Shakhov model near a global Maxwellian

Gi-Chan Bae (Joint work with Seok-Bae Yun)

Department of Mathematical Sciences
Seoul National University, KOREA

gcbae02@snu.ac.kr

August 23, 2021

Outline

1. Boltzmann equation
   - Boltzmann - BGK model
2. Shakhov model
   - Prandtl number
   - Shakhov model
3. The Shakhov model near a global Maxwellian
   - Linearization of the Shakhov model
   - Main theorem

Boltzmann equation

What is the kinetic theory

• What is kinetic theory?
$\rightarrow$ Modelling of a gas or plasma.
$\rightarrow$ Modelling of system made up of a large number of particles

What is Boltzmann equation?

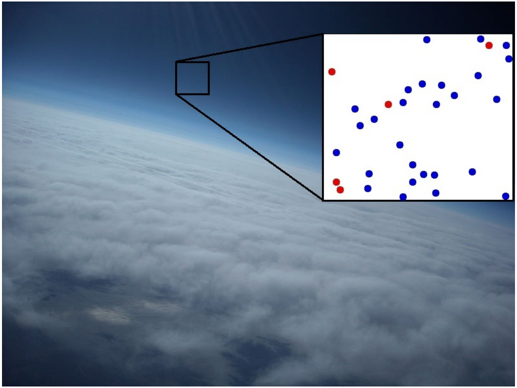


Figure: https://www.flickr.com/photos/23468143@N08/3332216506, https://en.wikipedia.org/wiki/Elasticcollision

Boltzmann Equation (Ludwig Boltzmann (1872))

Non-ionized monatomic gas

- Transport + collision → Boltzmann equation!

$$\partial_t F + \underbrace{v \cdot \nabla_x F}_{transport} = \underbrace{Q(F,F)}_{collision}.$$

$F(x,v,t)$: velocity distribution function in phase space $(x,v) \in (\mathbb{T}^3 \times \mathbb{R}^3)$ and $t \in \mathbb{R}_+$.

Construction of collision opeartor

Collision operator is given by

$$Q(F,F) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |v - v_*|)(F(v_*')F(v') - F(v_*)F(v))d\omega dv_*,$$

$$v' = v - [w\cdot(v - v_*)]w, \qquad v_*' = v_* + [w\cdot(v - v_*)]w$$

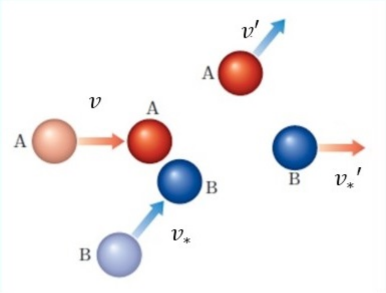
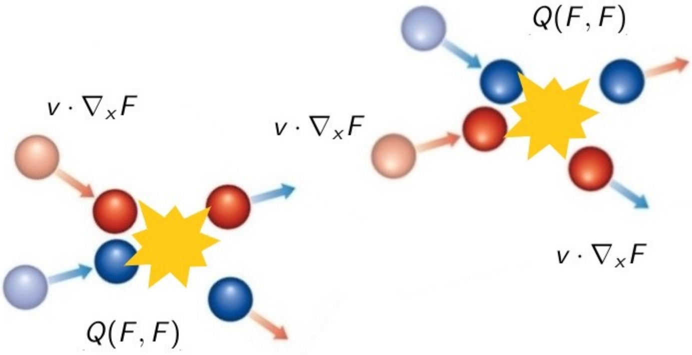


Figure: http://yjh-phys.tistory.com/1385

The Boltzmann equation

- Transport + collision → Boltzmann equation!

$$\partial_t F + \underbrace{v \cdot \nabla_x F}_{transport} = \underbrace{Q(F, F)}_{collision}.$$



Conservation laws and H-theorem

- Conservation law

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3} F\left(1, v, |v|^2\right) dvdx = 0.$$

- H-Theorem

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3} F \ln F dvdx \leq 0.$$

- Local equilibrium

$$\mathcal{M}(F) = \frac{\rho}{\sqrt{2\pi T}^3} e^{-\frac{|v-U|^2}{2T}}.$$

Boltzmann BGK model (Bhatnagar-Gross-Krook (1954))

Relaxation operator $Q(F,F) \to \mathcal{M}(F) - F$

$$\partial_t F + v \cdot \nabla_x F = \mathcal{M}(F) - F,$$

where $\mathcal{M}(F)$ is given by

$$\mathcal{M}(F) = \frac{\rho(x,t)}{\sqrt{2\pi T(x,t)}^3} \exp\left(-\frac{|v-U(x,t)|^2}{2T(x,t)}\right).$$

The macroscopic fields are defined by

$$\rho(x,t) = \int_{\mathbb{R}^3} F(x,v,t)dv,$$
$$\rho(x,t)U(x,t) = \int_{\mathbb{R}^3} F(x,v,t)vdv,$$
$$3\rho(x,t)T(x,t) = \int_{\mathbb{R}^3} F(x,v,t)|v-U(x,t)|^2 dv.$$

Effect of BGK operator

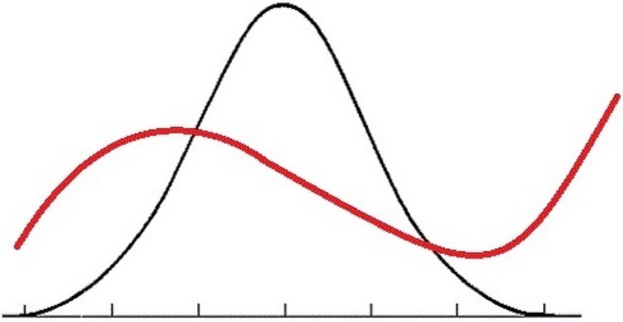


Figure: Operation of the BGK operator

Conservation laws and H-theorem

- Conservation law

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} F\left(1, v, |v|^2\right) dvdx = 0.$$

- H-Theorem

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} F\ln F dvdx = \frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3}(\mathcal{M} - F)\ln F dvdx \leq 0.$$

- Local equilibrium

$$\mathcal{M}(F) = \frac{\rho}{\sqrt{2\pi T}^3} e^{-\frac{|v-U|^2}{2T}}.$$

Shakhov model

The Prandtl number

- Prandtl number of the BGK model :

$$Pr = \frac{viscosity}{thermal\ conductivity} = 1.$$

However, in the case of monatomic gas, the Prandtl number is $2/3$.

Various types of the Prandtl numbers

- Molten potassium at 975 K, $Pr \approx 0.003$.
- Mercury, $Pr \approx 0.015$.
- Oxygen, $Pr \approx 0.63$.
- Monatomic gas (He, Ne, Ar,$\cdots$), $Pr \approx 0.67$.
- Air, $Pr \approx 0.71$.
- Water (At 18 degrees Celsius), $Pr \approx 7.56$.
- Engine oil, $Pr \approx 100\ 40,000$.
- Earth's mantle, $Pr \approx 10^{25}$.

To get the correct Prandtl number

- ES-BGK model (1965 Holway)
- Shakhov model (1968 Shakhov)
- Liu model (1990 Liu)
- BGK model with velocity-dependent collision frequency (1997 Struchtrup)

The Shakhov model

Shakhov model [1968 Shakhov, E. M.]

Shakhov model is obtained by modifying the heat flux:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{S}(F) - F,$$

where

$$\mathcal{S}(F) = \frac{\rho}{\sqrt{2\pi T}^3} \exp\left(-\frac{|v-U|^2}{2T}\right) \times \left[1 + \frac{1-Pr}{5} \frac{q \cdot (v-U)}{\rho T^2} \left(\frac{|v-U|^2}{2T} - \frac{5}{2}\right)\right],$$

$$q(x,t) = \int_{\mathbb{R}^3} F(x,v,t)(v - U(x,t))|v - U(x,t)|^2 dv.$$

- Substituting $Pr = 1$ gives the BGK model.

Defects of the Shakhov model

- The Shakohov model does not gurantee the non-negativity of the Shakhov operator $\mathcal{S}(F)$, and the solution F. Recall that

$$\mathcal{S}(F) = \frac{\rho}{\sqrt{2\pi T}^3} \exp\left(-\frac{|v-U|^2}{2T}\right) \times \left[1 + \frac{1-Pr}{5} \frac{q\cdot(v-U)}{\rho T^2}\left(\frac{|v-U|^2}{2T} - \frac{5}{2}\right)\right].$$

- H-theorem is only guaranteed near a local Maxwellian.

$$\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} F \ln F dv dx = \int_{\mathbb{T}^3\times\mathbb{R}^3} (\mathcal{S}(F) - F) \ln F dv dx \leq 0.$$

Numerical point of view

- Numerical point of view
  1. (2015 Chen et al.) In most case and under tough conditions such as the shock structure, the Shakhov model works better than the ES-BGK model.
  2. (2019 Zhang et al.) The Shakhov model can capture the velocity slip and the temperature jump near the wall more accurately.

The Shakhov model near a global Maxwellian

Linearization of the Shakhov model

- We substitute $F = m + \sqrt{m}f$ on the Shakhov model where

$$m(v) = \frac{1}{\sqrt{2\pi}^3} e^{-\frac{|v|^2}{2}},$$

then we have

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\tau}(P_c f + (1 - Pr)P_{nc} f - f + \Gamma(f, f)),$$

where $P_c + P_{nc}$ is L^2_v projection on

$$\left\{\sqrt{m}, v\sqrt{m}, |v|^2\sqrt{m}, v|v|^2\sqrt{m}\right\}.$$

Linearization of the Boltzmann-BGK model

• BGK model [2010 Yun, S.-B.]: If we substitute $F = m + \sqrt{m}f$ on the Boltzmann equation or BGK model, then we have

$$\partial_t f + v \cdot \nabla_x f = P_c f - f + \Gamma(f, f),$$

where P_c is L^2_v projection on

$$\left\{\sqrt{m}, v\sqrt{m}, |v|^2\sqrt{m}\right\}.$$

So that,

$$\langle Lf, f\rangle_{L^2_{x,v}} = -\|(I - P_c)f\|^2_{L^2_{x,v}},$$
$$KerL = span\{\sqrt{m}, v\sqrt{m}, |v|^2\sqrt{m}\}.$$

Kernel depending on the Prandtl number

Linear operator of the Shakhov model.

$$Lf = P_c f + (1 - Pr)P_{nc} f - f.$$

- When $Pr > 0$, the linear operator Lf satisfies the following property:

$$\langle Lf, f\rangle_{L^2_{x,v}} \leq -\min\{Pr, 1\}\|(I - P_c)f\|^2_{L^2_{x,v}},$$
$$KerL = span\{\sqrt{m}, v\sqrt{m}, |v|^2\sqrt{m}\}.$$

- When $Pr = 0$, the linear operator L satisfies following property:

$$\langle Lf, f\rangle_{L^2_{x,v}} = -\|(I - P_c - P_{nc})f\|^2_{L^2_{x,v}},$$
$$KerL = span\{\sqrt{m}, v\sqrt{m}, |v|^2\sqrt{m}, v|v|^2\sqrt{m}\}.$$

Theorem (Bae, G.-C., Yun, S.-B.)

We assume that $F_0(x,v) = m + \sqrt{m} f_0(x,v) \geq 0$,

$$N \geq 3, \quad Pr > 0, \quad \sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta f_0\|^2_{L^2_{x,v}} \leq M.$$

We choose sufficiently small M.

1. *There exist unique global-in-time classical solution.*
2. *The solution* F *and the Shakhov operator are non-negative:*

$$F(x,v,t) = m + \sqrt{m} f(x,v,t) \geq 0, \quad \mathcal{S}(F)(x,v,t) \geq 0.$$

3. *The perturbation* f *has exponential decay:*

$$\sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta f(t)\|_{L^2_{x,v}} \leq Ce^{-\delta t}.$$

Theorem (Bae, G.-C., Yun, S.-B.)

In the case of $Pr = 0$, if we futher assume that the total third moment of the initial data is equal to zero:

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F_0(x,v)v|v|^2 dvdx = 0,$$

then we can have same global-in-time existence result under the same assumption, where the energy norm is defined as

$$\mathcal{E}(f)(t) = \frac{1}{2}\sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta f(t)\|^2_{L^2_{x,v}} + \sum_{|\alpha|+|\beta|\leq N}\int_0^t \|\partial^\alpha_\beta f(s)\|^2_{L^2_{x,v}} ds.$$

Sketch of the proof

1. Coercivity when $Pr > 0$.
2. Coercivity when $Pr = 0$.

Coercivity estimate when $Pr > 0$

Once we substitute $f = P_c f + (I - P_c)f$ then by the conservation laws, we have

$$\sum_{|\alpha|\leq N} \|\partial^\alpha P_c f\|^2_{L^2_{x,v}} \leq C \sum_{|\alpha|\leq N} \|\partial^\alpha (I - P_c) f\|^2_{L^2_{x,v}},$$

for sufficiently small $\mathcal{E}(t)$.

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2_{L^2_{x,v}} \leq -\|(I-P_c)\partial^\alpha f\|^2_{L^2_{x,v}} + \langle \partial^\alpha \Gamma(f,f), \partial^\alpha f\rangle_{L^2_{x,v}}, \\ \Rightarrow\quad &\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2_{L^2_{x,v}} \leq -\delta\|\partial^\alpha f\|^2_{L^2_{x,v}} + \langle \partial^\alpha \Gamma(f,f), \partial^\alpha f\rangle_{L^2_{x,v}}. \end{aligned}$$

Coercivity estimate when $Pr = 0$

We substitute $f = (P_c + P_{nc})f + (I - P_c - P_{nc})f$. We write

$$(P_c + P_{nc})f = \left(a(x,t) + b(x,t)\cdot v + c(x,t)|v|^2 + d(x,t)\cdot v|v|^2\right)\sqrt{m}.$$

We note that $d(x,t)$ does not have a conservation law. Applying Poincaré inequality on d_i gives

$$\|d_i\|_{L^2_x} \leq \|\nabla_x d_i\|_{L^2_x} + C\left\|\int_{\mathbb{T}^3} d_i dx\right\|_{L^2_x}.$$

Sketch of the proof

Multiplying $v_i|v|^2$ and integrating each side of the Shakhov model with respect to dt gives

$$\int_{\mathbb{T}^3} d_i(x,t)dx - \int_{\mathbb{T}^3} d_i(x,0)dx = \frac{1}{5\tau}\int_0^t \int_{\mathbb{T}^3} U_i\rho(T-\Theta_{ii}) - \sum_{j\neq i} 2\rho U_j\Theta_{ij}dxdt.$$

For sufficiently small $\mathcal{E}(t) \leq M$, we have

$$\left|\int_{\mathbb{T}^3} d_i(x,t)dx - \int_{\mathbb{T}^3} d_i(x,0)dx\right| \leq C\int_0^t \|f\|^2_{L^2_{x,v}}dt \leq C\mathcal{E}(t),$$

which gives

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|_{L^2_{x,v}} \leq -\delta\|\partial^\alpha f\|^2_{L^2_{x,v}} + C\mathcal{E}^2(t) + \langle\partial^\alpha\Gamma(f,f),\partial^\alpha f\rangle_{L^2_{x,v}}.$$

Summary

- The Shakhov model is derived modifying heat flux from the BGK model to obtain the correct Prandtl number.
- We derived the full coercivity from the lack of dissipation even when $Pr = 0$ to construct the global-in-time classical solution.

Thank you for attention!