UNIVERSITY OF OXFORD
Mathematical Institute

Models for collective behavior: qualitative properties and model hierarchy

J.A. Carrillo

Mathematical Institute
University of Oxford

Virtual Summer school on Kinetic and fluid equations for collective dynamics

August 23-26 2021

France-Korea International Research Laboratory in Mathematics

Oxford
Mathematics

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 883363)

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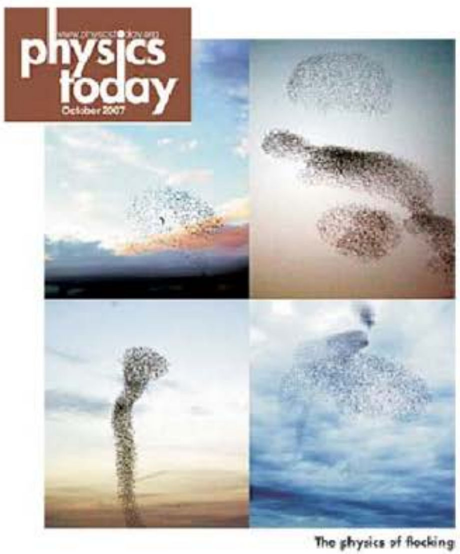
Outline

Swarming by Nature or by design?



Fish schools and Birds flocks.

Individual Based Models (Particle models)

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
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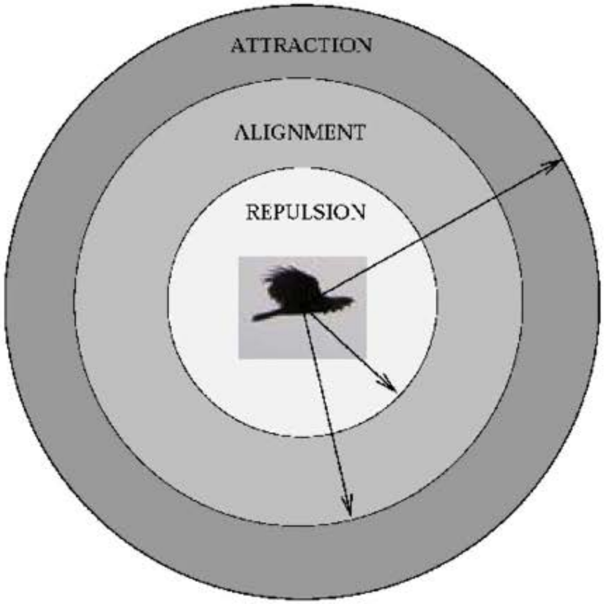

Metric versus Topological Interaction

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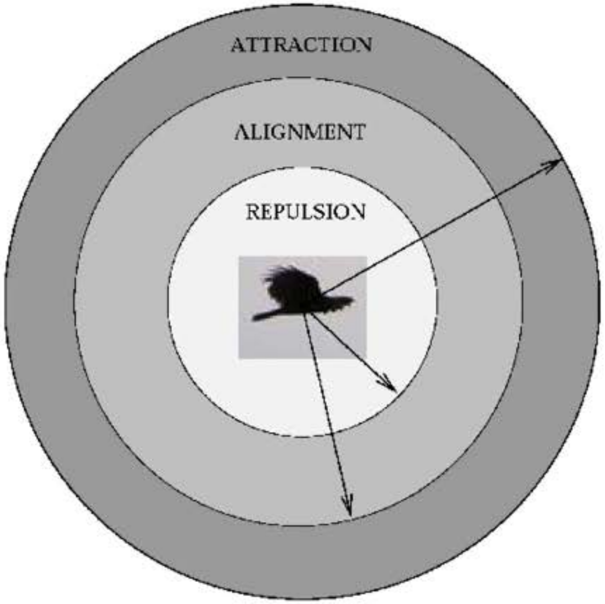

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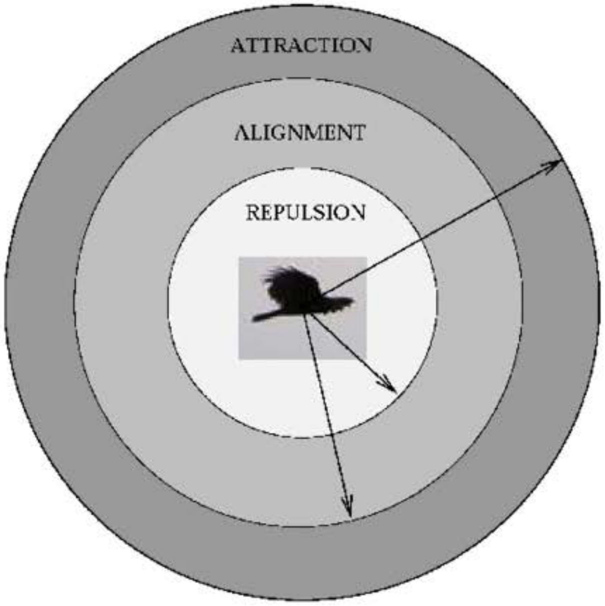

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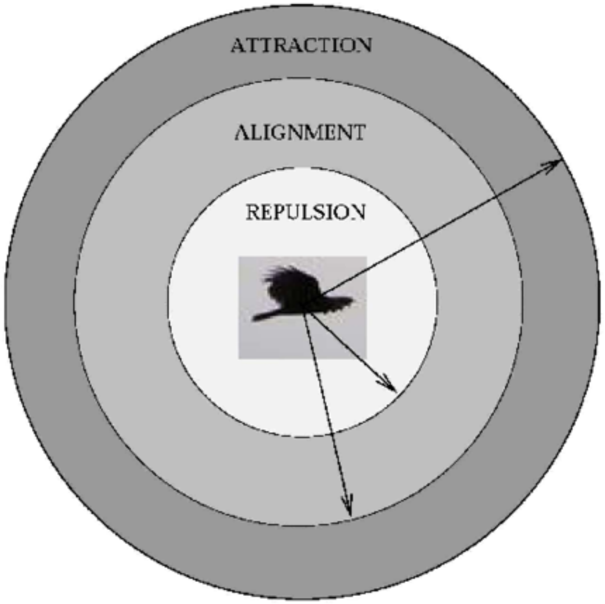


Metric versus Topological Interaction

2nd Order Model: Newton's like equations

D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

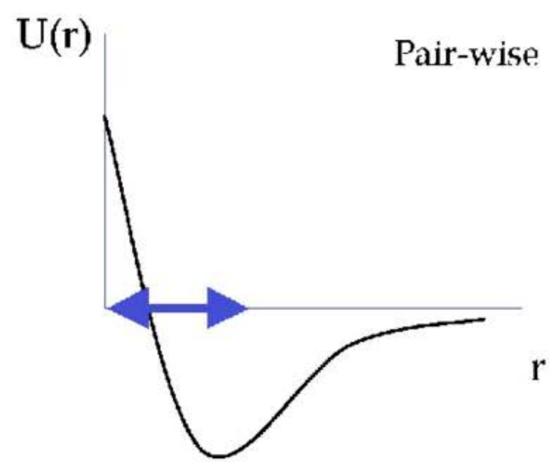
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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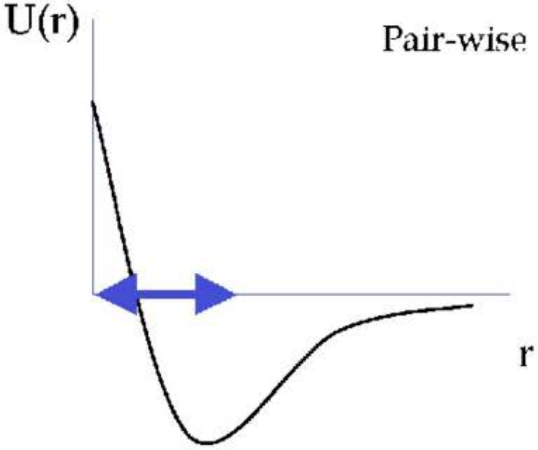

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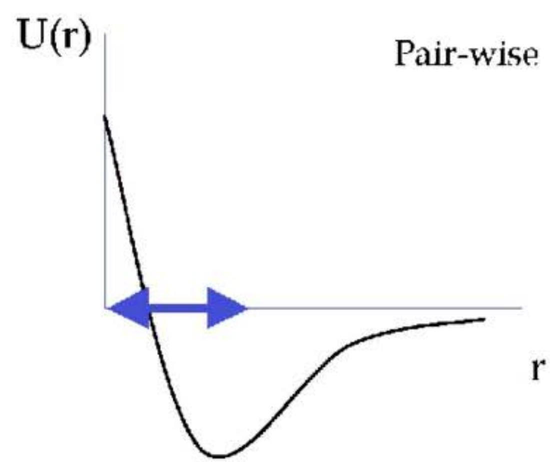

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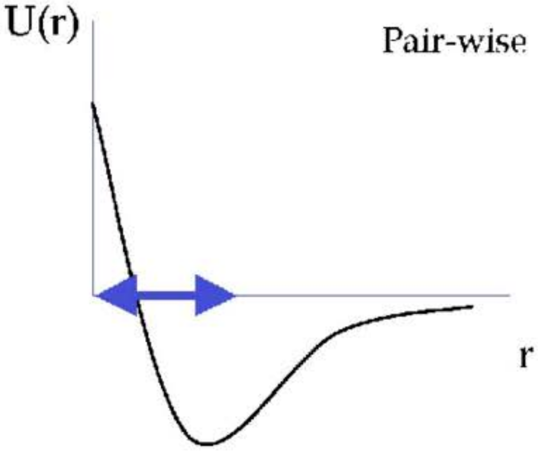
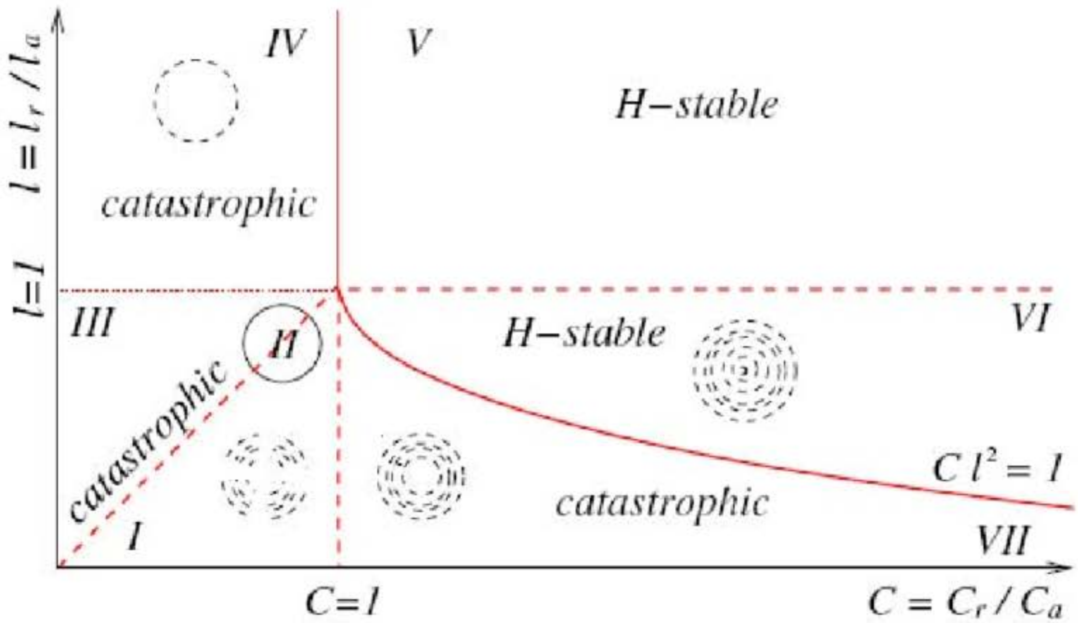


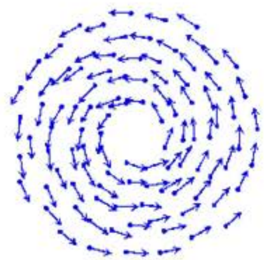
Model with an asymptotic velocity

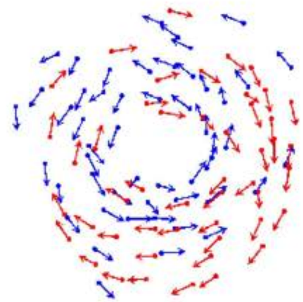
Classification of possible patterns: Morse potential. D'Orsogna, Bertozzi et al. model (PRL 2006).

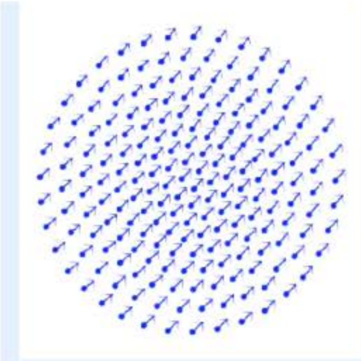


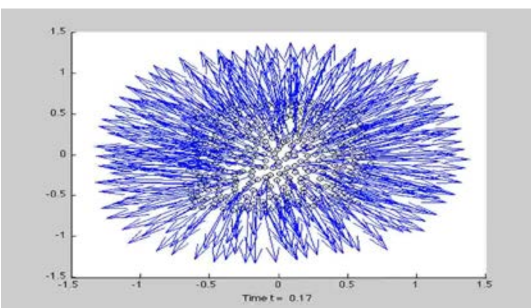
Model with an asymptotic speed

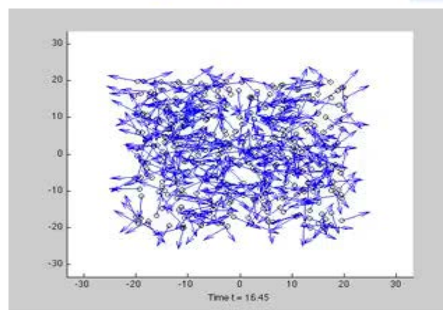
Typical patterns: milling, double milling or flocking:

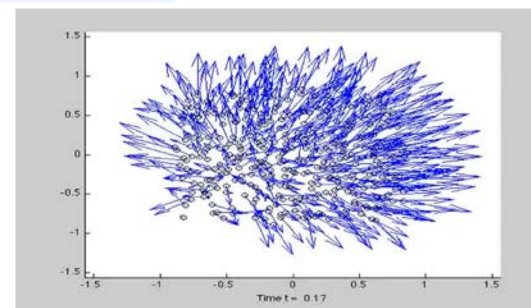


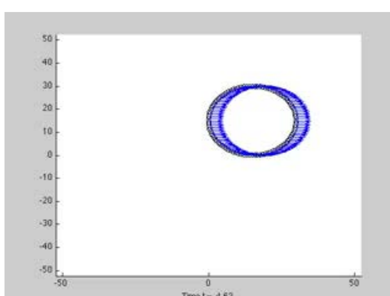


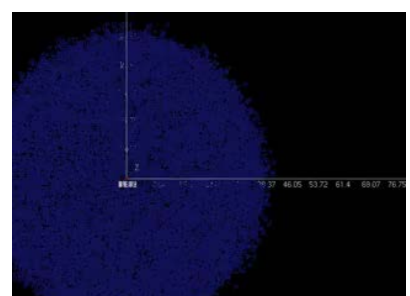












Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\left(v_j - v_i\right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Asymptotic flocking: $\gamma < 1/2$; Cucker-Smale.
General Proof for $0 < \gamma \leq 1/2$; C.-Fornasier-Rosado-Toscani.

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Leadership, Geometrical Constraints, and Cone of Influence

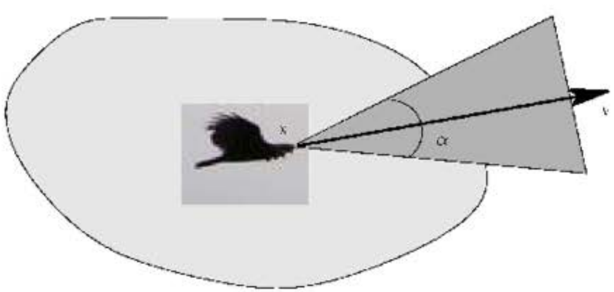
Cucker-Smale with local influence regions:

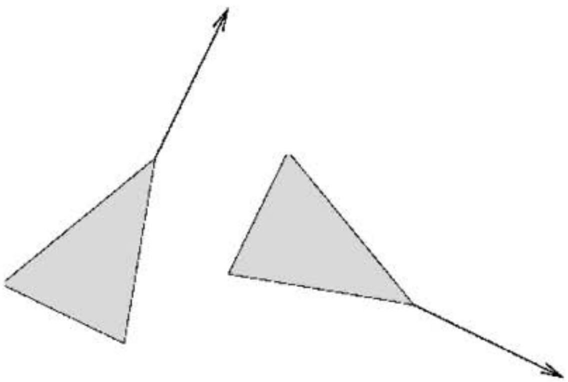
$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases}$$

where $\Sigma_i(t) \subset \{1, \ldots, N\}$ is the set of dependence, given by

$$\Sigma_i(t) := \left\{ 1 \le \ell \le N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \ge \alpha \right\}.$$

Cone of Vision:





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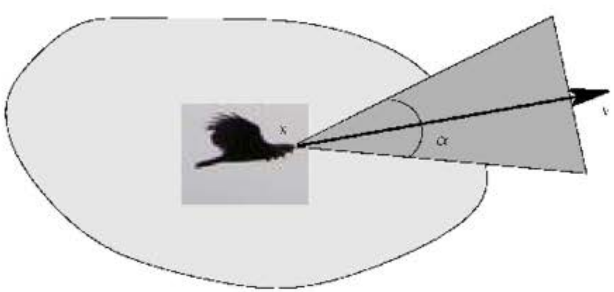
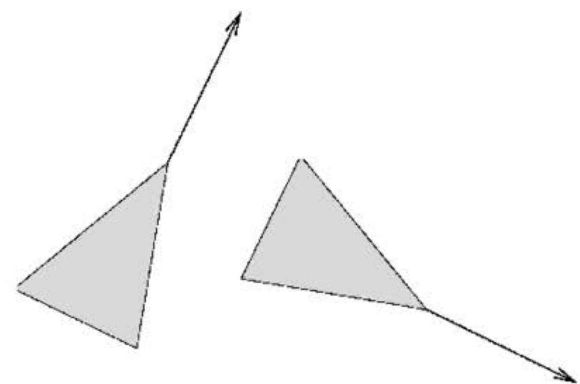




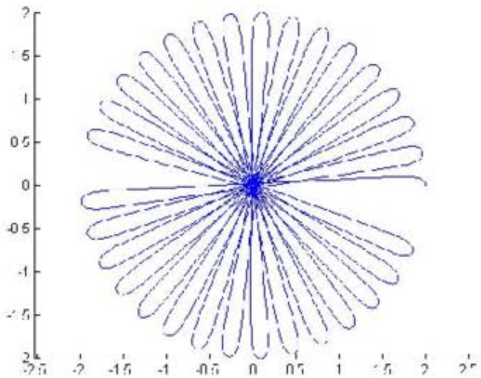
Roosting Forces

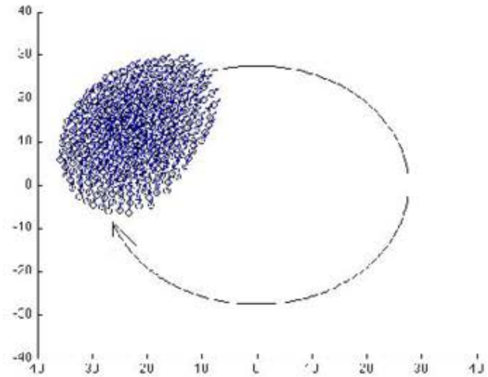
Adding a roosting area to the model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|) - v_i^{\perp}\,\nabla_{x_i}\left[\phi(x_i)\cdot v_i^{\perp}\right]. \end{cases}$$

with the roosting potential ϕ given by $\phi(x) := \dfrac{b}{4}\left(\dfrac{|x|}{R_{\mathrm{Roost}}}\right)^4$.

Roosting effect: milling flocks $N = 400$, $R_{\mathrm{roost}} = 20$.





Adding Noise

Self-Propelling/Friction/Interaction with Noise Particle Model:

$$\begin{cases} \dot{x}_i = v_i, \\ dv_i = \left[(\alpha - \beta |v_i|^2) v_i - \nabla_{x_i} \sum_{j \neq i} U(|x_i - x_j|) \right] dt + \sqrt{2\sigma}\, d\Gamma_i(t) \,, \end{cases}$$

where $\Gamma_i(t)$ are N independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. The Cucker–Smale Particle Model with Noise:

$$\begin{cases} dx_i = v_i dt \,, \\ dv_i = \sum_{j=1}^{N} a(|x_j - x_i|)(v_j - v_i)\, dt + \sqrt{2\sigma \sum_{j=1}^{N} a(|x_j - x_i|)}\, d\Gamma_i(t) \,. \end{cases}$$

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Vicsek's model

Assume N particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i\, dt, \\ dV_t^i = \sqrt{2}\, P(V_t^i) \circ dB_t^i - P(V_t^i)\left(\frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j)\right) dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2}.$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-C.

Main issue: phase transition? Degond-Liu-Frouvelle.

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1st Order Friction Model:

Edelshtein-Keshet, Mogilner (JMB 2000): Assume the variations of the velocity and speed are much smaller than spatial variations, then from Newton's equation:

$$m\frac{d^2x_i}{d^2t} + \alpha\frac{dx_i}{dt} + \sum_{j\neq i}\nabla U(|x_i - x_j|) = 0$$

$$\frac{dx_i}{dt} = -\sum_{j\neq i}\nabla U(|x_i - x_j|) \qquad \text{in the continuum setting} \Rightarrow \qquad \begin{cases} \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

Flock Solutions: stationary states x_i^s of the 1st order model are connected to particular solutions of the Bertozzi etal 2nd order model of the form

$$x_i(t) = x_i^s + t v_0$$

with v_0 fixed with $|v_0|^2 = \frac{\alpha}{\beta}$.
For which potentials do we evolve towards some nontrivial steady states/patterns?
Is there any implication of the stability from first to 2nd order models?

If repulsion is very strong and localized while attraction has a larger length-scale, we assume $U = U_a + \delta_0$, and thus

$$\frac{\partial\rho}{\partial t} = \operatorname{div}(\rho\nabla U_a * \rho) + \Delta\rho^2$$

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1st Order Friction Model:

Edelshtein-Keshet, Mogilner (JMB 2000): Assume the variations of the velocity and speed are much smaller than spatial variations, then from Newton's equation:

$$m\frac{d^2x_i}{d^2t} + \alpha\frac{dx_i}{dt} + \sum_{j\neq i}\nabla U(|x_i - x_j|) = 0$$

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Flock Solutions: stationary states x_i^s of the 1st order model are connected to particular solutions of the Bertozzi etal 2nd order model of the form

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Outline

Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Big[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v - w}{(1 + |x - y|^2)^{\gamma}} f(y, w, t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x, v, t) \Big]$$

Orientation, Attraction and Repulsion:

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Definition of the distance

Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d) \, .$$

Random variables:

Say that X is a random variable with law given by μ, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right] .$$

Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{ \mathbb{E}\left[|X - Y|^p\right] \right\}$$

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Well-posedness in probability measures[1]

Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0, +\infty); \mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with P^t the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions f and g with initial data f_0 and g_0, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \leq \alpha(t)\, W_1(f_0, g_0)$$

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Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta |v_i|^2) v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij} (v_j - v_i)}^{\text{orientation}}. \end{cases}$$

  then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

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- Convergence of approximations of measures by particles due to the stability at any given time T as an alternative derivation of the kinetic models.

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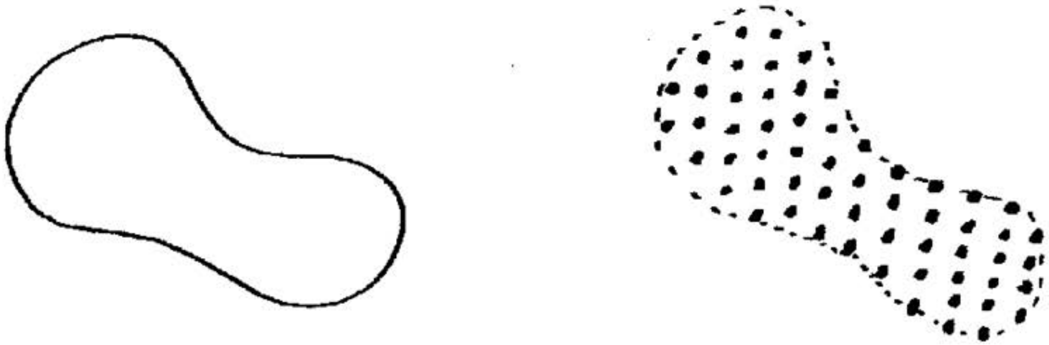
Mean-Field Limit

Just take as many particles as needed in order to have

$$W_1(f_t, f_t^N) \le \alpha(t)\, W_1(f_0, f_0^N) \to 0 \qquad \text{as } N \to \infty$$

by sampling the initial data in a suitable way.

The sequences of particle solutions becomes a Cauchy sequence with the distance W_1 converging to the solution of the kinetic equation.



Hauray-Jabin 2011: mean field limit for Vlasov with potentials such that $|\nabla U| \le r^{-\alpha}$, with $\alpha < 1$ with initial data for Vlasov in $L^1 \cap L^\infty$.

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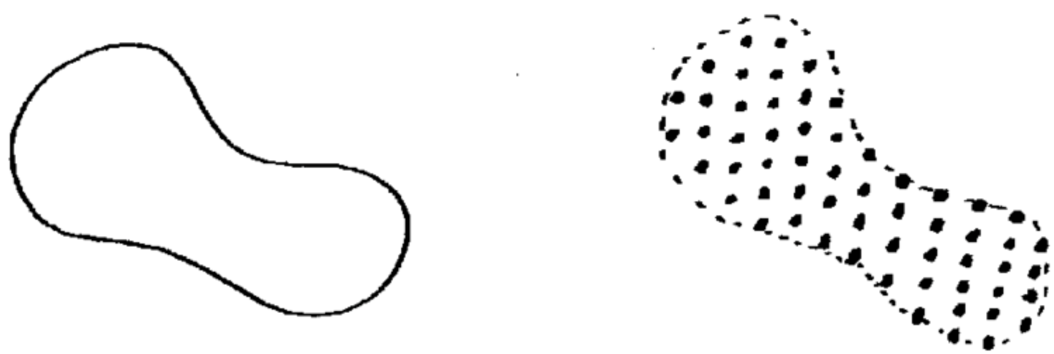

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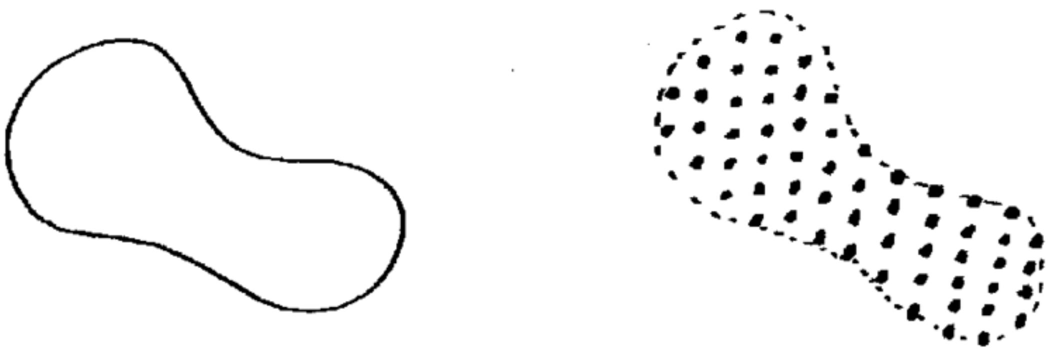

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Outline

1. Kinetic Models and measure solutions
   - Vlasov-like Models
   - Proof
2. Mean-Field Limit for 1st Order Model
   - Setting of the problem
3. Stochastic Mean-Field Limit
   - Setting of the problem
   - Proof
4. Conclusions

Proof of the Theorem

Conditions on E:

1. E is continuous on $[0, T] \times \mathbb{R}^d$,
2. For some $C > 0$,

$$|E(t,x)| \leq C_E(1 + |x|), \quad \text{for all } t, x \in [0,T] \times \mathbb{R}^d, \text{ and}$$

3. E is locally Lipschitz with respect to x, i.e., for any compact set $K \subseteq \mathbb{R}^d$ there is some $L_K > 0$ such that

$$|E(t,x) - E(t,y)| \leq L_K |x - y|, \qquad t \in [0,T], \quad x, y \in K.$$

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Proof of the Theorem

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta |v|^2) v f) = 0,$$

which is a linear first-order equation. The associated characteristic system of ODE's is

$$\frac{d}{dt} X = V,$$
$$\frac{d}{dt} V = E(t, X) + V(\alpha - \beta |V|^2).$$

Flow Map:

Given $(X_0, V_0) \in \mathbb{R}^d \times \mathbb{R}^d$ there exists a unique solution (X, V) to the ODE system in $\mathcal{C}^1([0, T]; \mathbb{R}^d \times \mathbb{R}^d)$ satisfying $X(0) = X_0$ and $V(0) = V_0$. In addition, there exists a constant C which depends only on T, $|X_0|$, $|V_0|$, α, β and the constant C_E, such that

$$|(X(t), V(t))| \le |(X_0, V_0)| \, e^{Ct} \quad \text{for all } t \in [0, T].$$

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We can thus consider the flow at time $t \in [0, T)$ of ODE's equations

$$\mathcal{T}_E^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d.$$

Again by basic results in ode's, the map $(t, x, v) \mapsto \mathcal{T}_E^t(x, v) = (X, V)$ with (X, V) the solution at time t to the ODE system with initial data (x, v), is jointly continuous in (t, x, v).

For a measure $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, the function

$$f : [0, T) \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d), \quad t \mapsto f_t := \mathcal{T}_E^t \# f_0$$

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Proof of the Theorem

Estimates on characteristics:

- Taking characteristics with initial data inside a fixed ball then there exists $R > 0$ depending on T, in which the whole trajectories are inside a possibly larger ball of radius R for all times $t \in [0, T]$.
- For some constant C which depends only on α, β, R and $\mathrm{Lip}_R(E^i)$, for all P^0 in B_R

$$\left|\mathcal{T}^t_{E^1}(P^0) - \mathcal{T}^t_{E^2}(P^0)\right| \leq \frac{e^{Ct} - 1}{C} \sup_{s \in [0,T)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)}.$$

- For some constant C as before

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Error on transported measures through different flows:

Let $\mathcal{T}_1, \mathcal{T}_2 : \mathbb{R}^d \to \mathbb{R}^d$ be two Borel measurable functions. Also, take $f \in \mathcal{P}_1(\mathbb{R}^d)$. Then,

$$W_1(\mathcal{T}_1\#f, \mathcal{T}_2\#f) \leq \|\mathcal{T}_1 - \mathcal{T}_2\|_{L^\infty(\text{supp} f)} .$$

Continuity in time for solutions of the linear transport:

$$W_1(\mathcal{T}_E^s\#f, \mathcal{T}_E^t\#f) \leq C\,|t-s|\,, \qquad \text{for any } t, s \in [0,T].$$

Error on transported measures through different initial data:

Take a locally Lipschitz map $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ and $f, g \in \mathcal{P}_1(\mathbb{R}^d)$, both with compact support contained in the ball B_R. Then,

$$W_1(\mathcal{T}\#f, \mathcal{T}\#g) \leq L\,W_1(f,g),$$

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&\leq C_3 \mathrm{Lip}_{2R}(\nabla U) \int_0^t e^{C_4(t-s)} W_1(f_s, g_s)\, ds + e^{C_1 t}\, W_1(f_0, g_0).
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Outline

2nd order models

The Bertozzi-D'Orsogna model:

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = (\alpha - \beta |v_j|^2) v_j + \frac{1}{N} \sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla U(x_l - x_j) \end{cases} , \quad j = 1, \ldots, N,$$

with $\alpha, \beta > 0$. Particular case $U(x) = k(|x|)$ with

$$k(r) = \frac{r^a}{a} - \frac{r^b}{b}, \qquad a > b > 0.$$

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = \frac{1}{N} \sum_{l=1}^{N} H(x_j - x_l)(v_l - v_j) + \frac{1}{N} \sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla U(x_l - x_j) \end{cases} , \quad j = 1, \ldots, N$$

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Asymptotic solutions

Definition

• We call a *flock ring*, the solution such that $\{x_j\}_{j=1}^N$ are equally distributed on a circle with a certain radius, R and $\{v_j\}_{j=1}^N = u_0$, with $|u_0| = \sqrt{\alpha/\beta}$.

• We call a *mill ring*, the solution such that $\{x_j\}_{j=1}^N$ are equally distributed on a circle with a certain radius, R and $\{v_j\}_{j=1}^N = \sqrt{\alpha/\beta}\, x_j^\perp/|x_j|$ with $x_j^\perp$ the orthogonal vector.

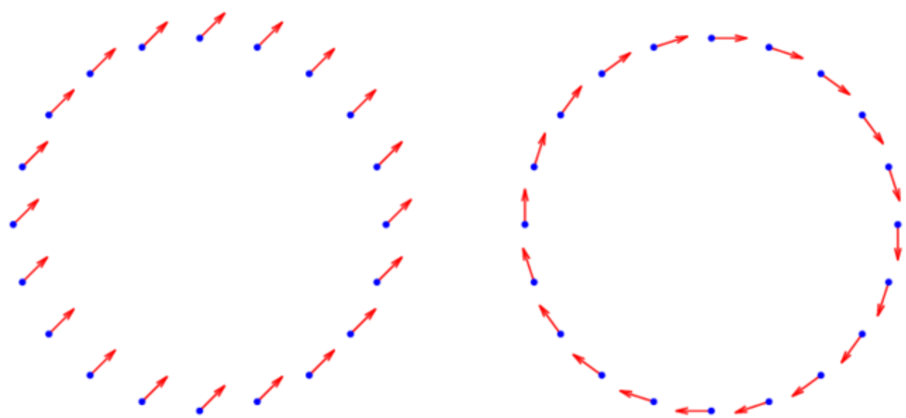

Figure : Flock and mill ring solutions.

Outline

Change of Variables

- Change of variables to the comoving frame:

$$\begin{cases} y_j = x_j(t) - u_0 t \\ z_j = v_j(t) - u_0 \end{cases} ,j = 1, \dots, N,$$

Then the system reads

$$\begin{cases} \dot{y}_j = z_j \\ \dot{z}_j = (\alpha - \beta |z_j|^2)(z_j + u_0) + \frac{1}{N} \sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla U(y_l - y_j) \end{cases} ,j = 1, \dots, N.$$

Write the stationary ring $(y_j^0, z_j^0) = (Re^{i\theta_j}, 0)$ where $\theta_j = \frac{2\pi j}{N}$, for $j = 1, \dots, N$. A general flock spatial profile will be denoted by $(\hat{x}_j, 0)$.

- Consider the following type of perturbations:

$$\tilde{y}_j(t) = \hat{x}_j + h_j(t), \quad \text{with} \quad |h_j| \ll 1.$$

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Analysis of the stability of flocks (I)

- Write the matrix of the linearized system for these perturbations

$$L = \begin{pmatrix} 0_{2N} & \mathrm{Id}_{2N} \\ \mathbf{M} & -2\beta\mathcal{U}_0 \end{pmatrix},$$

where **M** is symmetric and represents the $2N \times 2N$ Jacobian that results from linearizing the first order model, $M = (G_{ij})$ with G_{ij} being the 2×2-blocks defined as

$$G_{ij} = \begin{cases} -\sum_{j \neq i} \mathrm{Hess}\, U(\hat{x}_i - \hat{x}_j) & \text{for } i = j \\ \mathrm{Hess}\, U(\hat{x}_i - \hat{x}_j) & \text{for } i \neq j \end{cases},$$

with Hess U denoting the Hessian matrix of the interaction potential U.

$\mathcal{U}_0$ is the diagonal matrix with N blocks of the 2×2 matrix $u_0 u_0^T$ along the diagonal. Assume that $u_0 = e_1 = (1, 0)$ by rotational symmetry.

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Analysis of the stability of flocks (II)

Symmetries & Linear Instability

Due to translational invariance and rotational invariance of the velocity configuration, zero is always an eigenvalue of the linearized matrix L.

Moreover, there is always a generalized eigenvector associated to the zero eigenvalue generated from the eigenvector due to rotational invariance of the velocity configuration.

Therefore, a flock solution is always linearly unstable.

Instability Result - Spectral Equivalence

The linearized second order system around the flock solution has an eigenvalue with positive real part if and only if the linearized first order system around the flock solution has a positive eigenvalue.

(Albi, Balagué, C., von Brecht; submitted)

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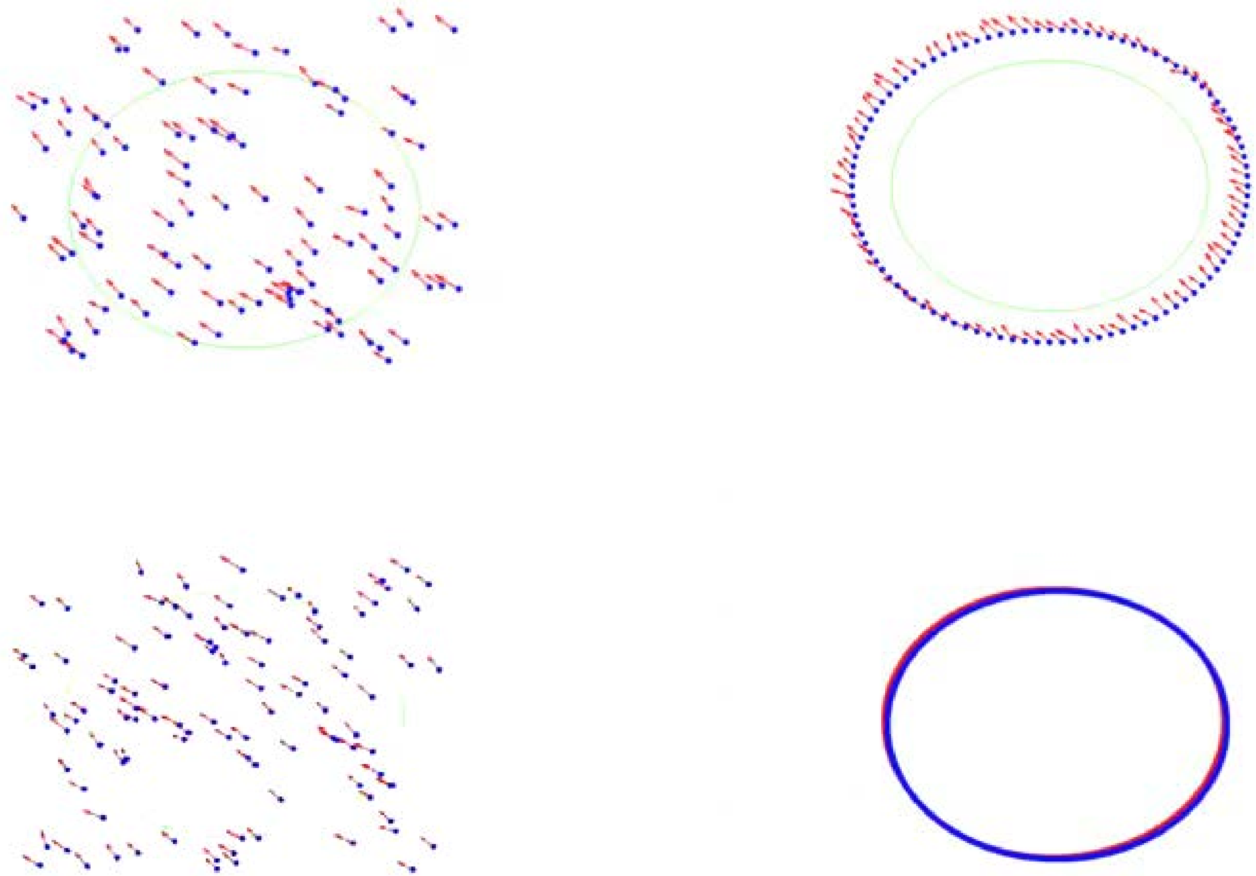
Particle Simulations: Perturbation of flocks



Outline

Stability: New change of variables

- Original coordinates: flock transversal profile
- New coordinates: relative to $m(t) = \frac{1}{N}\sum_i v_i(t)$.

→ all flocks are stationary, $4N+2$-dimensional dynamics $z \overset{t}{\mapsto} \mathcal{F}(z)$

- Reduce dynamics to mean-velocity consistent states, by choosing a invariant base B: $\mathcal{F}_B^B := \mathcal{F}|_{\text{span}\,B} \to \text{span}\,B$.

→ Study the linearisation $z \approx z_F + F_B^B(z - z_f)$

$$\begin{pmatrix} 0_{2N\times 2N} & \begin{matrix} I_{2N-2} \\ -1_{N-1}^T \otimes I_2 \end{matrix} & 0_{2N\times 2} \\ [G(\hat{x})] & -I_{N-1}\otimes 2\beta(m\otimes m^T) & 0_{2N-2\times 2} \\ 0_{2\times 2N} & 0_{2\times 2N-2} & -2\beta(m\otimes m^T) \end{pmatrix}.$$

$$\dot{x}_1 = \dots$$
$$\vdots$$
$$\dot{x}_N = \dots$$
$$\dot{v}_1 = \dots$$
$$\vdots$$
$$\dot{v}_N = \dots$$

flock solution:

$$z_F = (\hat{x} + v_0 t, v_0)^T,\ |v_0| = \sqrt{\alpha/\beta}.$$

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$$\begin{pmatrix} 0_{2N\times 2N} & \begin{matrix} I_{2N-2} \\ -1_{N-1}^T \otimes I_2 \end{matrix} & 0_{2N\times 2} \\ [G(\hat{x})] & -I_{N-1}\otimes 2\beta(m\otimes m^T) & 0_{2N-2\times 2} \\ 0_{2\times 2N} & 0_{2\times 2N-2} & -2\beta(m\otimes m^T) \end{pmatrix}.$$

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$$z_F = (\hat{x}, 0, m)\,,\ |m| = \sqrt{\alpha/\beta}.$$

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Result

Suppose the first-order aggregation system

$$\frac{dx_i}{dt} = -\sum_{i \neq j} \nabla U(x_i - x_j)\,,$$

is linearly stable except for translational and rotational invariance at a stationary profile $\hat{x}$.
Then the transformed second-order system behaves well:

- F_B^B has no generalised eigenvector for eigenvalue zero.
- $\dim(\mathrm{eig}(F_B^B, 0)) = 4$ with 4 eigenvectors that all represent linearised flow within the set of stationary flock solutions.
  2 $\rightsquigarrow$ translation in space, 1 $\rightsquigarrow$ rotation in space, 1 $\rightsquigarrow$ rotation in mean velocity
- All non-zero eigenvalues of F_B^B have negative real-part.

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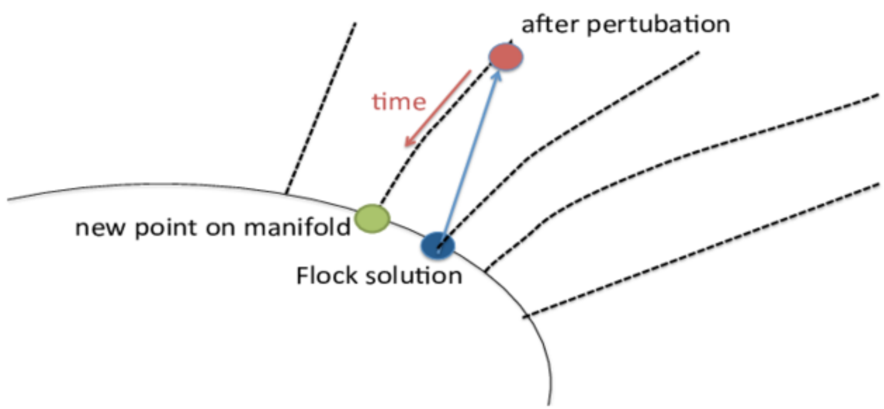
Stability Theorem

This is sufficient to establish that the family of flock solutions

$$Z_F = \left\{ (x^*, 0, m)\ ,\ x^* = T_x R[\phi]\hat{x},\ |m| = \sqrt{\alpha/\beta} \right\}$$

is a normally hyperbolic invariant manifold with a purely stable tangent-bundle splitting and exponentially decaying local stability (T_x translation, $R[\phi]$ rotation).

(C., Huang, Martin; Nonlinear Analysis: Real World Applications 2014)



Outline

Convergence of the particle method

Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta |v_i|^2) v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij} (v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f_N : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))} \qquad \text{with} \sum_{i=1}^{N} m_i = 1\,,$$

is expected to be the solution corresponding to initial atomic measures.

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Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Big[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\,dy\,dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \Big]$$

Orientation, Attraction and Repulsion:

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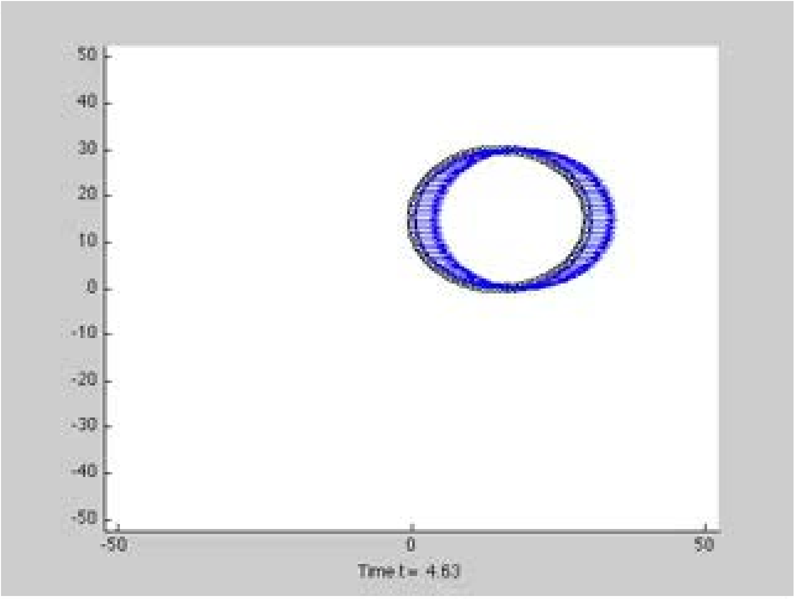
Outline

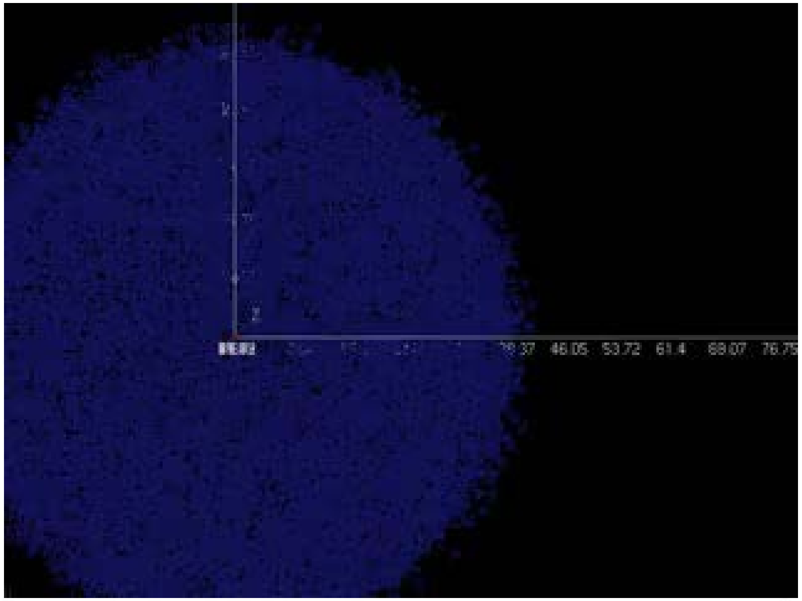
Macroscopic equations

Monokinetic Solutions

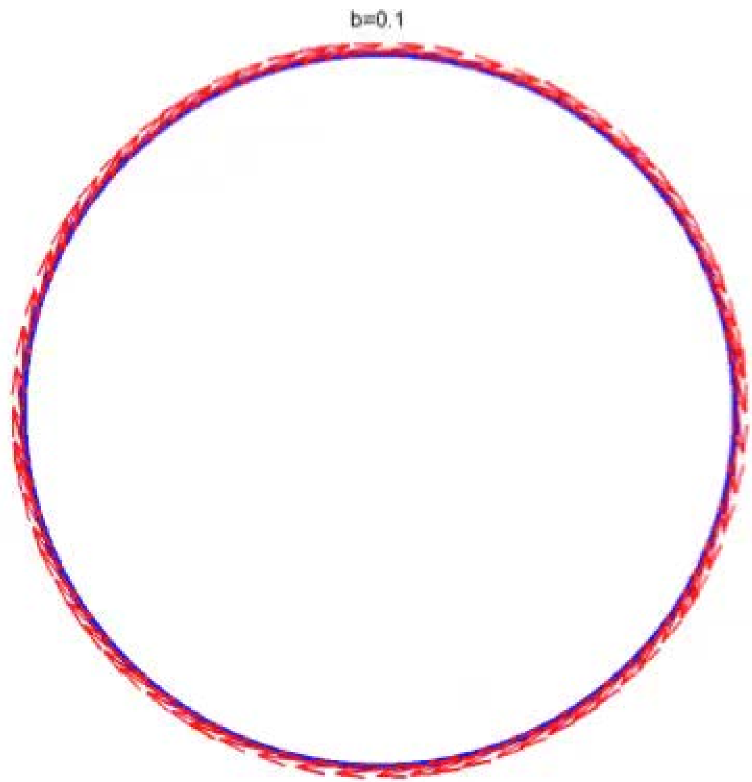
Assuming that there is a deterministic velocity for each position and time, $f(x,v,t) = \rho(x,t)\,\delta(v-u(x,t))$ is a distributional solution if and only if,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) = 0, \\ \rho\,\dfrac{\partial u}{\partial t} + \rho\,(u\cdot\nabla_x)u = \rho\,(\alpha - \beta|u|^2)u - \rho\,(\nabla_x U \star \rho). \end{cases}$$





What about mills?



Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns.
  More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Stability of flocks is understood. However, mill's stability remains unknown.
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- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Rosado-Toscani (SIMA 2010).
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