (\rho^\alpha u) &= 0, \\
\partial_t (\rho^\alpha u) + \nabla \cdot (\rho^\alpha u \otimes u) + \nabla p^\alpha - \frac{q^\alpha}{m^\alpha} E \rho^\alpha &= 0, \\
\partial_t \mathcal{E}^e + \nabla \cdot (u(\mathcal{E}^e + p^e)) - q^e m^e \rho^e u \cdot E &= \nu^{ei}(T^i - T^e), \\
\partial_t \mathcal{E}^i + \nabla \cdot (u(\mathcal{E}^i + p^i)) - q^i m^i \rho^i u \cdot E &= -\nu^{ei}(T^i - T^e).
\end{aligned}$$

Ohm's law

$$\frac{\rho^i q^i}{m^i} E = -\frac{\rho^e q^e}{m^e} E = -c^i \nabla p^e + c^e \nabla p^i.$$

Kinetic entropies

The maxwellian functions are of form

$$M_l^\alpha(U^\alpha) = \theta_l U^\alpha + \zeta \cdot F^\alpha(U^\alpha), \qquad 1 \leq l \leq 4, \qquad \alpha = e, i,$$

with $\theta_l \in \mathbb{R}$ and $\zeta \in \mathbb{R}^2$. Let (η^α, Q^α) be an entropy-entropy flux pair and set :

$$G_l^\alpha(U) = \theta_l \eta^\alpha(U) + \zeta_l \cdot Q^\alpha(U).$$

If the subcharacteristic condition is satisfied then the kinetic entropies are

$$H_l^\alpha(f_l^\alpha) = G_l^\alpha((M_l^\alpha)^{-1}(f_l^\alpha)).$$

- H_l^α is convex. (E0)
- $\sum_{l=1}^{4} H_l^\alpha(M_l^\alpha(U^\alpha)) = \eta^\alpha(U^\alpha).$ (E1)
- $\sum_{l=1}^{4} V_l H_l^\alpha(M_l^\alpha(U^\alpha)) = Q^\alpha(U^\alpha).$ (E2)
- si $U_f = P(f)$, $\sum_{l=1}^{4} H_l^\alpha(M_l^\alpha(U_f)) \leq \sum_{l=1}^{4} H_l^\alpha(f_l).$ (E3)

Admissibility

If $\mathcal{U}$ is a solution of the Euler bitemperature equation obtained by hydrodynamic limit of the discrete BGK model then it is an admissible solution:

$$\partial_t \eta(\mathcal{U}) + \mathrm{div}_x \cdot Q(\mathcal{U}) \leq -\frac{\nu_{ei}}{k_B T_i T_e}(T_i - T_e)^2.$$

Ingredients of the proof:

- Multiply the equation on f_l^α by $H_l^{\alpha\prime}(f_l^\alpha)$
- Use the fact that $H_l^{\alpha\prime}(f_l^\alpha) N(E) f_l^\alpha = 0$
- Use convexity and minimization property
- Pass to the limit

Obtention of the first order scheme

The scheme

Δx_1 and Δx_2: space steps, Δt: time step, $j = (j_1, j_2) \in \mathbb{Z}^2$.
For all unknown, $v(x_1, x_2, t)$, v_j^n: approximation at time t^n on cell $C_j =]x_{1,j_1-\frac{1}{2}}, x_{1,j_1+\frac{1}{2}}[\times]x_{2,j_2-\frac{1}{2}}, x_{2,j_2+\frac{1}{2}}[$. Suppose $\mathcal{U}_j^n = (\rho_j^n, \rho_j^n u_j^n, \mathcal{E}_{e,j}^n, \mathcal{E}_{i,j}^n)$ is known.

Step 1: Definition of $f_j^{\alpha,n}$ as

$$U_j^{\alpha,n} = (c^\alpha \rho_j^n, c^\alpha \rho_j^n u_j^n, \mathcal{E}_j^{\alpha,n}), \quad f_j^{\alpha,n} = M^\alpha(U_j^{\alpha,n}), \qquad j \in \mathbb{Z}^2, \qquad \alpha = e, i.$$

Step 2: Resolution of transport equations by upwind scheme

$$\partial_t f^\alpha + \sum_{d=1}^{2} \Lambda_d \partial_{x_d} f^\alpha = 0$$

One obtains for each species α:

$$U^{\alpha,n+\frac{1}{2}} = P f_j^{\alpha,n+\frac{1}{2}} = U_j^{\alpha,n} - \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d} \left(F_{j+e_d/2}^{\alpha,n} - F_{j-e_d/2}^{\alpha,n} \right)$$

Force and source terms

Step 3: Implicit scheme

$$f_{j,l}^{\alpha,n+\frac{3}{4}} = f_{j,l}^{\alpha,n+\frac{1}{2}} - \Delta t \frac{q^\alpha}{m^\alpha} N(E_j^{n+1}) f_{j,l}^{\alpha,n+1} + \Delta t B_l^{\alpha\beta}(f_j^{\alpha,n+1}, f_j^{\cdot,n+1}), \quad 1 \leq l \leq 4$$

and

$$U_j^{\alpha,n+1} = P(f_j^{\alpha,n+\frac{3}{4}}).$$

Quasineutrality:

$$\rho_j^{\alpha,n+1} = c_\alpha \rho_j^{n+1}, \quad \alpha = e, i$$

Equations on $\rho u^\alpha \implies$ Discrete Ohm's law

$$u \cdot \nabla_x (c_e p_i - c_i p_e) \iff u_j^{n+1} \cdot \sum_{d=1}^{2} \frac{1}{\Delta x_d} \left(\delta_{j+\frac{e_d}{2}}^n - \delta_{j-\frac{e_d}{2}}^n \right)$$

where

$$\delta_{j+\frac{e_d}{2}}^n = -c^i F_{j+\frac{e_d}{2},2}^{e,n} + c^e F_{j+\frac{e_d}{2},2}^{i,n} \in \mathbb{R}^2.$$

Approximation of nonconservative terms

$$\delta^n_{j+\frac{e_d}{2}} = \left| \begin{array}{l} \left(-c_i p^{e,n}_{j+e_d} + c_e p^{i,n}_{j+e_d}\right) e_d \;\; \text{si} \quad \lambda_d^- < \lambda_d^+ \leq 0, \\ \left(-c_i p^{e,n}_{j} + c_e p^{i,n}_{j}\right) e_d \; \text{si} \quad 0 \leq \lambda_d^- < \lambda_d^+, \\ \left(\dfrac{\lambda_d^+}{\lambda_d^+ - \lambda_d^-}(-c^i p^{e,n}_j + c^e p^{i,n}_j) - \dfrac{\lambda_d^-}{\lambda_d^+ - \lambda_d^-}(-c^i p^{e,n}_{j+e_d} + c^e p^{i,n}_{j+e_d})\right) e_d \\ \hfill \text{if} \quad \lambda_d^- < 0 < \lambda_d^+ \,. \end{array} \right.$$

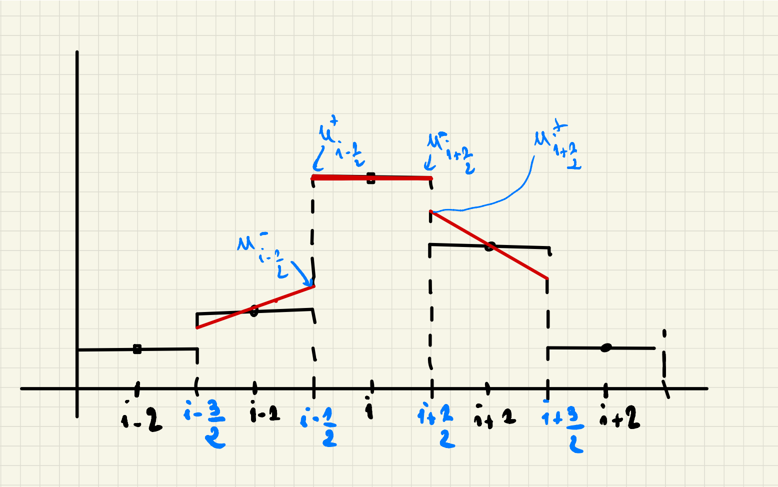
Consistent with $c^e p^i - c^i p^e$.

Properties

1. If the subcharacteristic and CFL conditions are satisfied then a discrete entropy inequality holds.
2. If $\gamma_e = \gamma_i$ then $(\rho_j^n, \rho_j^n u_j^n, \mathcal{E}_j^{e,n} + \mathcal{E}_j^{i,n})$ is solution of HLL scheme.
   Consequence:
$$\rho > 0, \quad p_e + p_i > 0.$$

Second order scheme

Affine reconstruction in 1D



1D case for conservation laws

$$\partial_t U + \partial_x F(U) = 0.$$

Starting point: first order scheme

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

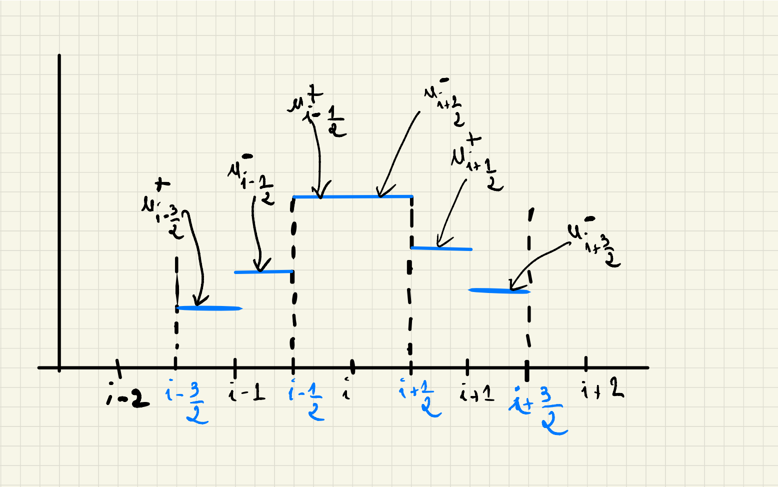
Step 1: affine reconstruction

$$\forall x \in C_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[, \qquad U^n(x) = U_j^n + \sigma_j^n(x - x_j), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}).$$

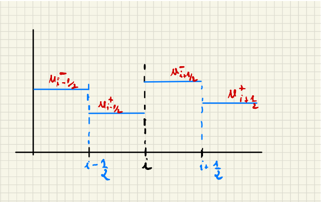
Step 2: Interface values

$$U_{j+\frac{1}{2}}^+ = (U^n(x_{j+\frac{1}{2}}))^+ = U_{j+1}^n - \sigma_{j+1}^n \frac{\Delta x}{2}, \qquad U_{j+\frac{1}{2}}^- = (U^n(x_{j+\frac{1}{2}}))^- = U_j^n + \sigma_j^n \frac{\Delta x}{2}.$$

1D Affine reconstruction



The scheme for conservation laws



$$U_i^{n+1,-} = U^+_{i-\frac{1}{2}} - \frac{2\Delta t}{\Delta x}\Big(\mathcal{F}(U^+_{i-\frac{1}{2}}, U^-_{i+\frac{1}{2}}) - \mathcal{F}(U^-_{i-\frac{1}{2}}, U^+_{i-\frac{1}{2}})\Big)$$

$$U_i^{n+1,+} = U^-_{i+\frac{1}{2}} - \frac{2\Delta t}{\Delta x}\Big(\mathcal{F}(U^-_{i+\frac{1}{2}}, U^+_{i+\frac{1}{2}}) - \mathcal{F}(U^+_{i-\frac{1}{2}}, U^-_{i+\frac{1}{2}})\Big).$$

Finally

$$U_j^{n+1} = \frac{1}{2}\Big(U_j^{n+1,-} + U_j^{n+1,+}\Big) = U_i^n - \frac{\Delta t}{\Delta x}\Big(\mathcal{F}(U^-_{i+\frac{1}{2}}, U^+_{i+\frac{1}{2}}) - \mathcal{F}(U^-_{i-\frac{1}{2}}, U^+_{i-\frac{1}{2}})\Big)$$

The implementation does not necessitate the fluxes at cell centers

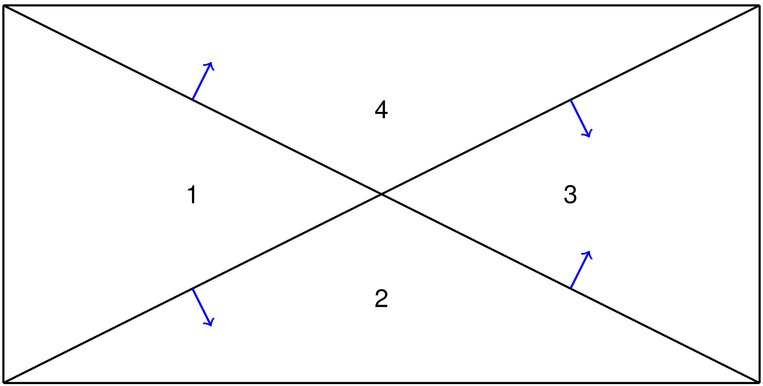
Multi-D conservative case

The 1D procedure can be generalized by subdividing each cell into triangles.

[Perthame, Shu 1996], [Bouchut 2004]: Positivity and, partially, entropy properties are preserved.

Limitation procedure: [Perthame, Qiu 1994], [Berthon 2006], [Calgaro, Creusé, Goudon, Penel 2013]

Subcells in the cartesian case



Implementation in the nonconservative case

$(\mathcal{U}_j^n)_j$ approximate solution at t^n. Reconstruction :

$$\forall x \in C_j, \qquad \mathcal{U}(x) = \mathcal{U}_j^n + (x - x_j) \cdot \sigma_j^n.$$

4 constant states:

$$\mathcal{U}_j^{(1)} = \mathcal{U}_j^n - \frac{\Delta x_1}{2}\sigma_{1,j}^n, \qquad \mathcal{U}_j^{(2)} = \mathcal{U}_j^n - \frac{\Delta x_2}{2}\sigma_{2,j}^n,$$
$$\mathcal{U}_j^{(3)} = \mathcal{U}_j^n + \frac{\Delta x_1}{2}\sigma_{1,j}^n, \qquad \mathcal{U}_j^{(4)} = \mathcal{U}_j^n + \frac{\Delta x_2}{2}\sigma_{2,j}^n.$$

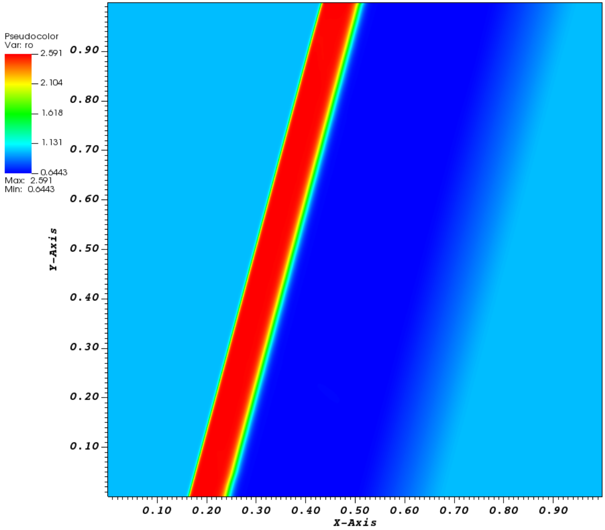
The first order scheme is applied on each triangle. All is explicit (upwind).

Practical difference with the conservative case: no simplification.

Numerical results

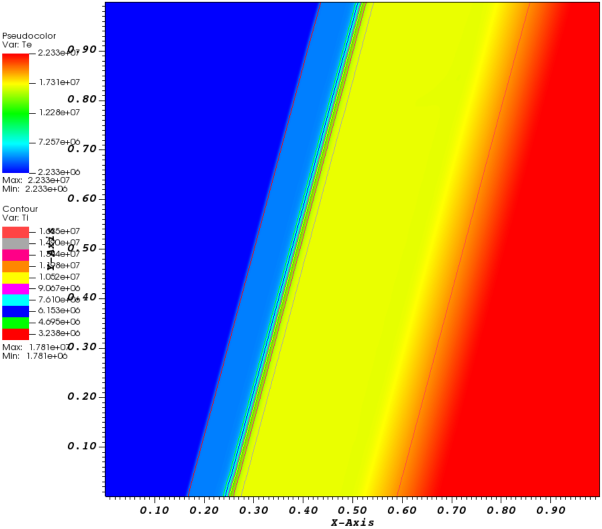
Oblique Sod test case: total density

$\nu^{ei} = 4 \times 10^9$. 800×800 grid



Oblique Sod test case: temperatures

$\nu^{ei} = 4 \times 10^9$, 800 by 800 points



1D: first order vs second order

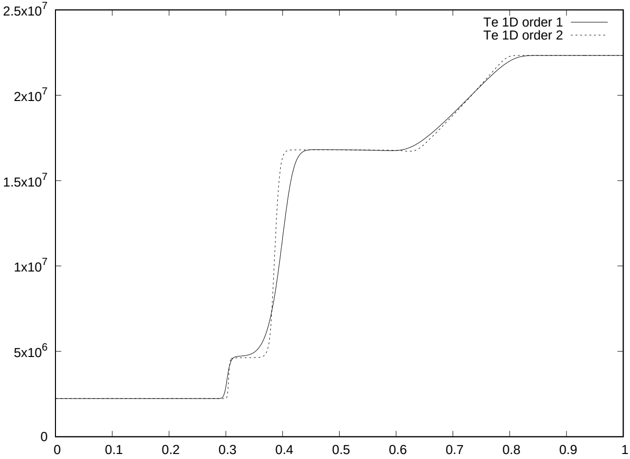


Figure: electronic temperature

Comparison 1D/2D

Sod test case avec $\nu^{ei} = 4 \times 10^9$, 800×800 grid.

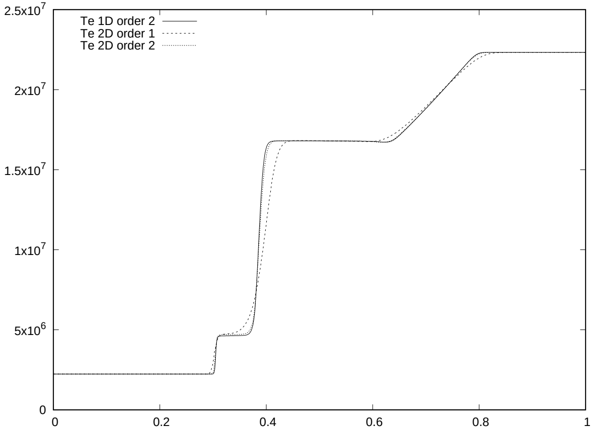


Figure: Left: electronic temperature, droite: ionic temperature

4 interfaces Riemann problem

$[0,1]\times[0,1]$, partitionés en 4 quadrants de tailles identiques

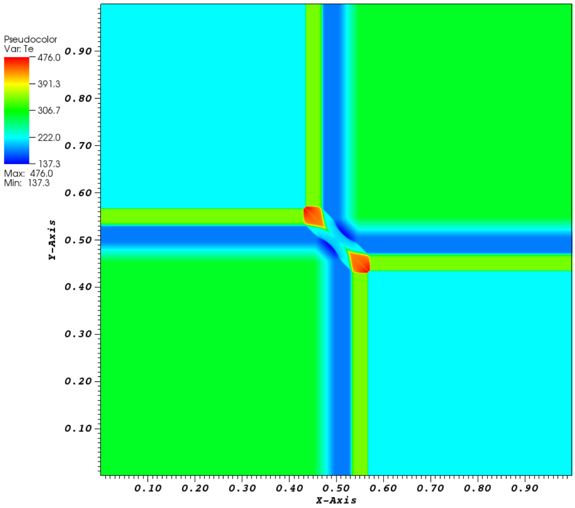
$$\begin{aligned}
\rho(x_1,x_2,0) &= 1\ \text{kg.m}^{-3},\ \text{si } x_1 < 0.5 \text{ and } x_2 < 0.5,\\
\rho(x_1,x_2,0) &= 0.125\ \text{kg.m}^{-3},\ \text{si } x_1 < 0.5 \text{ and } x_2 > 0.5,\\
\rho(x_1,x_2,0) &= 0.125\ \text{kg.m}^{-3},\ \text{si } x_1 > 0.5 \text{ and } x_2 < 0.5,\\
\rho(x_1,x_2,0) &= 1\ \text{kg.m}^{-3},\ \text{si } x_1 > 0.5 \text{ and } x_2 > 0.5,
\end{aligned}$$

Températures électronique and ioniques:

$$\begin{gathered}
T^e(x_1,x_2,0) = 293\ \text{K}\ ,\ T^i(x_1,x_2,0) = 273\ \text{K}\ ,\ \text{si } x_1 < 0.5 \text{ and } x_2 < 0.5,\\
T^e(x_1,x_2,0) = 220\ \text{K}\ ,\ T^i(x_1,x_2,0) = 200\ \text{K}\ ,\ \text{si } x_1 < 0.5 \text{ and } x_2 > 0.5,\\
T^e(x_1,x_2,0) = 220\ \text{K}\ ,\ T^i(x_1,x_2,0) = 200\ \text{K}\ ,\ \text{si } x_1 > 0.5 \text{ and } x_2 < 0.5,\\
T^e(x_1,x_2,0) = 293\ \text{K}\ ,\ T^i(x_1,x_2,0) = 273\ \text{K}\ ,\ \text{si } x_1 > 0.5 \text{ and } x_2 > 0.5,
\end{gathered}$$

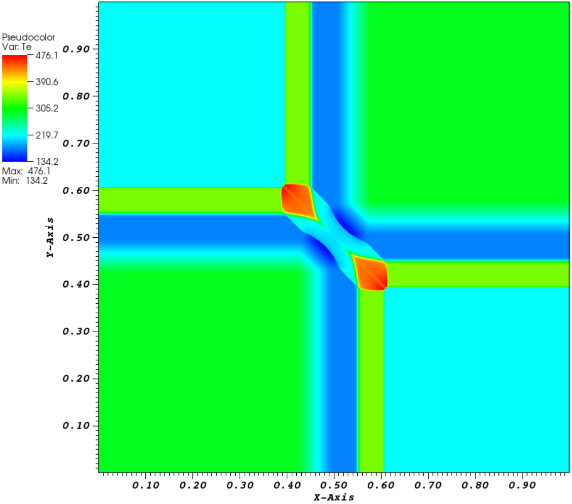
4 interfaces Riemann problem: electronic temperature

$\nu^{ei} = 100\ \mathrm{s}^{-1}$, 2000×2000 grid



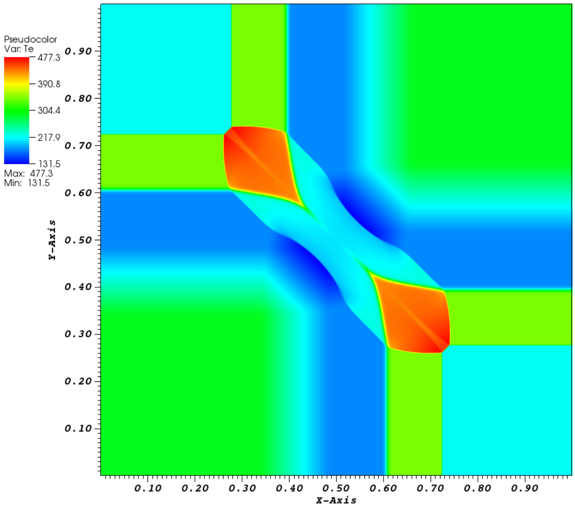
4 interfaces Riemann problem: electronic temperature

$\nu^{ei} = 100\ \mathrm{s}^{-1}$, 2000×2000 grid



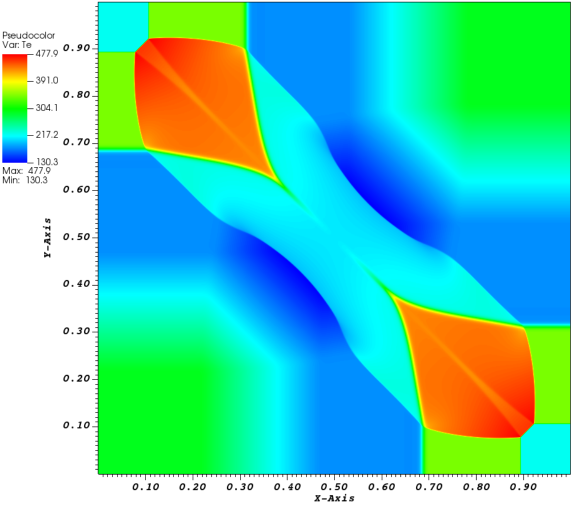
4 interfaces Riemann problem: electronic temperature

$\nu^{ei} = 100\ \mathrm{s}^{-1}$, 2000×2000 grid



4 interfaces Riemann problem: electronic temperature

$\nu^{ei} = 100\ \mathrm{s}^{-1}$, 2000×2000 grid



Implosion problem

Riemann initial condition: $\rho = 1$ kg.m^{-3}, $u = 0$ m.s^{-1}

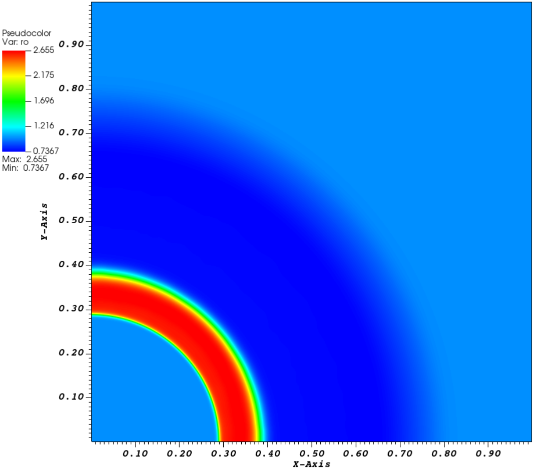
Temperatures:

$$T^e(x_1, x_2, 0) = 2,3 \times 10^6 K, \quad T^i(x_1, x_2, 0) = 1.7406 \times 10^6 K \quad \text{if } (x_1)^2 + (x_2)^2 < \frac{1}{4},$$
$$T^e(x_1, x_2, 0) = 2,3 \times 10^7 K, \quad T^i(x_1, x_2, 0) = 1.7406 \times 10^7 K \quad \text{sinon}$$

Final simulation time: $t = 4.0901 \times 10^{-7}$ s. ν^{ei} is physically given (NRL formulary)

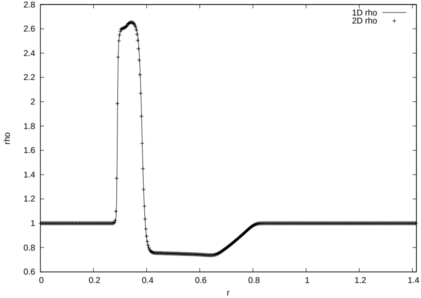
Total density, 500×500 grid

,



Comparison with a 1D computation

Total density and velocity along the first bissector



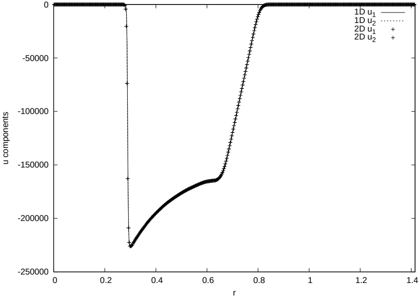
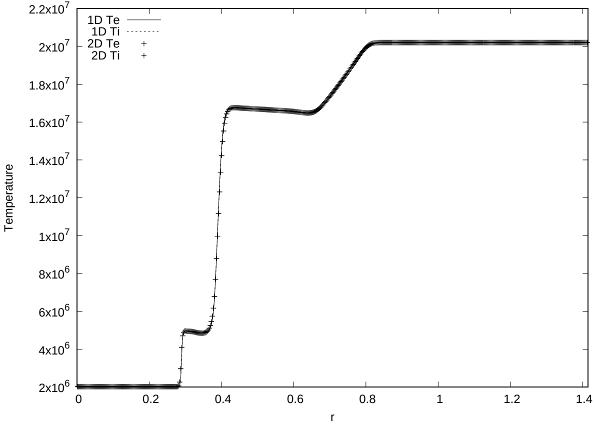


Figure: Total density (left) and velocity (right)

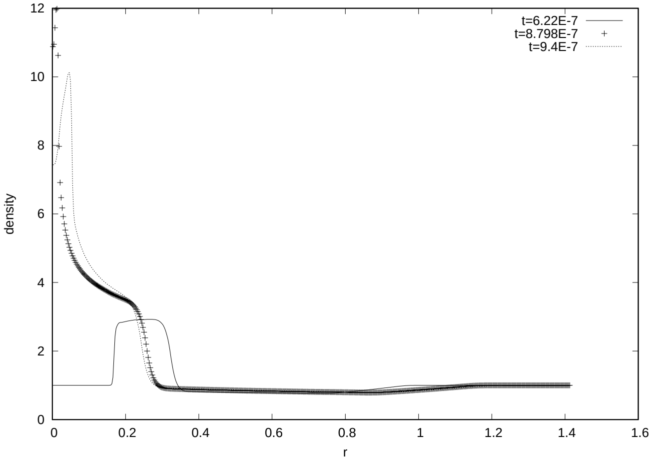
Comparison with a 1D computation

Electronic and ionic temperatures

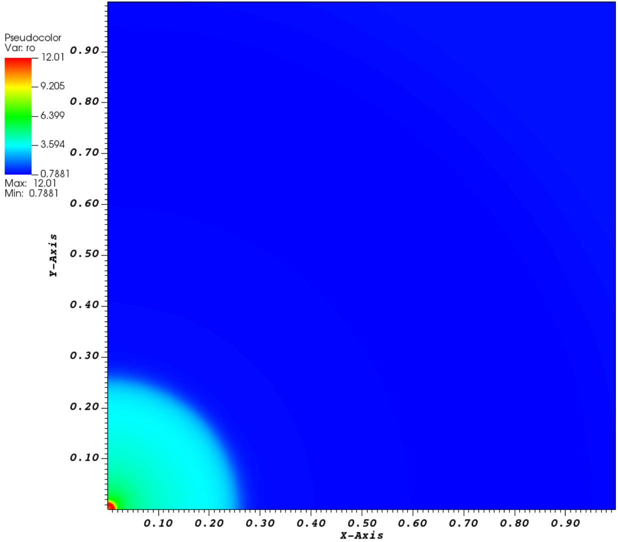


Comparison with a 1D computation

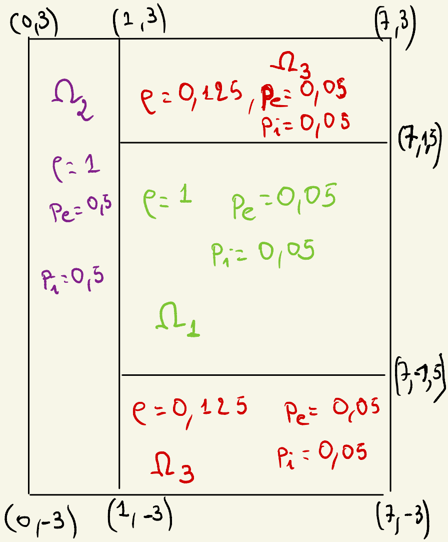
Three successive times. Peak: $t = 8.798 \times 10^{-7}$sec.



Density when peak occurs

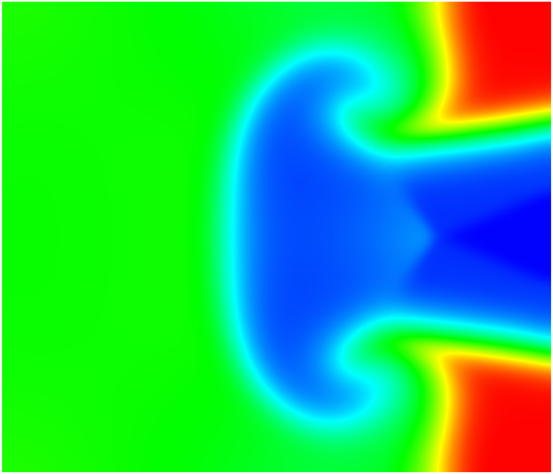


Triple point



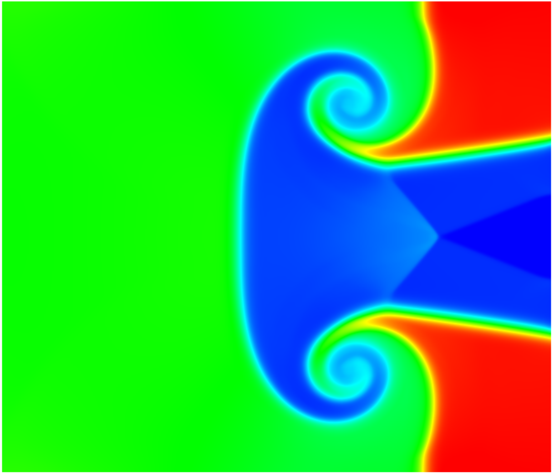
Triple point order 1

Electronic temperature: 500×500 points, $\nu_{ei} = 0$



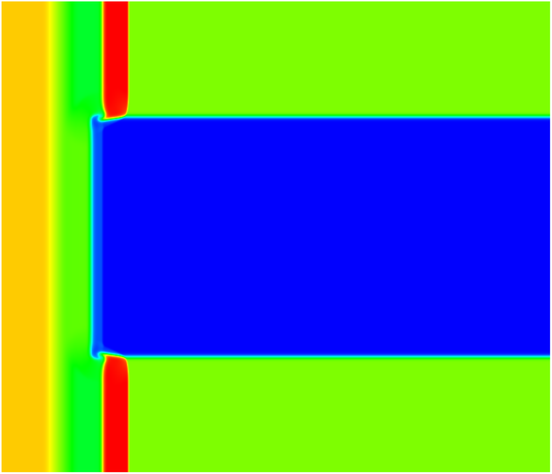
Triple point order 2

Electronic temperature: 500×500 points, $\nu_{ei} = 0$



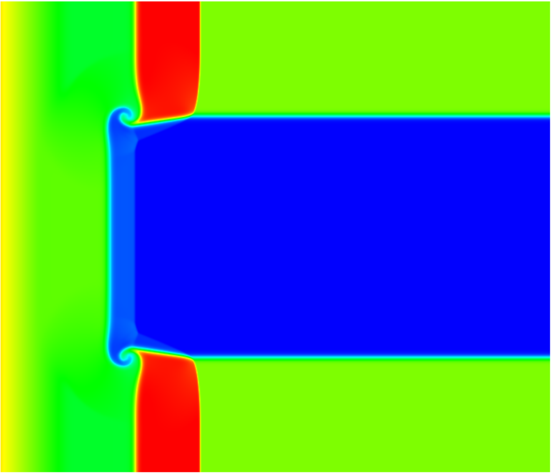
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



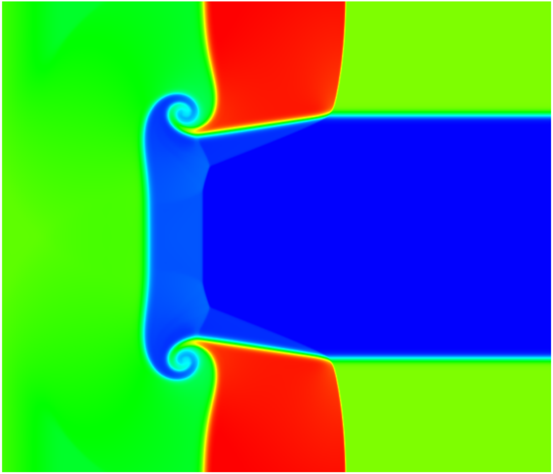
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



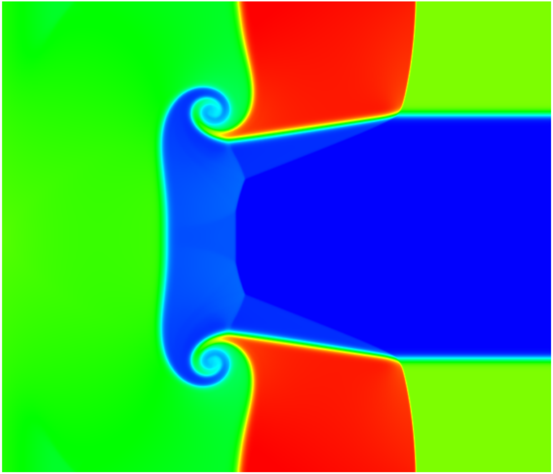
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



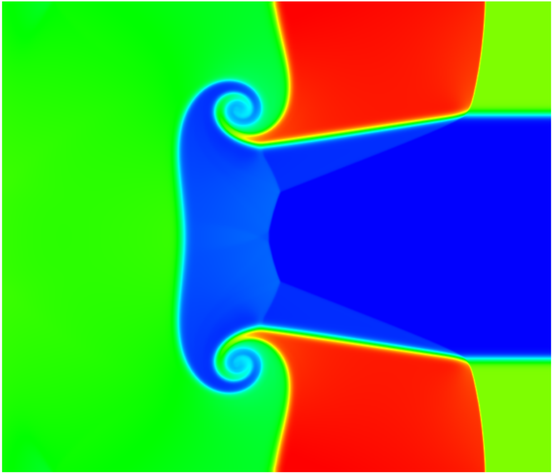
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



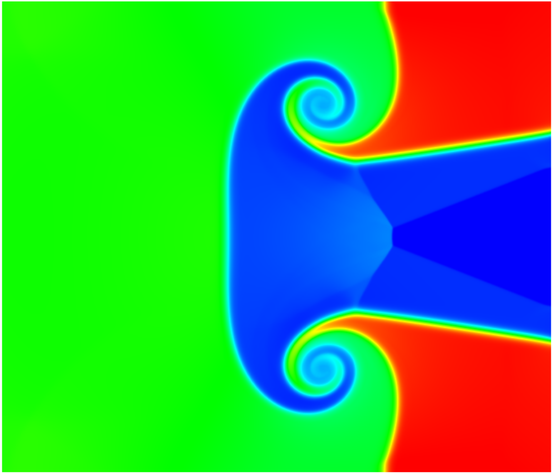
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



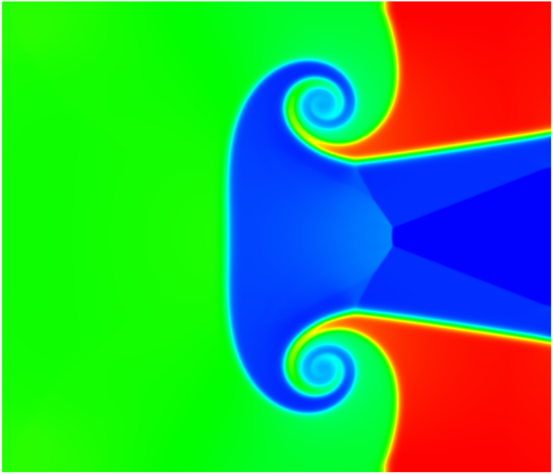
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



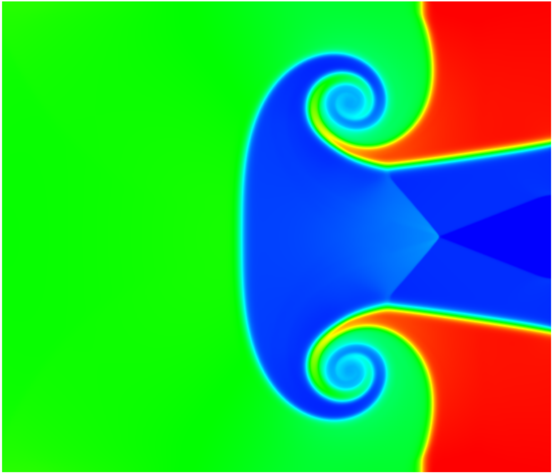
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



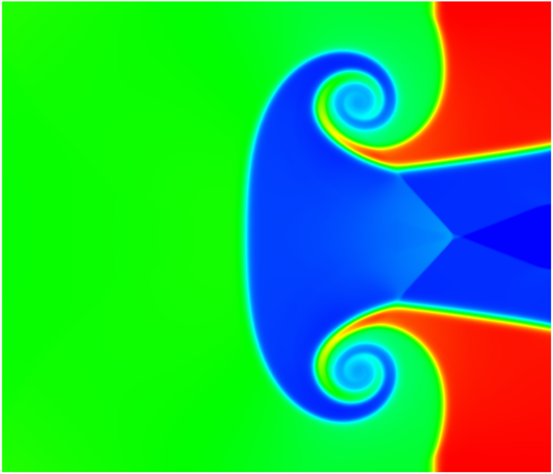
Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



Triple point

Electronic temperature: 1000×1000 points, $\nu_{ei} = 0$



Conclusions

Conclusions

- 2D discrete BGK scheme on a nonconservative Euler model for plasmas
- Second order extension

Perspectives

- Other BGK models (FDM)
- Magnetic fields: discrete BGK models
- Comparison with [Estibals, Guillard, Sangam]

Thank you for your attention