Rigorous derivation of the continuum hydrodynamic equations

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(based on the work with José A. Carrillo)

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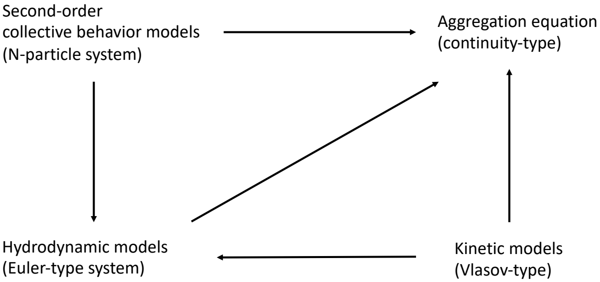
Outline of talk



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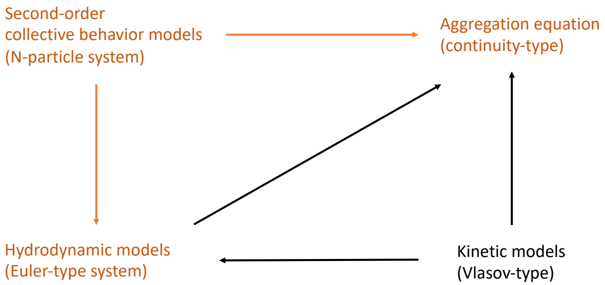

Setting of Problem: Main Equation

Newtonian dynamics:

$$\frac{d}{dt}x_i = v_i, \quad i = 1, \dots, N, \quad t > 0,$$

$$\varepsilon_N \frac{d}{dt}v_i = \underbrace{-\gamma v_i}_{Damping} \underbrace{-\nabla V(x_i)}_{Confinement} \underbrace{-\frac{1}{N}\sum_{j=1}^{N} \nabla W(x_i - x_j)}_{Attraction/Repulsion} + \underbrace{\frac{1}{N}\sum_{j=1}^{N} \psi(x_i - x_j)(v_j - v_i)}_{Alignment}$$

- $\gamma > 0$: strength of the linear damping in velocity
- $V : \mathbb{R}^d \to \mathbb{R}$: confinement potential
- $W : \mathbb{R}^d \to \mathbb{R}$: interaction potential
- $\psi : \mathbb{R}^d \to \mathbb{R}_+$: communication weight

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Examples: Newton's equations(Vlasov–Poisson system), Cucker–Smale model, Second-order aggregation equations, ...

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Goal: study the behavior of solutions as $N \to \infty$

Setting of Problem: Main Goal

Derivation of pressureless Euler system ($\varepsilon_N = O(1)$):

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d, \quad t > 0,$$

$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u)$$

$$= -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x-y)(u(y) - u(x))\, \rho(y)\, dy$$

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Derivation of continuity-type equation ($\varepsilon_N \to 0$ as $N \to \infty$):

$$\partial_t \bar{\rho} + \nabla_x \cdot (\bar{\rho}\bar{u}) = 0,$$

where

$$\gamma \bar{\rho}\bar{u} = -\bar{\rho}\nabla_x V - \bar{\rho}\nabla_x W \star \bar{\rho} + \bar{\rho} \int \psi(x-y)(\bar{u}(y) - \bar{u}(x))\bar{\rho}(y)\, dy$$

Ref.- Ha, Carrillo, Tadmor, Shvydkoy, Tan, ...

Mean-field limit: from particle to kinetic

As $N \to \infty$, (at the formal level) we can derive the following Vlasov-type equation from the particle system:

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot ((\gamma v + \nabla_x V + \nabla_x W \star \rho) f) + \nabla_v \cdot (F(f) f) = 0$$

- $\rho = \rho(x, t)$: local particle density

$$\rho(x, t) := \int f(x, v, t)\, dv$$

- $F(f) = F(f)(x, v, t)$: nonlocal velocity alignment force

$$F(f)(x, v, t) := \iint \psi(x - y)(w - v) f(y, w, t)\, dy dw$$

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Empirical measure: μ^N associated to a solution to the particle system

$$\mu^N_t(x,v) := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))}$$

As long as there exists a solution to the particle system, μ^N satisfies the kinetic equation in the sense of distributions.

Ref.- Golse, Ha, Carrillo, Serfaty, Jabin, Hauray, Samir, Pickl, Wang, ...

Hydrodynamic limit: from kinetic to hydrodynamic

Macroscopic observables:

- ρu: local momentum, ρE: local energy

$$\rho u := \int vf\,dv, \quad \rho E := \int |v|^2 f\,dv$$

- P: strain tensor, q: heat flux

$$P := \int (u-v)\otimes(u-v)f\,dv, \quad q := \int |v-u|^2(v-u)f\,dv$$

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Local balanced laws:

$$\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
&\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \cdot P \\
&\qquad = -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x - y)(u(y) - u(x)) \rho(y) \, dy, \\
&\partial_t (\rho E) + \nabla_x \cdot (\rho E u + P u + q) \\
&\qquad = -\gamma \rho E - \rho u \cdot \nabla_x V - \rho u \cdot \nabla_x W \star \rho \\
&\qquad\quad + \rho \int \psi(x - y) \, (u(x) \cdot u(y) - E(x)) \, \rho(y) \, dy.
\end{aligned}$$

Hydrodynamic limit: from kinetic to hydrodynamic

$$\boxed{\text{mono-kinetic closure: } f(x,v,t)\,dxdv \simeq \rho(x,t)\,dx \otimes \delta_{u(x,v)}(dv)}$$

Pressureless Euler-type system:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$

$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u)$$

$$= -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x-y)(u(y)-u(x))\rho(y)\,dy$$

Hydrodynamic limit: from kinetic to hydrodynamic

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\end{aligned}
$$

Maxwellian closure: $f(x,v,t) \simeq \rho(x,t)\exp(-|u(x,t)-v|^2/2)$

Isothermal Euler-type system:

$$
\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
&\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \rho \\
&\qquad = -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x-y)(u(y)-u(x))\rho(y)\,dy
\end{aligned}
$$

Hydrodynamic limit: from kinetic to hydrodynamic

Kinetic equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot ((\gamma v + \nabla_x V + \nabla_x W \star \rho) f) + \nabla_v \cdot (F(f) f)$$
$$= \frac{1}{\varepsilon} \nabla_v \cdot (\sigma \nabla_v f - (u - v) f)$$

- $\sigma = 0$ and $\varepsilon \to 0$: mono-kinetic closure
- $\sigma = 1$ and $\varepsilon \to 0$: Maxwellian closure

Hydrodynamic limit: from kinetic to hydrodynamic

Kinetic equation:

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Main mathematical tool: Relative entropy (or modulated energy) method

$\Rightarrow$ Lecture 4

Hydrodynamic limit: from kinetic to hydrodynamic

Kinetic equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot ((\gamma v + \nabla_x V + \nabla_x W \star \rho) f) + \nabla_v \cdot (F(f) f) = \frac{1}{\varepsilon} \nabla_v \cdot (\sigma \nabla_v f - (u - v) f)$$

- $\sigma = 0$ and $\varepsilon \to 0$: mono-kinetic closure
- $\sigma = 1$ and $\varepsilon \to 0$: Maxwellian closure

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Question: derivation from the particle system?

Ref.- Karper-Mellet-Trivisa(2015), Figalli-Kang(2019), Carrillo-C.-Jung(2021), ...

Key observation

Mono-kinetic ansatz: $\rho(x,t)\delta_{u(x,t)}(v)$ is a solution to the kinetic equation in the sense of distributions as long as $(\rho, u)(x,t)$ is a strong solution to the pressureless Euler-type system:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$

$$\rho \partial_t u + \rho(u \cdot \nabla_x) u = -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x-y)(u(y) - u(x))\rho(y)\, dy$$

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Indeed, for any $\varphi \in \mathcal{C}_0^1(\mathbb{R}^d \times \mathbb{R}^d)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\iint \varphi(x,v)\rho(x,t)\,\delta_{u(x,t)}(dv)\,dx \\
&\quad = \frac{d}{dt}\int \varphi(x,u(x,t))\rho(x,t)\,dx \\
&\quad = \int \varphi(x,u(x,t))\partial_t \rho\,dx + \int (\nabla_v \varphi)(x,u(x,t)) \cdot (\partial_t u)\rho\,dx \\
&\quad =: (I) + (II).
\end{aligned}$$

Using the continuity equation, (I) can be easily rewritten as

$$\begin{aligned}
(I) &= \int \nabla_x(\varphi(x,u(x,t))) \cdot (\rho u)\,dx \\
&= \iint (\nabla_x \varphi)(x,v) \cdot (\rho v)\delta_{u(x,t)}(dv)\,dx + \int (\nabla_v \varphi)(x,u(x,t)) \cdot \rho(u \cdot \nabla_x)u\,dx.
\end{aligned}$$

Key observation

For (II), we use the momentum equation to obtain

$$
\begin{aligned}
(II) = &- \int (\nabla_v \varphi)(x, u(x,t)) \cdot \rho (u \cdot \nabla_x) u \, dx \\
&- \int (\nabla_v \varphi)(x, u(x,t)) \cdot (\gamma u + \nabla_x V + \nabla_x W \star \rho) \, \rho \, dx \\
&+ \iint (\nabla_v \varphi)(x, u(x,t)) \cdot (u(y) - u(x)) \psi(x-y) \rho(x) \rho(y) \, dxdy
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&+ \iint (\nabla_v \varphi)(x, u(x,t)) \cdot (u(y) - u(x)) \psi(x-y) \rho(x) \rho(y) \, dxdy \\
= &- \int (\nabla_v \varphi)(x, u(x,t)) \cdot \rho (u \cdot \nabla_x) u \, dx \\
&- \iint (\nabla_v \varphi)(x, v) \cdot (\gamma v + \nabla_x V + \nabla_x W \star \rho) \, \rho \delta_{u(x,t)}(dv) \, dx \\
&+ \iiiint (\nabla_v \varphi)(x, v) \cdot (w - v) \psi(x-y) \rho(x) \delta_{u(x,t)}(dv) \rho(y) \delta_{u(y,t)}(dw) \, dxdy.
\end{aligned}
$$

Key observation

Thus, we have

$$\begin{aligned}
&\frac{d}{dt}\iint \varphi(x,v)\rho\delta_{u(x,t)}(dv)\,dx \\
&\quad = \iint ((\nabla_x\varphi)(x,v)\cdot v)\rho\delta_{u(x,t)}(dv)\,dx \\
&\qquad - \int (\nabla_v\varphi)(x,v)\cdot(\gamma v + \nabla_x V + \nabla_x W \star \rho)\,\rho\delta_{u(x,t)}(dv)\,dx \\
&\qquad + \iiiint (\nabla_v\varphi)(x,v)\cdot(w-v)\psi(x-y)\rho(x)\delta_{u(x,t)}(dv)\rho(y)\delta_{u(y,t)}(dw)\,dxdy.
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$$= \iint ((\nabla_x\varphi)(x,v)\cdot v)\rho\delta_{u(x,t)}(dv)\,dx$$
$$- \int (\nabla_v\varphi)(x,v)\cdot(\gamma v + \nabla_x V + \nabla_x W\star\rho)\,\rho\delta_{u(x,t)}(dv)\,dx$$
$$+ \iiiint (\nabla_v\varphi)(x,v)\cdot(w-v)\psi(x-y)\rho(x)\delta_{u(x,t)}(dv)\rho(y)\delta_{u(y,t)}(dw)\,dxdy.$$

Note that

$$\iiiint (\nabla_v\varphi)(x,v)\cdot(w-v)\psi(x-y)\rho(x)\delta_{u(x,t)}(dv)\rho(y)\delta_{u(y,t)}(dw)\,dxdy$$
$$= \iint (\nabla_v\varphi)(x,v)\cdot F(\rho\delta_u)\rho(x)\delta_{u(x,t)}(dv)\,dx.$$

This shows that $\rho(x,t)\delta_{u(x,t)}(v)$ satisfies the kinetic equation:

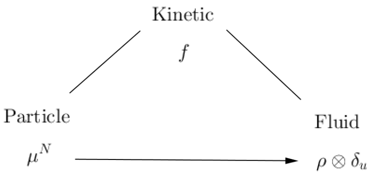
$$\partial_t f + v\cdot\nabla_x f - \nabla_v\cdot((\gamma v + \nabla_x V + \nabla_x W\star\rho)f) + \nabla_v\cdot(F(f)f) = 0$$

in the sense of distributions.

Ref.- Carrillo-D'Orsogna-Panferov(2009), Kang-Kim(2020), ...

Key observation

Observation: both the empirical measure $\mu^N(t)$ associated to the particle system and the monokinetic solutions $\rho(x,t) \otimes \delta_{u(x,t)}(v)$, with $(\rho, u)(x,t)$ satisfying the pressureless Euler-type equations in the strong sense, are distributional solutions of the "same" kinetic equation.



Mathematical tools

Modulated kinetic energy:

$$\frac{1}{2}\iint f|v-u|^2\,dxdv$$

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$$\underbrace{\frac{1}{2}\iint f|v-u|^2\,dxdv}_{\text{modulated mesoscopic energy}} = \underbrace{\frac{1}{2}\int \rho_f|u_f-u|^2\,dx}_{\text{modulated macroscopic energy}} + \frac{1}{2}\iint f|v-u_f|^2\,dxdv$$

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Discrete version of the modulated kinetic energy:

$$\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) := \frac{1}{2}\iint |u-v|^2\,\mu_t^N(dxdv) = \frac{1}{2N}\sum_{i=1}^{N}|u(x_i(t),t)-v_i(t)|^2$$

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Bounded Lipschitz distance: Let $\mu,\nu \in \mathcal{M}(\mathbb{R}^d)$ be two Radon measures. Then the bounded Lipschitz distance, which is denoted by $\mathrm{d}_{BL} : \mathcal{M}(\mathbb{R}^d)\times\mathcal{M}(\mathbb{R}^d)\to\mathbb{R}_+$, between μ and ν is defined by

$$\mathrm{d}_{BL}(\mu,\nu) := \sup_{\phi\in\Omega}\left|\int \phi(x)(\mu(dx)-\nu(dx))\right|,$$

where the admissible set Ω of test functions are given by

$$\Omega := \left\{\phi:\mathbb{R}^d\to\mathbb{R} : \|\phi\|_{L^\infty}\le 1,\ Lip(\phi) := \sup_{x\neq y}\frac{|\phi(x)-\phi(y)|}{|x-y|}\le 1\right\}.$$

Theorem A. Let $T > 0$, $\varepsilon_N = 1$, and $\mathcal{Z}^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system, and let (ρ, u) be the unique classical solution of the pressureless Euler-type system satisfying $\rho > 0$ on $\mathbb{R}^d \times [0, T)$, $\rho \in \mathcal{C}([0, T]; \mathcal{P}(\mathbb{R}^d))$ and $u \in L^\infty(0, T; \mathcal{W}^{1,\infty}(\mathbb{R}^d))$ up to time $T > 0$ with initial data (ρ_0, u_0). Suppose that the interaction potential W and the communication weight function ψ satisfy $\nabla_x W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$ and $\psi \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$, respectively. up to time $T > 0$ with initial data (ρ_0, u_0). Then we have

$$\iint |v - u(x,t)|^2 \mu_t^N(dxdv) + \mathrm{d}_{BL}^2(\rho_t^N(\cdot), \rho(\cdot, t))$$
$$\leq C \left(\iint |v - u_0(x)|^2 \mu_0^N(dxdv) + \mathrm{d}_{BL}^2(\rho_0^N, \rho_0) \right),$$

where $C > 0$ only depends on $\|u\|_{L^\infty \cap Lip}$, $\|\psi\|_{L^\infty \cap Lip}$, $\|\nabla W\|_{\mathcal{W}^{1,\infty}}$, and T.

Main Theorem

In particular, if the initial data are chosen such that the right hand side of the above inequality goes to zero as $N \to \infty$, then the following consequences hold:

$$\int \mu^N(dv) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \rightharpoonup \rho \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}(\mathbb{R}^d)),$$

$$\int v\, \mu^N(dv) = \frac{1}{N}\sum_{i=1}^{N} v_i\, \delta_{x_i} \rightharpoonup \rho u \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}(\mathbb{R}^d)),$$

$$\int (v \otimes v)\, \mu^N(dv) = \frac{1}{N}\sum_{i=1}^{N} (v_i \otimes v_i)\, \delta_{x_i} \rightharpoonup \rho u \otimes u \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}(\mathbb{R}^d)),$$

and

$$\mu^N \rightharpoonup \rho \otimes \delta_u \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$$

as $N \to \infty$.

Proof of Theorem A: convergence

For the convergence estimates, we find

(i) convergence of local moment:

$$\mathrm{d}_{BL}\left(\int v\,\mu^N(dv),\,\rho u\right) \le \left(\iint |v-u(x)|^2\mu^N(dxdv)\right)^{1/2} + C\,\mathrm{d}_{BL}(\rho^N,\rho),$$

(ii) convergence of local energy:

$$\begin{aligned}
&\mathrm{d}_{BL}\left(\int (v\otimes v)\,\mu^N(dv),\,\rho u\otimes u\right)\\
&\le \iint |v-u(x)|^2\mu^N(dxdv) + C\left(\iint |v-u(x)|^2\mu^N(dxdv)\right)^{1/2}\\
&\quad + C\,\mathrm{d}_{BL}(\rho^N,\rho),
\end{aligned}$$

(iii) convergence of empirical measure:

$$\mathrm{d}_{BL}^2(\mu^N,\rho\delta_u) \le C\iint |v-u(x)|^2\,\mu^N(dxdv) + C\,\mathrm{d}_{BL}^2(\rho^N,\rho).$$

Here $C>0$ is independent of N.

Estimate of the modulated kinetic energy

Proposition A. Let $T > 0$, $\mathcal{Z}^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system, and let (ρ, u) be the unique classical solution of the pressureless Euler-type system up to time $T > 0$. Then we have

$$\frac{d}{dt}\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + 2\gamma\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + \frac{1}{N^2}\sum_{i,j=1}^N \psi(x_i - x_j)|v_i - u(x_i)|^2$$

$$\leq C\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + C\,\mathrm{d}_{BL}^2(\rho_t^N(\cdot), \rho(\cdot, t)),$$

where

$$\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) = \frac{1}{2}\iint |u - v|^2\, \mu_t^N(dxdv) = \frac{1}{2N}\sum_{i=1}^N |u(x_i(t), t) - v_i(t)|^2,$$

$$\rho_t^N = \int \mu_t^N\, dv,$$

and $C > 0$ is independent of N and γ.

In particular, this implies

$$\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) \leq C\mathcal{E}^N(\mathcal{Z}_0^N|U_0) + C\int_0^t \mathrm{d}_{BL}^2(\rho_s^N(\cdot), \rho(\cdot, s))\, ds.$$

Proof of Proposition A

Straightforward computations yield

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + \frac{\gamma}{N}\sum_{i=1}^N |u(x_i(t),t) - v_i(t)|^2 \\
&= \frac{1}{N}\sum_{i=1}^N (u(x_i(t),t) - v_i(t)) \cdot ((v_i(t) - u(x_i(t),t)) \cdot \nabla_x) u(x_i(t),t) \\
&\quad - \frac{1}{N}\sum_{i=1}^N (u(x_i(t),t) - v_i(t)) \cdot \Big((\nabla_x W \star \rho)(x_i) - (\nabla_x W \star \rho^N)(x_i) \Big) \\
&\quad + \frac{1}{N}\sum_{i=1}^N (u(x_i(t),t) - v_i(t)) \cdot \mathrm{F}(x_i(t), v_i(t)) \\
&=: (I) + (II) + (III),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{F}(x_i(t), v_i(t)) := &\int \psi(x_i(t) - y)(u(y,t) - u(x_i(t),t))\rho(y,t)\,dy \\
&- \frac{1}{N}\sum_{j=1}^N \psi(x_i(t) - x_j(t))(v_j(t) - v_i(t)).
\end{aligned}
$$

Proof of Proposition A (conti.)

For (I),

$$(I) \leq \|\nabla_x u(\cdot,t)\|_{L^\infty} \frac{1}{N} \sum_{i=1}^{N} |u(x_i(t),t) - v_i(t)|^2$$
$$= 2\|\nabla_x u(\cdot,t)\|_{L^\infty} \mathcal{E}^N(\mathcal{Z}^N(t)|U(t)).$$

For (II), we notice that

$$\|(\nabla_x W \star (\rho - \rho^N))(\cdot,t)\|_{L^\infty} \leq \|\nabla_x W\|_{\mathcal{W}^{1,\infty}} \, \mathrm{d}_{BL}(\rho^N,\rho),$$

thus

$$(II) = \frac{1}{N} \sum_{i=1}^{N} (v_i(t) - u(x_i(t),t)) \cdot (\nabla_x W \star (\rho - \rho^N))(x_i(t),t)$$
$$\leq \|\nabla_x W\|_{\mathcal{W}^{1,\infty}} \, \mathrm{d}_{BL}(\rho^N,\rho) \left(\frac{1}{N} \sum_{i=1}^{N} |v_i(t) - u(x_i(t),t)|^2 \right)^{1/2}$$
$$= \|\nabla_x W\|_{\mathcal{W}^{1,\infty}} \, \mathrm{d}_{BL}(\rho^N,\rho) \sqrt{2\mathcal{E}^N(\mathcal{Z}^N(t)|U(t))}.$$

Proof of Proposition A (conti.)

$$
\begin{aligned}
(III) &= \frac{1}{N}\sum_{i=1}^{N}(u(x_i)-v_i)\cdot\frac{1}{N}\sum_{j=1}^{N}\psi(x_i-x_j)(u(x_j)-v_j) \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}(u(x_i)-v_i)\cdot\Big(\int\psi(x_i-y)(u(y)-u(x_i))\rho(y)\,dy \\
&\qquad\qquad - \int\psi(x_i-y)(u(y)-v_i)\rho^N(y)\,dy\Big) \\
&=: (III)_1+(III)_2.
\end{aligned}
$$

Here

$$
(III)_1 \le \|\psi\|_{L^\infty}\frac{1}{N}\sum_{i=1}^{N}|u(x_i)-v_i|^2 = 2\|\psi\|_{L^\infty}\mathcal{E}^N(\mathcal{Z}^N(t)|U(t))
$$

and

$$
(III)_2 \le -\frac{1}{N^2}\sum_{i,j=1}^{N}\psi(x_i-x_j)|v_i-u(x_i)|^2 + C\sqrt{\mathcal{E}^N(\mathcal{Z}^N(t)|U(t))}\,\mathrm{d}_{BL}(\rho^N,\rho),
$$

where $C>0$ only depends on $\|\psi\|_{\mathcal{W}^{1,\infty}}$ and $\|u\|_{\mathcal{W}^{1,\infty}}$.

Lemma A. Let ρ^N and ρ be defined as above. Then we have

$$\mathrm{d}_{BL}^2(\rho^N(\cdot,t),\rho(\cdot,t)) \le C\,\mathrm{d}_{BL}^2(\rho_0^N,\rho_0) + C\int_0^t \mathcal{E}^N(\mathcal{Z}^N(s)|U(s))\,ds,$$

where $C > 0$ depends only on $\|u\|_{L^\infty(0,T;Lip)}$ and T.

Proof of Lemma A

Consider a forward characteristics $\eta = \eta(x,t)$ for the pressureless Euler-type system:

$$\frac{d\eta(x,t)}{dt} = u(\eta(x,t),t)$$

subject to the initial data: $\eta(x,0) = x \in \mathbb{R}^d$.

- Lipschitz continuous regularity of u implies that
  - the characteristic η is well-defined,
  - η is Lipschitz continuous in $\mathbb{R}^d$:

    $$|\eta(x,t) - \eta(y,t)| \leq C|x-y|,$$

    where $C > 0$ depends only on $\|u\|_{L^\infty(0,T;Lip)}$ and T

- Note that

$$\int \phi(\eta(x,t))\rho_0(x)\,dx = \int \phi(x)\rho(x,t)\,dx$$

for $\phi \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$.

Proof of Lemma A (conti.)

We also get

$$|x_i(t) - \eta(x,t)| \leq |x_i(0) - x| + \int_0^t |v_i(s) - u(\eta(x,s),s)|\, ds.$$

Here,

$$\begin{aligned}
&\int_0^t |v_i(s) - u(\eta(x,s),s)|\, ds \\
&\quad \leq \int_0^t |v_i(s) - u(x_i(s),s)|\, ds + \int_0^t |u(x_i(s),s) - u(\eta(x,s),s)|\, ds \\
&\quad \leq \int_0^t |v_i(s) - u(x_i(s),s)|\, ds + \|u\|_{Lip} \int_0^t |x_i(s) - \eta(x,s)|\, ds.
\end{aligned}$$

This yields

$$|x_i(t) - \eta(x,t)| \leq C|x_i(0) - x| + C \int_0^t |v_i(s) - u(x_i(s),s)|\, ds,$$

where C depends only on $\|u\|_{L^\infty(0,T;Lip)}$ and T. In particular, by taking $x = x_i(0)$, we get

$$|x_i(t) - \eta(x_i(0),t)| \leq C \int_0^t |v_i(s) - u(x_i(s),s)|\, ds.$$

Proof of Lemma A (conti.)

Then for any $\phi \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$ we estimate

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^d} \phi(x)(\rho^N - \rho)\, dx \right| \\
&= \left| \frac{1}{N} \sum_{i=1}^N \phi(x_i(t)) - \int \phi(\eta(x,t))\rho_0\, dx \right| \\
&= \left| \frac{1}{N} \sum_{i=1}^N (\phi(x_i(t)) - \phi(\eta(x_i(0),t))) + \frac{1}{N} \sum_{i=1}^N \phi(\eta(x_i(0),t)) - \int \phi(\eta(x,t))\rho_0\, dx \right| \\
&\le \frac{1}{N} \sum_{i=1}^N |\phi(x_i(t)) - \phi(\eta(x_i(0),t))| + \left| \frac{1}{N} \sum_{i=1}^N \phi(\eta(x_i(0),t)) - \int \phi(\eta(x,t))\rho_0\, dx \right| \\
&=: (I) + (II).
\end{aligned}
$$

Here,

$$
\begin{aligned}
(I) &\le \frac{\|\phi\|_{Lip}}{N} \sum_{i=1}^N |x_i(t) - \eta(x_i(0),t)| \le \frac{\|\phi\|_{Lip}}{N} \int_0^t \sum_{i=1}^N |v_i(s) - u(x_i(s),s)|\, ds \\
&\le \|\phi\|_{Lip} \sqrt{T} \left(\int_0^t \mathcal{E}^N(\mathcal{Z}^N(s)|U(s))\, ds \right)^{1/2}.
\end{aligned}
$$

Proof of Lemma A (conti.)

For the estimate of (II), we notice that

$$\frac{1}{N}\sum_{i=1}^{N}\phi(\eta(x_i(0),t)) = \int \phi(\eta(x,t))\rho_0^N\,dx.$$

Using this identity, the Lipschitz estimate for η, and the fact $\phi \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$, we find

$$(II) = \left|\int \phi(\eta(x,t))(\rho_0^N - \rho_0)\,dx\right| \le C\,\mathrm{d}_{BL}(\rho_0^N,\rho_0)$$

for some $C > 0$ depending on $\|\phi\|_{\mathcal{W}^{1,\infty}}$ and $\|\eta\|_{Lip}$. Hence,

$$\mathrm{d}_{BL}(\rho_t^N(\cdot),\rho(\cdot,t)) \le C\,\mathrm{d}_{BL}(\rho_0^N,\rho_0) + C\left(\int_0^t \mathcal{E}^N(\mathcal{Z}^N(s)|U(s))\,ds\right)^{1/2}$$

for $0 \le t \le T$, where $C > 0$ depends only on $\|u\|_{L^\infty(0,T;Lip)}$ and T.

Proof of Theorem A

Applying Grönwall's lemma and Young's inequality to the differential inequality in **Proposition A** yields

$$\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) \le C\mathcal{E}^N(\mathcal{Z}_0^N|U_0) + C\int_0^t \mathrm{d}_{BL}^2(\rho_s^N(\cdot), \rho(\cdot, s))\, ds,$$

where $C > 0$ is independent of N. We then use **Lemma A** to have

$$\begin{aligned}
&\mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + \mathrm{d}_{BL}^2(\rho_t^N(\cdot), \rho(\cdot, t)) \\
&\quad \le C\mathcal{E}^N(\mathcal{Z}_0^N|U_0) + C\,\mathrm{d}_{BL}^2(\rho_0^N, \rho_0) \\
&\qquad + C\int_0^t \mathrm{d}_{BL}^2(\rho_s^N(\cdot), \rho(\cdot, s))\, ds + C\int_0^t \mathcal{E}^N(\mathcal{Z}^N(s)|U(s))\, ds.
\end{aligned}$$

We finally apply Grönwall's to the above to conclude the desired result.

Mean-field limit: singular interaction potential case

Let $d \geq 1$ and consider a potential $\widetilde{W}$ has the form of

$$\widetilde{W}(x) = |x|^{-\alpha} \quad \max\{d-2, 0\} \leq \alpha < d \quad \forall\, d \geq 1$$

or

$$\widetilde{W}(x) = -\log|x| \quad \text{for } d = 1 \text{ or } 2.$$

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or

$$\widetilde{W}(x) = -\log|x| \quad \text{for } d = 1 \text{ or } 2.$$

Theorem A′. Let $T > 0$, $\varepsilon_N = 1$, and $\mathcal{Z}^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system, and let (ρ, u) be the unique classical solution of the pressureless Euler-type system with nonlocal interaction forces $\widetilde{W}$ up to time $T > 0$ with initial data (ρ_0, u_0). Assume that the classical solution (ρ, u) satisfies $\rho \in L^\infty(0, T; (\mathcal{P} \cap L^\infty)(\mathbb{R}^d))$ and $u \in L^\infty(0, T; \mathcal{W}^{1,\infty}(\mathbb{R}^d))$. In the case $s \geq d-1$, we further assume that $\rho \in L^\infty(0, T; \mathcal{C}^\sigma(\mathbb{R}^d))$ for some $\sigma > \alpha - d + 1$. Then there exists $\beta < 2$ such that

$$\begin{aligned}
&\iint |v - u(x,t)|^2 \, \mu_t^N(dxdv) + \mathrm{d}_{BL}^2(\rho_t^N(\cdot), \rho(\cdot, t)) \\
&\quad + \iint_{\Delta^c} \widetilde{W}(x-y)(\rho^N - \rho)(x)(\rho^N - \rho)(y)\, dxdy \\
&\leq C \iint |v - u_0(x)|^2 \, \mu_0^N(dxdv) + C\, \mathrm{d}_{BL}^2(\rho_0^N, \rho_0) \\
&\quad + C \iint_{\Delta^c} \widetilde{W}(x-y)(\rho_0^N - \rho_0)(x)(\rho_0^N - \rho_0)(y)\, dxdy + C N^{\beta-2},
\end{aligned}$$

where $C > 0$ is independent of N.

Ref.- Serfaty(2020), C.-Jeong(preprint, 2020).

Mean-field/small inertia limit: formal derivation

Newtonian dynamics:

$$\frac{d}{dt}x_i = v_i, \quad i = 1, \dots, N, \quad t > 0,$$

$$\varepsilon_N \frac{d}{dt} v_i = -\gamma v_i - \nabla V(x_i) - \frac{1}{N}\sum_{j=1}^{N} \nabla W(x_i - x_j) + \frac{1}{N}\sum_{j=1}^{N} \psi(x_i - x_j)(v_j - v_i)$$

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By **Theorem A**, we expect that for sufficiently large $N \gg 1$, the above particle system can be well *approximated* by

$$\partial_t\bar\rho + \nabla\cdot(\bar\rho\bar u) = 0,$$
$$\varepsilon_N\left(\partial_t(\bar\rho\bar u) + \nabla\cdot(\bar\rho\bar u\otimes\bar u)\right)$$
$$= -\gamma\bar\rho\bar u - \bar\rho\nabla V - \bar\rho\nabla W\star\bar\rho + \bar\rho\int\psi(x-y)(\bar u(y) - \bar u(x))\,\bar\rho(y)\,dy.$$

Mean-field/small inertia limit: formal derivation

Newtonian dynamics:

$$\frac{d}{dt}x_i = v_i, \quad i = 1, \ldots, N, \quad t > 0,$$

$$\varepsilon_N \frac{d}{dt} v_i = -\gamma v_i - \nabla V(x_i) - \frac{1}{N}\sum_{j=1}^{N} \nabla W(x_i - x_j) + \frac{1}{N}\sum_{j=1}^{N} \psi(x_i - x_j)(v_j - v_i)$$

By **Theorem A**, we expect that for sufficiently large $N \gg 1$, the above particle system can be well *approximated* by

$$\partial_t \bar\rho + \nabla \cdot (\bar\rho \bar u) = 0,$$

$$\varepsilon_N \left(\partial_t(\bar\rho\bar u) + \nabla \cdot (\bar\rho \bar u \otimes \bar u)\right)$$
$$= -\gamma\bar\rho\bar u - \bar\rho\nabla V - \bar\rho\nabla W \star \bar\rho + \bar\rho \int \psi(x-y)(\bar u(y) - \bar u(x))\,\bar\rho(y)\,dy.$$

At the formal level, as $\varepsilon_N \to 0$, we have

$$\partial_t \bar\rho + \nabla \cdot (\bar\rho \bar u) = 0,$$

$$0 = -\gamma\bar\rho\bar u - \bar\rho\nabla V - \bar\rho\nabla W \star \bar\rho + \bar\rho \int \psi(x-y)(\bar u(y) - \bar u(x))\,\bar\rho(y)\,dy.$$

Mean-field/Small inertia limit

We rewrite the continuity-type equations as

$$\partial_t \bar\rho + \nabla \cdot (\bar\rho \bar u) = 0,$$

$$\varepsilon_N \partial_t (\bar\rho \bar u) + \varepsilon_N \nabla \cdot (\bar\rho \bar u \otimes \bar u)$$

$$= -\gamma \bar\rho \bar u - \bar\rho \nabla V - \bar\rho \nabla W \star \bar\rho + \bar\rho \int \psi(x-y)(\bar u(y) - \bar u(x))\, \bar\rho(y)\, dy + \varepsilon_N \bar\rho \bar e,$$

where $\bar e := \partial_t \bar u + \bar u \cdot \nabla \bar u$.

Mean-field/Small inertia limit

We rewrite the continuity-type equations as

$$\partial_t \bar\rho + \nabla\cdot(\bar\rho\bar u) = 0,$$

$$\varepsilon_N \partial_t(\bar\rho\bar u) + \varepsilon_N \nabla\cdot(\bar\rho\bar u\otimes\bar u)$$

$$= -\gamma\bar\rho\bar u - \bar\rho\nabla V - \bar\rho\nabla W\star\bar\rho + \bar\rho\int \psi(x-y)(\bar u(y)-\bar u(x))\,\bar\rho(y)\,dy + \varepsilon_N\bar\rho\bar e,$$

where $\bar e := \partial_t \bar u + \bar u\cdot\nabla\bar u$.

Theorem B. Let $T>0$ and $d\ge 1$. Let $\mathcal{Z}^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system, and let $(\bar\rho,\bar u)$ be the unique classical solution of the aggregation-type equation satisfying $\bar\rho\in\mathcal{C}([0,T];\mathcal{P}(\mathbb{R}^d))$ and $\bar\rho>0$ on $\mathbb{R}^d\times[0,T)$, $\bar u\in L^\infty(0,T;\mathcal{W}^{1,\infty}(\mathbb{R}^d))$ and $\partial_t\bar u\in L^\infty(\mathbb{R}^d\times(0,T))$ up to time $T>0$ with the initial data $\bar\rho_0$. Suppose that the strength of damping $\gamma>0$ is large enough. If the initial datum are chosen such that

$$\iint |v-\bar u_0(x)|^2\mu_0^N(dxdv) + d_{BL}(\rho_0^N,\bar\rho_0)\to 0 \quad \text{as} \quad N\to\infty,$$

then we have

$$\int v\,\mu^N(dv) = \frac{1}{N}\sum_{i=1}^N v_i\,\delta_{x_i} \rightharpoonup \bar\rho\bar u \quad \text{weakly in } L^1(0,T;\mathcal{M}(\mathbb{R}^d))$$

and

$$\mu^N \rightharpoonup \bar\rho\delta_{\bar u} \quad \text{weakly in } L^1(0,T;\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d))$$

as $N\to\infty$ (and thus $\varepsilon_N\to 0$).

Mean-field/Small inertia limit

In fact, we have the following quantitative bound estimate:

$$d_{BL}^2(\rho_t^N(\cdot), \bar{\rho}(\cdot, t)) + \int_0^t \iint |v - \bar{u}(x,s)|^2 \mu_s^N(dxdv)\, ds$$
$$\leq C\varepsilon_N \iint |v - \bar{u}_0(x)|^2 \mu_0^N(dxdv) + C d_{BL}^2(\rho_0^N, \bar{\rho}_0) + C\varepsilon_N^2$$

and

$$\frac{1}{\varepsilon_N} d_{BL}^2(\rho_t^N(\cdot), \bar{\rho}(\cdot, t)) + \iint |v - \bar{u}(x,t)|^2 \mu_t^N(dxdv)$$
$$\leq C(1+\varepsilon_N) \iint |v - \bar{u}_0(x)|^2 \mu_0^N(dxdv) + \frac{C}{\varepsilon_N} d_{BL}^2(\rho_0^N, \bar{\rho}_0) + C\varepsilon_N$$

for all $t \in [0, T]$, where $C > 0$ is independent of both ε_N and N but depending on $\|\bar{u}\|_{L^\infty(0,T;\mathcal{W}^{1,\infty})}$, $\|\partial_t \bar{u}\|_{L^\infty}$, $\|\nabla_x W\|_{\mathcal{W}^{1,\infty}}$, $\|\psi\|_{\mathcal{W}^{1,\infty}}$, and γ.

Mean-field/Small inertia limit

In fact, we have the following quantitative bound estimate:

$$
\begin{aligned}
d_{BL}^2(\rho_t^N(\cdot), \bar\rho(\cdot,t)) &+ \int_0^t \iint |v - \bar u(x,s)|^2 \mu_s^N(dxdv)\, ds \\
&\le C\varepsilon_N \iint |v - \bar u_0(x)|^2 \mu_0^N(dxdv) + C d_{BL}^2(\rho_0^N, \bar\rho_0) + C\varepsilon_N^2
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\varepsilon_N} d_{BL}^2(\rho_t^N(\cdot), \bar\rho(\cdot,t)) &+ \iint |v - \bar u(x,t)|^2 \mu_t^N(dxdv) \\
&\le C(1+\varepsilon_N) \iint |v - \bar u_0(x)|^2 \mu_0^N(dxdv) + \frac{C}{\varepsilon_N} d_{BL}^2(\rho_0^N, \bar\rho_0) + C\varepsilon_N
\end{aligned}
$$

for all $t \in [0, T]$, where $C > 0$ is independent of both ε_N and N but depending on $\|\bar u\|_{L^\infty(0,T;\mathcal{W}^{1,\infty})}$, $\|\partial_t \bar u\|_{L^\infty}$, $\|\nabla_x W\|_{\mathcal{W}^{1,\infty}}$, $\|\psi\|_{\mathcal{W}^{1,\infty}}$, and γ.

Proof of Theorem B. By using a similar argument as before, we find

$$
\begin{aligned}
\mathcal{E}^N(\mathcal{Z}^N(t)|\bar U(t)) &+ \frac{2\gamma - C}{\varepsilon_N} \int_0^t \mathcal{E}^N(\mathcal{Z}^N(s)|\bar U(s))\, ds \\
&+ \frac{1}{\varepsilon_N N^2} \sum_{i,j=1}^N \int_0^t \psi(x_i(s) - x_j(s)) |v_i(s) - \bar u(x_i(s), s)|^2\, ds \\
&\le \mathcal{E}^N(\mathcal{Z}_0^N|\bar U_0) + \frac{C}{\varepsilon_N} d_{BL}^2(\rho_0^N, \bar\rho_0) + C\varepsilon_N.
\end{aligned}
$$

Mean-field/Small inertia limit: singular interaction potential case

Theorem B′. Let $T > 0$ and $\mathcal{Z}^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system, and let $(\bar\rho, \bar u)$ be the unique strong solution of the continuity-type equation with $\widetilde{W}$ up to time $T > 0$ with the initial data $\bar\rho_0$. Suppose that the strength of damping $\gamma > 0$ is large enough and $(\bar\rho, \bar u)$ satisfies $\bar\rho \in L^\infty(\mathbb{R}^d \times (0, T))$. We further assume that $\bar\rho \in L^\infty(0, T; \mathcal{C}^\sigma(\mathbb{R}^d))$ for some $\sigma > \alpha - d + 1$ in the case $s \geq d - 1$. Then there exists $\beta < 2$ such that

$$
\begin{aligned}
&\mathrm{d}_{BL}^2(\rho_t^N(\cdot), \bar\rho(\cdot, t)) + \iint_{\Delta^c} \widetilde{W}(x-y)(\rho^N - \bar\rho)(x)(\rho^N - \bar\rho)(y)\,dxdy \\
&\quad + \int_0^t \iint |v - \bar u(x,s)|^2 \mu_s^N(dxdv)\,ds \\
&\qquad \leq C\,\mathrm{d}_{BL}^2(\rho_0^N, \bar\rho_0) + C \iint_{\Delta^c} \widetilde{W}(x-y)(\rho_0^N - \bar\rho_0)(x)(\rho_0^N - \bar\rho_0)(y)\,dxdy \\
&\qquad\quad + C\varepsilon_N \iint |v - \bar u_0(x)|^2 \mu_0^N(dxdv) + C\varepsilon_N^2 + CN^{\beta-2}
\end{aligned}
$$

for all $t \in [0, T]$, where $C > 0$ is independent of ε_N and N.

Conclusion

Summary:

- Quantitative mean-field limit of Newton dynamics: derivation of pressureless Euler system with nonlocal interaction forces

- Quantitative mean-field/small inertia limit of Newton dynamics: derivation of continuity-type equation

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- Quantitative mean-field limit of Newton dynamics: derivation of pressureless Euler system with nonlocal interaction forces
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Reference:

- J. A. Carrillo and Y.-P. Choi, Mean-field limits: from particle descriptions to macroscopic equations, Arch. Ration. Mech. Anal., 241, (2021), 1529–1573.

Conclusion

Summary:

- Quantitative mean-field limit of Newton dynamics: derivation of pressureless Euler system with nonlocal interaction forces
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Thank you for your attention.