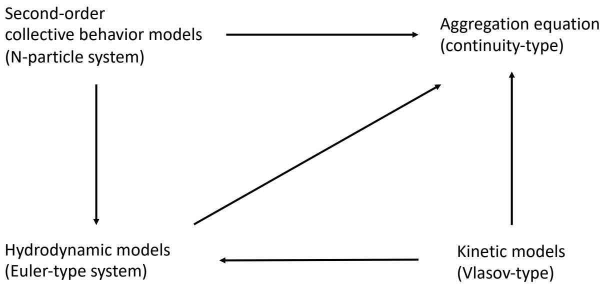
Asymptotic limits connecting: kinetic, hydrodynamic and aggregation equations

Young-Pil Choi
(based on the works with José A. Carrillo, Jinwook Jung, and Oliver Tse)

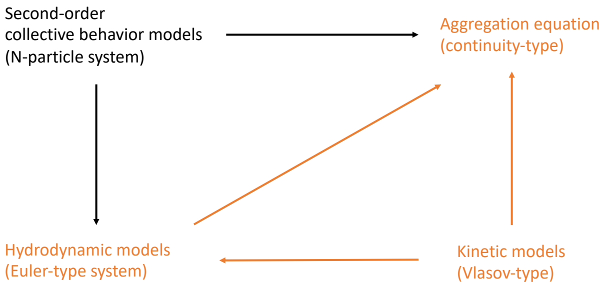
Department of Mathematics
Yonsei University

Virtual Summer school on Kinetic and fluid equations for collective dynamics
France-Korea International Research Laboratory in Mathematics
Aug. 26, 2021

Outline of talk



Outline of talk



Review: formal derivation of hydrodynamic collective behavior models

Kinetic collective behavior models(VE):

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot ((\gamma v + \nabla_x V + \nabla_x W \star \rho) f) + \nabla_v \cdot (F(f) f) = 0,$$

where F is the nonlocal velocity alignment force given by

$$F(f)(x, v, t) := \iint \psi(x - y)(w - v) f(y, w, t)\, dy dw$$

Macroscopic observables:

- $\rho = \rho(x, t)$: local particle density, ρu: local momentum

$$\rho := \int f\, dv, \quad \rho u := \int v f\, dv$$

Local balanced laws:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$

$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \cdot \left(\int (u - v) \otimes (u - v) f\, dv \right)$$

$$= -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x - y)(u(y) - u(x)) \rho(y)\, dy$$

Review: formal derivation of hydrodynamic collective behavior models

$$\boxed{\text{mono-kinetic closure: } f(x,v,t) \simeq \rho(x,t) \otimes \delta_{u(x,v)}(v)}$$

Pressureless Euler-type System(PES):

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u)$$
$$= -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x-y)(u(y) - u(x))\rho(y)\, dy$$

Review: formal derivation of hydrodynamic collective behavior models

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Pressureless Euler-type System(PES):

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&\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
&\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) \\
&\qquad = -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x-y)(u(y)-u(x))\rho(y)\, dy
\end{aligned}$$

Maxwellian closure: $f(x,v,t) \simeq \rho(x,t) \exp(-|u(x,t) - v|^2/2)$

Isothermal Euler-type System(IES):

$$\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
&\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \rho \\
&\qquad = -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \star \rho + \rho \int \psi(x-y)(u(y)-u(x))\rho(y)\, dy
\end{aligned}$$

Part I: From kinetic to Euler

Kinetic collective behavior models:

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot ((v + (\nabla_x V + \nabla_x W \star \rho))f) + \nabla_v \cdot (F[f]f) = \mathcal{N}_{FP}[f],$$

where $\mathcal{N}_{FP}$ is nonlinear Fokker–Planck operator given by

$$\mathcal{N}_{FP}[f](x,v) := \nabla_v \cdot (\beta(v-u)f + \sigma \nabla_v f) = \sigma \nabla_v \cdot \left(f \nabla_v \log \frac{f}{M_u} \right)$$

with the local Maxwellian

$$M_u := \frac{\beta^{d/2}}{(2\pi\sigma)^{d/2}} \exp\left(-\frac{\beta |u-v|^2}{2\sigma} \right),$$

and positive constants β and σ.

Part I: From kinetic to Euler

Kinetic collective behavior models:

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$$M_u := \frac{\beta^{d/2}}{(2\pi\sigma)^{d/2}}\exp\left(-\frac{\beta|u-v|^2}{2\sigma}\right),$$

and positive constants β and σ.

Asymptotic regimes:

- Strong local alignment and diffusion: $\sigma = \beta = \varepsilon^{-1}$
  $\Longrightarrow$ Isothermal Euler-type system
- Strong local alignment without diffusion: $\sigma = 0$, $\beta = \varepsilon^{-1}$
  $\Longrightarrow$ Pressureless Euler-type system

Ref.- Villani(2002), Karper-Mellet-Trivisa(2015), Figalli-Kang(2019), ...

Main assumptions

(H1) The initial data related to the entropy are well-prepared:

$$\int \left(\rho_0^\varepsilon \left(\log \rho_0^\varepsilon - \log \rho_0\right) + \left(\rho_0 - \rho_0^\varepsilon\right)\right) dx = \mathcal{O}(\sqrt{\varepsilon})$$

and

$$\int \left(\int f_0^\varepsilon \log f_0^\varepsilon \, dv - \rho_0 \log \rho_0 \right) dx = \mathcal{O}(\sqrt{\varepsilon}).$$

(H2) The initial data related to the kinetic energy part in the entropy are well-prepared:

$$\int \rho_0^\varepsilon |u_0 - u_0^\varepsilon|^2 \, dx = \mathcal{O}(\sqrt{\varepsilon}) \quad \text{and} \quad \int \left(\int f_0^\varepsilon |v|^2 \, dv - \rho_0 |u_0|^2 \right) dx = \mathcal{O}(\sqrt{\varepsilon}).$$

(H3) The bounded Lipschitz distance between initial local densities satisfies

$$\mathrm{d}_{BL}^2(\rho_0^\varepsilon, \rho_0) = \mathcal{O}(\sqrt{\varepsilon}).$$

Main results

Theorem A. Let f^ε be a weak solution to the equation **(VE)** with $\beta = \sigma = 1/\varepsilon$ in "some" sense and (ρ, u) be a strong solution to the system **(IES)** in "some" sense up to the time $T^* > 0$. Suppose that $\psi \in L^\infty$ and the assumptions **(H1)–(H2)** hold. Then we have the following inequalities for $0 < \varepsilon \leq 1$ and $t \leq T^*$:

(i) Coulomb case $\Delta W = -\delta_0$:

$$\frac{1}{2}\int \rho^\varepsilon |u^\varepsilon - u|^2\,dx + \int \mathcal{H}(\rho^\varepsilon|\rho)\,dx + \frac{1}{2}\int |\nabla W \star (\rho - \rho^\varepsilon)|^2\,dx$$
$$+ \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2\,dxdyds$$
$$\leq C\sqrt{\varepsilon} + C\int |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2\,dx,$$

(ii) Irregular case $\nabla W \in L^\infty$:

$$\frac{1}{2}\int \rho^\varepsilon |u^\varepsilon - u|^2\,dx + \int \mathcal{H}(\rho^\varepsilon|\rho)\,dx + \int_0^t \int \rho^\varepsilon(x)|u^\varepsilon(x) - u(x)|^2\,dxds$$
$$+ \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2\,dxdyds$$
$$\leq C\sqrt{\varepsilon}.$$

Here $C > 0$ is a positive constant independent of ε and $\mathcal{H}$ denotes the classical relative entropy between two probability densities ρ_1 and ρ_2:

$$\mathcal{H}(\rho_1|\rho_2) = \int_{\rho_2}^{\rho_1} \frac{\rho_1 - z}{z}\,dz = \rho_1 \log \rho_1 - \rho_2 \log \rho_2 - (1 + \log \rho_2)(\rho_1 - \rho_2).$$

Main results

Corollary A. Suppose that all the assumptions in **Theorem A** hold. Then we have the following convergences hold for the *weakly regular case (ii)*:

$$\rho^\varepsilon \to \rho, \quad \rho^\varepsilon u^\varepsilon \to \rho u, \quad \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon \to \rho u \otimes u \quad \text{a.e. and in} \quad L^\infty(0,T^*;L^1),$$

$$\int f^\varepsilon v \otimes v \, dv \to \rho u \otimes u + \rho \mathbb{I}_{d\times d} \quad \text{a.e.} \quad \text{and in} \quad L^p(0,T^*;L^1) \quad \text{for} \quad 1 \le p \le 2$$

as $\varepsilon \to 0$. The same convergences for the *Coulomb case (i)* can be obtained if

$$\int |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Moreover, we assume that the confinement potential V satisfies $|\nabla V(x)|^2 \le C|V(x)|$ for some $C > 0$. Then for $t \le T^*$, we have

$$\|f^\varepsilon - M_{\rho,u}\|_{L^1} \le C\left(\iint \mathcal{H}(f_0^\varepsilon | M_{\rho_0,u_0})\, dxdv\right)^{1/2} + C\varepsilon^{1/8}$$

for the *weakly regular case (ii)*, and

$$\begin{aligned}\|f^\varepsilon - M_{\rho,u}\|_{L^1} &\le C\left(\iint \mathcal{H}(f_0^\varepsilon | M_{\rho_0,u_0})\, dxdv\right)^{1/2} + C\varepsilon^{1/8} \\ &\quad + C\left(\min\left\{1, \int |\nabla W \star (\rho_0^\varepsilon - \rho_0)|^2 \, dx\right\}\right)^{1/4}\end{aligned}$$

for the *Coulomb case (i)*, where $C > 0$ is independent of $\varepsilon > 0$ and

$$M_{\rho,u} := \frac{\rho}{(2\pi)^{d/2}} e^{-\frac{|u-v|^2}{2}}.$$

Main results

Theorem B. Let f^ε be a weak solution to the equation **(VE)** with $\beta = 1/\varepsilon$ and $\sigma = 0$ in "some" sense and (ρ, u) be a strong solution to the system **(PES)** in "some" sense up to the time $T^* > 0$. Suppose that $\psi \in \mathcal{W}^{1,\infty}$ and the assumptions **(H2)–(H3)** hold. Then we have the following inequalities for $0 < \varepsilon \le 1$ and $t \le T^*$:

(i) Coulomb case $\Delta W = -\delta_0$:

$$\int \frac{\rho^\varepsilon}{2}|u^\varepsilon - u|^2\,dx + \frac{1}{2}\int |\nabla W \star (\rho - \rho^\varepsilon)|^2\,dx + \mathrm{d}^2_{BL}(\rho^\varepsilon,\rho)$$
$$+ \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dxdyds$$
$$\le C\sqrt{\varepsilon} + C\int |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2\,dx,$$

(ii) Regular case $\nabla W \in \mathcal{W}^{1,\infty}$:

$$\int \frac{\rho^\varepsilon}{2}|u^\varepsilon - u|^2\,dx + \mathrm{d}^2_{BL}(\rho^\varepsilon,\rho) + \int_0^t \int \rho^\varepsilon(x)|u^\varepsilon(x) - u(x)|^2\,dxds$$
$$+ \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2\,dxdyds$$
$$\le C\sqrt{\varepsilon}.$$

Here $C > 0$ is a positive constant independent of ε.

Main results

Corollary B. Suppose that all the assumptions in **Theorem B** hold. If

$$\int |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0$$

for *Coulomb case*, then the following convergences hold:

$$\rho^\varepsilon \rightharpoonup \rho, \quad \rho^\varepsilon u^\varepsilon \rightharpoonup \rho u, \quad \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon \rightharpoonup \rho u \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}),$$

$$\int f^\varepsilon v \otimes v \, dv \rightharpoonup \rho u \otimes u \quad \text{weakly in } L^1(0, T^*; \mathcal{M}), \quad \text{and}$$

$$f^\varepsilon \rightharpoonup \rho \otimes \delta_u \quad \text{weakly in } L^p(0, T^*; \mathcal{M})$$

as $\varepsilon \to 0$, for $1 \le p \le 2$. Here $\mathcal{M}$ is the space of nonnegative Radon measures.

Main ideas: Relative entropy method

We rewrite **(IES)** as a conservative form:

$$\partial_t U + \nabla \cdot A(U) = F(U),$$

where

$$U := \begin{pmatrix} \rho \\ m \end{pmatrix} \quad \text{with} \quad m = \rho u, \quad A(U) := \begin{pmatrix} m & 0 \\ \frac{m \otimes m}{\rho} & \rho \mathbb{I}_{d \times d} \end{pmatrix},$$

and

$$F(U) := \begin{pmatrix} 0 \\ \rho \int \psi(x-y)(u(y) - u(x))\rho(y)\, dy - \rho u - \rho\left(\nabla V + \nabla W \star \rho\right) \end{pmatrix}.$$

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and

$$F(U) := \begin{pmatrix} 0 \\ \rho \int \psi(x-y)(u(y)-u(x))\rho(y)\,dy - \rho u - \rho\left(\nabla V + \nabla W \star \rho\right) \end{pmatrix}.$$

The entropy of the above system is given by

$$E(U) := \frac{|m|^2}{2\rho} + \rho \log \rho$$

and the relative entropy is the quantity:

$$\mathcal{E}(\bar{U}|U) := E(\bar{U}) - E(U) - DE(U)(\bar{U} - U) \quad \text{with} \quad \bar{U} := \begin{pmatrix} \bar{\rho} \\ \bar{m} \end{pmatrix}, \quad \bar{m} = \bar{\rho}\bar{u},$$

where $DE(U)$ denotes the derivation of E with respect to ρ, m.

Main ideas: Relative entropy method

Relative entropy:

$$\mathcal{E}(\bar{U}|U) = \frac{\bar{\rho}}{2}|\bar{u} - u|^2 + \mathcal{H}(\bar{\rho}|\rho)$$

Main ideas: Relative entropy method

Relative entropy:

$$\mathcal{E}(\bar{U}|U) = \frac{\bar{\rho}}{2}|\bar{u} - u|^2 + \mathcal{H}(\bar{\rho}|\rho)$$

Lemma A. The relative entropy $\mathcal{E}$ satisfies the following equality:

$$\begin{aligned}
&\frac{d}{dt}\int \mathcal{E}(\bar{U}|U)\,dx + \frac{1}{2}\iint \bar{\rho}(x)\bar{\rho}(y)\psi(x-y)|(\bar{u}(x)-u(x)) - (\bar{u}(y)-u(y))|^2 dxdy \\
&\quad = \int \partial_t E(\bar{U})\,dx - \int \nabla(DE(U)) : A(\bar{U}|U)\,dx \\
&\qquad - \int DE(U)\left[\partial_t \bar{U} + \nabla\cdot A(\bar{U}) - F(\bar{U})\right] dx \\
&\qquad + \frac{1}{2}\iint \bar{\rho}(x)\bar{\rho}(y)\psi(x-y)|\bar{u}(x)-\bar{u}(y)|^2\,dxdy \\
&\qquad - \iint \bar{\rho}(x)(\rho(y)-\bar{\rho}(y))\psi(x-y)(\bar{u}(x)-u(x))\cdot(u(y)-u(x))\,dxdy \\
&\qquad - \int \bar{\rho}|\bar{u}-u|^2 - \bar{\rho}|\bar{u}|^2\,dx + \int \nabla V\cdot\bar{\rho}\bar{u}\,dx \\
&\qquad + \int \bar{\rho}(\bar{u}-u)\cdot\nabla W \star (\rho-\bar{\rho}) + \bar{\rho}\bar{u}\cdot\nabla W\star\bar{\rho}\,dx,
\end{aligned}$$

where $A(\bar{U}|U)$ is the relative flux functional given by

$$A(\bar{U}|U) := A(\bar{U}) - A(U) - DA(U)(\bar{U}-U).$$

Main ideas: Relative entropy method

Free energy estimate: Set

$$\mathcal{F}(f) := \iint f \log f \, dxdv + \frac{1}{2}\iint |v|^2 f \, dxdv + \frac{1}{2}\iint W(x-y)\rho(x)\rho(y)\, dxdy + \int V\rho \, dx$$

and

$$\mathcal{D}(f) := \iint \frac{1}{f}\left|\nabla_v f - f(u-v)\right|^2 dxdv$$

Then we have

$$\mathcal{F}(f^\varepsilon) + \int_0^t \left(\frac{1}{2\varepsilon}\mathcal{D}(f^\varepsilon) + \frac{1}{2}\iint \psi(x-y)|u^\varepsilon(x) - u^\varepsilon(y)|^2 \rho^\varepsilon(x)\rho^\varepsilon(y)\, dxdy \right) ds$$
$$+ \int_0^t \int \rho^\varepsilon |u^\varepsilon|^2 \, dxds \le \mathcal{F}(f_0^\varepsilon) + C\varepsilon,$$

where $C > 0$ depends only T and $\|\psi\|_{L^\infty}$.

Main ideas: Relative entropy method

Free energy estimate: Set

$$\mathcal{F}(f) := \iint f \log f \, dxdv + \frac{1}{2} \iint |v|^2 f \, dxdv + \frac{1}{2} \iint W(x-y)\rho(x)\rho(y) \, dxdy + \int V\rho \, dx$$

and

$$\mathcal{D}(f) := \iint \frac{1}{f} |\nabla_v f - f(u-v)|^2 \, dxdv$$

Then we have

$$\mathcal{F}(f^\varepsilon) + \int_0^t \left(\frac{1}{2\varepsilon}\mathcal{D}(f^\varepsilon) + \frac{1}{2} \iint \psi(x-y)|u^\varepsilon(x) - u^\varepsilon(y)|^2 \rho^\varepsilon(x)\rho^\varepsilon(y) \, dxdy \right) ds$$
$$+ \int_0^t \int \rho^\varepsilon |u^\varepsilon|^2 \, dxds \le \mathcal{F}(f_0^\varepsilon) + C\varepsilon,$$

where $C > 0$ depends only T and $\|\psi\|_{L^\infty}$.

To Do: Global-in-time existence of weak solution satisfying the above free energy inequality

Main ideas: Relative entropy method

Proposition A. Let f^ε be a global weak solution to the equation **(VE)** and (ρ, u) be a strong solution to the system **(IES)** on the time interval $[0, T]$. Then we have

$$
\begin{aligned}
&\int \mathcal{E}(U^\varepsilon | U)\, dx + \int_0^t \int \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2\, dxds \\
&\quad + \frac{1}{2} \int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dxdyds \\
&\qquad \le C\sqrt{\varepsilon} + C \int_0^t \int \mathcal{E}(U^\varepsilon | U)\, dxds \\
&\qquad\quad + \int_0^t \int \rho^\varepsilon(x)(u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x)\, dxds
\end{aligned}
$$

for $0 < \varepsilon \le 1$, where $C > 0$ is independent of ε.

Proof of Proposition A

It follows from **Lemma A** that

$$
\begin{aligned}
&\int \mathcal{E}(U^\varepsilon|U)\,dx + \int_0^t \int \rho^\varepsilon(x)|u^\varepsilon(x) - u(x)|^2\,dxds \\
&\quad + \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x)-u(x)) - (u^\varepsilon(y)-u(y))|^2 dxdyds \\
&\quad = \int \mathcal{E}(U_0^\varepsilon|U_0)\,dx + \int E(U^\varepsilon) - E(U_0^\varepsilon)\,dx - \int_0^t \int \nabla(DE(U)) : A(U^\varepsilon|U)\,dxds \\
&\qquad - \int_0^t \int DE(U)\,[\partial_s U^\varepsilon + \nabla\cdot A(U^\varepsilon) - F(U^\varepsilon)]\,dxds \\
&\qquad + \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|u^\varepsilon(x) - u^\varepsilon(y)|^2\,dxdyds \\
&\qquad - \int_0^t \iint \rho^\varepsilon(x)(\rho(y) - \rho^\varepsilon(y))\psi(x-y)(u^\varepsilon(x) - u(x))\cdot(u(y) - u(x))\,dxdyds \\
&\qquad + \int_0^t \int \rho^\varepsilon(x)|u^\varepsilon(x)|^2\,dxds + \int_0^t \int \nabla V(x)\cdot\rho^\varepsilon(x)u^\varepsilon(x)\,dxds \\
&\qquad + \int_0^t \int \rho^\varepsilon(x)(u^\varepsilon(x) - u(x))\cdot(\nabla W \star (\rho - \rho^\varepsilon))(x) + \rho^\varepsilon(x)u^\varepsilon(x)\cdot(\nabla W \star \rho^\varepsilon)(x)\,dxds \\
&\quad =: \sum_{i=1}^{9} I_i^\varepsilon.
\end{aligned}
$$

Proof of Proposition A

- Assumptions **(H1)** & **(H2)**:

$$I_1^\varepsilon = \int \mathcal{E}(U_0^\varepsilon | U_0)\, dx = \frac{1}{2} \int \rho_0^\varepsilon |u_0^\varepsilon - u_0|^2 \, dx + \int \mathcal{H}(\rho_0^\varepsilon | \rho_0)\, dx = \mathcal{O}(\sqrt{\varepsilon}).$$

- Free energy estimates:

$$\sum_{i \in \{2,4,5,7,8,9\}} I_i^\varepsilon \le \mathcal{O}(\sqrt{\varepsilon}) + C\varepsilon + \int_0^t \int \rho^\varepsilon(x)(u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x)\, dxds.$$

- Relative flux:

$$I_3^\varepsilon = \int_0^t \int \nabla u : \rho^\varepsilon (u^\varepsilon - u) \otimes (u^\varepsilon - u)\, dxds \le \|\nabla u\|_{L^\infty} \int_0^t \int \mathcal{E}(U^\varepsilon | U)\, dxds.$$

- Velocity alignment:

$$\begin{aligned} I_6^\varepsilon &\le 2\alpha \|u\|_{L^\infty} \|\psi\|_{L^\infty} \int_0^t \iint \rho^\varepsilon(x) |\rho(y) - \rho^\varepsilon(y)| |u^\varepsilon(x) - u(x)|\, dxdyds \\ &= 2\alpha \|u\|_{L^\infty} \|\psi\|_{L^\infty} \int_0^t \|\rho - \rho^\varepsilon\|_{L^1} \int \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|\, dxds \\ &\le C \int_0^t \int \mathcal{E}(U^\varepsilon | U)\, dxds. \end{aligned}$$

Proof of Theorem A: Coulomb case

Lemma A′. Suppose that the interaction potential W satisfies $\Delta W = -\delta_0$. Then we have

$$\frac{1}{2}\frac{d}{dt}\int |\nabla W \star (\rho - \rho^\varepsilon)|^2 \, dx = \int \nabla W \star (\rho - \rho^\varepsilon) \cdot ((\rho u) - (\rho^\varepsilon u^\varepsilon)) \, dx$$

for $t \in [0, T]$.

Proof of Theorem A. Note that

$$\begin{aligned}
&\int \rho^\varepsilon (u^\varepsilon - u) \cdot (\nabla W \star (\rho - \rho^\varepsilon)) \, dx + \int \nabla W \star (\rho - \rho^\varepsilon) \cdot ((\rho u) - (\rho^\varepsilon u^\varepsilon)) \, dx \\
&= \int \nabla W \star (\rho - \rho^\varepsilon) \cdot u(\rho - \rho^\varepsilon) \, dx \\
&= -\int \nabla W \star (\rho - \rho^\varepsilon) \cdot u \, (\Delta W \star (\rho - \rho^\varepsilon)) \, dx \\
&= -\frac{1}{2}\int |\nabla W \star (\rho - \rho^\varepsilon)|^2 \nabla \cdot u \, dx + \int \nabla W \star (\rho - \rho^\varepsilon) \otimes \nabla W \star (\rho - \rho^\varepsilon) : \nabla u \, dx,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left| \int \rho^\varepsilon (u^\varepsilon - u) \cdot (\nabla W \star (\rho - \rho^\varepsilon)) \, dx + \int \nabla W \star (\rho - \rho^\varepsilon) \cdot ((\rho u) - (\rho^\varepsilon u^\varepsilon)) \, dx \right| \\
&\quad \le \frac{3}{2} \|\nabla u\|_{L^\infty} \int |\nabla W \star (\rho - \rho^\varepsilon)|^2 \, dx.
\end{aligned}$$

Proof of Theorem A: Coulomb case

This together with **Lemma A′** and **Proposition A** yields

$$\begin{aligned}
&\int \mathcal{E}(U^\varepsilon|U)\,dx + \frac{1}{2}\int |\nabla W \star (\rho-\rho^\varepsilon)|^2\,dx + \int_0^t \int \rho^\varepsilon(x)|u^\varepsilon(x)-u(x)|^2\,dxds \\
&\quad + \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x)-u(x))-(u^\varepsilon(y)-u(y))|^2 dxdyds \\
&\quad \le C\sqrt{\varepsilon} + \frac{1}{2}\int |\nabla W \star (\rho_0-\rho_0^\varepsilon)|^2\,dx \\
&\qquad + C\int_0^t \int \mathcal{E}(U^\varepsilon|U)\,dxds + C\int_0^t \int |\nabla W \star (\rho-\rho^\varepsilon)|^2\,dxds.
\end{aligned}$$

We finally apply Grönwall's lemma to the above to conclude the desired result.

Remark. The convergence

$$\int |\nabla W \star (\rho-\rho^\varepsilon)|^2\,dx \to 0 \quad \text{as} \quad \varepsilon \to 0$$

implies

$$\rho^\varepsilon \to \rho \quad \text{in} \quad L^\infty(0,T;H^{-1}).$$

Indeed, we can easily find

$$\|\rho^\varepsilon-\rho\|_{H^{-1}} \le \|\nabla W \star (\rho-\rho^\varepsilon)\|_{L^2}.$$

Proof of Theorem A: Irregular case

Lemma A″. Suppose that the interaction potential W satisfies $\nabla W \in L^\infty(\Omega)$. Then we have

$$\left|\int \rho^\varepsilon(x)(u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x)\, dx\right| \le 2\|\nabla W\|_{L^\infty} \int \mathcal{E}(U^\varepsilon|U)\, dx.$$

Proof of Theorem A. By combining **Lemma A″** and **Proposition A**, we find

$$\begin{aligned}
&\int \mathcal{E}(U^\varepsilon|U)\, dx + \gamma \int_0^t \int \rho^\varepsilon(x)|u^\varepsilon(x) - u(x)|^2\, dxds \\
&\quad + \frac{\alpha}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dxdyds \\
&\qquad \le C\sqrt{\varepsilon} + C(1+\gamma+\alpha)\int_0^t \int \mathcal{E}(U^\varepsilon|U)\, dxds.
\end{aligned}$$

We complete the proof by using the Grönwall inequality to the above.

Remark. The modulated interaction energy

$$\int |\nabla W \star (\rho - \rho^\varepsilon)|^2\, dx$$

is not required.

Pressureless case

- Conservative form of **(PES)**:

$$\partial_t U + \nabla \cdot \hat{A}(U) = F(U),$$

where

$$m = \rho u, \quad U := \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad \hat{A}(U) := \begin{pmatrix} m \\ \frac{m \otimes m}{\rho} \end{pmatrix},$$

and

$$F(U) := \begin{pmatrix} 0 \\ \rho \int \psi(x-y)(u(y)-u(x))\rho(y)\,dy - \rho u - \rho(\nabla V + \nabla W \star \rho) \end{pmatrix}.$$

- Entropy (kinetic energy):

$$\hat{E}(U) := \frac{|m|^2}{2\rho}.$$

- Relative entropy (modulated kinetic energy):

$$\begin{aligned} \hat{\mathcal{E}}(\bar{U}|U) :=& \hat{E}(\bar{U}) - \hat{E}(U) - D\hat{E}(U)(\bar{U} - U) \\ =& \frac{\bar{\rho}}{2}|\bar{u} - u|^2 \quad \text{with} \quad \bar{U} := \begin{pmatrix} \bar{\rho} \\ \bar{m} \end{pmatrix}. \end{aligned}$$

Modulated kinetic energy estimate

Proposition B. Let $T > 0$, f^ε be a global weak solution to the **(VE)** with $\sigma = 0$, and let (ρ, u) be a strong solution to the **(PES)** on the time interval $[0, T]$. Then we have

$$
\begin{aligned}
&\int \hat{\mathcal{E}}(U^\varepsilon|U)\,dx + \int_0^t \int \hat{\mathcal{E}}(U^\varepsilon|U)\,dxds \\
&\quad + \frac{1}{2}\int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x)-u(x))-(u^\varepsilon(y)-u(y))|^2 dxdyds \\
&\qquad \le \int \hat{\mathcal{E}}(U_0^\varepsilon|U_0)\,dx + \hat{K}(f_0^\varepsilon) - \int \hat{E}(U_0^\varepsilon)\,dx + C\int_0^t \int \hat{\mathcal{E}}(U^\varepsilon|U)\,dxds + C\varepsilon \\
&\qquad\quad + C\int_0^t \mathrm{d}_{BL}^2(\rho^\varepsilon,\rho)\,ds + \int_0^t \int \rho^\varepsilon(x)(u^\varepsilon(x)-u(x))\cdot(\nabla W \star (\rho-\rho^\varepsilon))(x)\,dxds
\end{aligned}
$$

for $t \in [0, T]$, where $\hat{K}(f)$ denotes the kinetic energy for the kinetic equation, i.e.,

$$
\hat{K}(f) := \frac{1}{2}\iint |v|^2 f\,dxdv.
$$

Proof of Proposition B

For the term with the communication weight function ψ, we denoted it by I^ε and split into two terms:

$$
\begin{aligned}
I^\varepsilon &= -\int_0^t \iint \rho^\varepsilon(x)(\rho(y) - \rho^\varepsilon(y))\psi(x-y)(u^\varepsilon(x) - u(x)) \cdot u(y)\, dxdyds \\
&\quad + \int_0^t \iint \rho^\varepsilon(x)(\rho(y) - \rho^\varepsilon(y))\psi(x-y)(u^\varepsilon(x) - u(x)) \cdot u(x)\, dxdyds \\
&=: I_1^\varepsilon + I_2^\varepsilon,
\end{aligned}
$$

where I_1^ε can be estimated as

$$
\begin{aligned}
|I_1^\varepsilon| &= \left| \int_0^t \int \left(\int (\rho(y) - \rho^\varepsilon(y))\psi(x-y)u(y)\, dy \right) \cdot \rho^\varepsilon(x)(u^\varepsilon(x) - u(x))\, dxds \right| \\
&\le C \int_0^t \mathrm{d}_{BL}(\rho^\varepsilon, \rho) \int \rho^\varepsilon(x)|u^\varepsilon(x) - u(x)|\, dxdt \\
&\le C \int_0^t \mathrm{d}_{BL}^2(\rho^\varepsilon, \rho)\, ds + C \int_0^t \int \hat{\mathcal{E}}(U^\varepsilon | U)\, dxds.
\end{aligned}
$$

Here we used the fact that $y \mapsto \psi(\cdot, y)u(y)$ is bounded and Lipschitz continuous. Similarly, I_2^ε can be estimated, and thus

$$
|I^\varepsilon| \le C \int_0^t \mathrm{d}_{BL}^2(\rho^\varepsilon, \rho)\, ds + C \int_0^t \int \hat{\mathcal{E}}(U^\varepsilon | U)\, dxds,
$$

where $C > 0$ is independent of $\varepsilon > 0$.

Relation between $\mathrm{d}_{BL}(\rho, \rho^\varepsilon)$ & $\hat{\mathcal{E}}(U^\varepsilon | U)$

Lemma B. Let $T > 0$, f^ε be a global weak solution to the **(VE)** with $\sigma = 0$, and let (ρ, u) be a strong solution to the **(PES)** on the time interval $[0, T]$. Then we have

$$\mathrm{d}_{BL}(\rho(t), \rho^\varepsilon(t)) \le C\,\mathrm{d}_{BL}(\rho_0, \rho_0^\varepsilon) + C\left(\int_0^t \int \hat{\mathcal{E}}(U^\varepsilon | U)\, dxds\right)^{1/2}$$

for $0 \le t \le T$, where $C > 0$ is independent of $\varepsilon > 0$.

Ref.- Figalli-Kang(2019), Carrillo-C.(2020), C.(2021)

Proof of Theorem B

(i) Coulomb case: **Lemma A** + **Lemma B** + **Proposition B** $\Longrightarrow$

$$\int \hat{\mathcal{E}}(U^\varepsilon|U)\,dx + \int |\nabla W \star (\rho - \rho^\varepsilon)|^2\,dx + \mathrm{d}^2_{BL}(\rho^\varepsilon,\rho) + \int_0^t \int \hat{\mathcal{E}}(U^\varepsilon|U)\,dxds$$
$$+ \int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x)-u(x)) - (u^\varepsilon(y)-u(y))|^2 dxdyds$$
$$\le C\sqrt{\varepsilon} + C\int |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2\,dx + C\,\mathrm{d}^2_{BL}(\rho_0^\varepsilon,\rho_0)$$
$$+ C\int_0^t \int \hat{\mathcal{E}}(U^\varepsilon|U)\,dxds + C\int_0^t \int |\nabla W \star (\rho - \rho^\varepsilon)|^2\,dxds + C\int_0^t \mathrm{d}^2_{BL}(\rho^\varepsilon,\rho)\,ds.$$

(ii) Regular case: Note that

$$\left|\int \rho^\varepsilon(x)(u^\varepsilon(x)-u(x))\cdot(\nabla W \star (\rho-\rho^\varepsilon))(x)\,dx\right|$$
$$\le C\,\mathrm{d}^2_{BL}(\rho^\varepsilon,\rho) + C\int \rho^\varepsilon|u^\varepsilon - u|^2\,dx.$$

The above observation + **Lemma B** + **Proposition B** $\Longrightarrow$

$$\int \hat{\mathcal{E}}(U^\varepsilon|U)\,dx + \mathrm{d}^2_{BL}(\rho^\varepsilon,\rho) + \int_0^t \int \hat{\mathcal{E}}(U^\varepsilon|U)\,dxds$$
$$+ \int_0^t \iint \rho^\varepsilon(x)\rho^\varepsilon(y)\psi(x-y)|(u^\varepsilon(x)-u(x)) - (u^\varepsilon(y)-u(y))|^2\,dxdyds$$
$$\le C\sqrt{\varepsilon} + C\int_0^t \mathrm{d}^2_{BL}(\rho^\varepsilon,\rho)\,ds + C\int_0^t \int \hat{\mathcal{E}}(U^\varepsilon|U)\,dxds.$$

Part II: From kinetic to aggregation

Vlasov–Fokker–Planck(VFP) equation:

$$\partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x f^\varepsilon + \varepsilon^{-1} \nabla_v \cdot \left(f^\varepsilon (\mathsf{F}(x, \rho^\varepsilon) - \varepsilon^{-1} v) \right) = \varepsilon^{-2} \Delta_v f^\varepsilon$$

- $\varepsilon^{-1} > 0$: strength of the linear damping in velocity and diffusion
- $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon \, dv$
- $\mathsf{F} : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$: driving force of the system given by

$$\mathsf{F}(x, \rho) = -(\nabla V)(x) - (\nabla W \star \rho)(x) \qquad \text{for } (x, \rho) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d),$$

where $V : \mathbb{R}^d \to \mathbb{R}$ and $W : \mathbb{R}^d \to \mathbb{R}$ are given functions

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Example: Vlasov–Poisson–Fokker–Planck system; $\nabla W = \zeta x / |x|^d$, $d \geq 1$. Here, the constant ζ can be chosen $\zeta = \pm 1$ according to applications in either plasma physics or astrophysics.

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Goal: study the behaviors of solutions of **VFP** equations when $\varepsilon \to 0$

Part II: From kinetic to aggregation

Aggregation-Diffusion(AD) equation:

$$\partial_t \rho + \nabla_x \cdot \big(\rho \mathsf{F}(\cdot,\rho)\big) = \Delta_x \rho$$

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Gradient flow structure: AD equation can be written as

$$\partial_t \rho - \nabla_x \cdot \big(\rho \nabla_x \delta_\rho \mathcal{E}(\rho)\big) = 0,$$

where

$$\mathcal{E}(\rho) := \int (\log \rho + V + \frac{1}{2} W \star \rho)\, d\rho$$

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Goal: establish the quantified overdamped limit from **VFP** to **AD** equations as $\varepsilon \to 0$

Previous works

- $W \equiv 0$ **case:**
  - seminal work of Kramers(1940); formal discussion, now known as *Smoluchowski-Kramers limit*, coarse-graining map
  - Nelson(1967); rigorous derivation, SDEs
  - ...
  - Duong-Lamacz-Peletier-Schlichting-Sharma(2018); first *quantitative* result

- $W \not\equiv 0$ **case:**
  - Poupaud-Soler(2000), Goudon(2005), El Ghani-Masmoudi(2010); Vlasov–Poisson–Fokker–Planck system, *qualitative*
  - Duong-Lamacz-Peletier-Sharma(2017); $W \in \mathcal{C}^2 \cap \mathcal{W}^{1,1}(\mathbb{R}^d)$ & $\nabla W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$, *qualitative*

Motivation: no results on the quantified overdamped limit for VFP equation even with smooth interaction potentials

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Ref.- Jabin(2000), Fetecau-Sun(2015), Carrillo-C.(2020), ...

Remark: from Euler to AD

AD equation can also be derived from compressible Euler equations with nonlocal interactions.

• C.-Jeong(preprint): from Euler–Riesz to the fractional porous medium equations

$$\partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0,$$

$$\partial_t (\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{1}{\varepsilon} c_p \nabla p(\rho^\varepsilon) = -\frac{1}{\varepsilon}\rho^\varepsilon u^\varepsilon + \frac{1}{\varepsilon} c_W \rho^\varepsilon \nabla \Lambda^{\alpha-d} \rho^\varepsilon$$

$$\downarrow \quad \varepsilon \to 0$$

$$\partial_t \rho + c_W \nabla \cdot (\rho \nabla \Lambda^{\alpha-d}\rho) = c_P \Delta p(\rho).$$

- $\Lambda^{\alpha-d} = (-\Delta)^{\frac{\alpha-d}{2}}$: Riesz operator, $d-2 < \alpha < d$
- $\Lambda^{\alpha-d}\rho = c_{\alpha,d} W \star \rho$ with $W = |x|^{-\alpha}$, $c_W \in \mathbb{R}$
- $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$, $c_p \geq 0$.

Ref.- Lattanzio, Tzavaras, Carrillo, Soler, Huang, Pan, Coulombel, Goudon, ...

Formal derivation of AD from VFP

- Rewrite **VFP** equation as

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon}\nabla_v \cdot (\mathrm{F}(x,\rho^\varepsilon) f^\varepsilon) = \frac{1}{\varepsilon^2}\nabla_v \cdot (\nabla_v f^\varepsilon + v f^\varepsilon)$$

- Note that

$$\textbf{RHS} = \frac{1}{\varepsilon^2}\nabla_v \cdot \left(f^\varepsilon \nabla_v \log \frac{f^\varepsilon}{\mathscr{N}^d} \right),$$

  where $\mathscr{N}^d$ is the standard d-dimensional normal distribution (or Maxwellian)

$$\mathscr{N}^d(v) = \frac{1}{(2\pi)^{d/2}} e^{-|v|^2/2}$$

- **RHS** has the order ε^{-2}, and thus

$$f^\varepsilon(x,v) \simeq \rho^\varepsilon(x)\mathscr{N}^d(v) \qquad \text{for } \varepsilon \ll 1\,. \tag{1}$$

Formal derivation of AD from VFP

- If we set $m^\varepsilon = \int v f^\varepsilon \, dv$, then we find

$$\begin{aligned} \partial_t \rho^\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot m^\varepsilon &= 0\,, \\ \partial_t m^\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot \left(\int v \otimes v \, f^\varepsilon \, dv \right) &= \frac{1}{\varepsilon} \rho^\varepsilon \mathrm{F}(\cdot, \rho^\varepsilon) - \frac{1}{\varepsilon^2} m^\varepsilon\,. \end{aligned} \tag{2}$$

- We then use (1) to obtain

$$\frac{1}{\varepsilon} \nabla_x \cdot \left(\int v \otimes v \, f^\varepsilon \, dv \right) \simeq \frac{1}{\varepsilon} \nabla_x \rho^\varepsilon \qquad \text{for } \varepsilon \ll 1\,,$$

- $\partial_t m^\varepsilon + \varepsilon^{-1} \nabla_x \rho^\varepsilon \simeq \varepsilon^{-1} \rho^\varepsilon \mathrm{F}(\cdot, \rho^\varepsilon) - \varepsilon^{-2} m^\varepsilon$ for $\varepsilon \ll 1$
- $\varepsilon^{-1} m^\varepsilon \simeq \rho^\varepsilon \mathrm{F}(\cdot, \rho^\varepsilon) - \nabla_x \rho^\varepsilon$ for $\varepsilon \ll 1$
- Inserting this into the continuity equation (2) yields

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \left(\rho^\varepsilon \mathrm{F}(\cdot, \rho^\varepsilon) \right) \simeq \Delta_x \rho^\varepsilon\,,$$

  which is our limiting equation.

Main result

Concerning the potential function V, we assume throughout this paper that $0 \leq V \in \mathrm{Lip}_{loc}(\mathbb{R}^d)$,

$(\mathbf{A}_V^1)$ there exists $c_V > 0$ such that

$$|(\nabla V)(x)| \leq c_V(1 + |x|) \quad \text{for all } x \in \mathbb{R}^d \quad \text{and} \quad \|\nabla V\|_{\mathrm{Lip}} \leq c_V\,;$$

$(\mathbf{A}_V^2)$ for any $r \in [1, \infty)$:

$$c_{V,r} := \sup_{x \in \mathbb{R}^d} |(\nabla V)(x)|^r e^{-V(x)} < \infty\,.$$

Remark. quadratic confinement potential $V = |x|^2/2$

2-**Wasserstein distance:**

$$\mathrm{d}_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left(\iint_{\mathbb{R}^m \times \mathbb{R}^m} |x - y|^2 \, \pi(dxdy) \right)^{1/2}$$

for any Borel probability measures μ and ν on $\mathbb{R}^m$, $m \in \mathbb{N}$, where $\Pi(\mu, \nu)$ is the set of all probability measures on $\mathbb{R}^m \times \mathbb{R}^m$ with first and second marginals μ and ν, respectively, i.e. for $\varphi\,, \psi \in \mathcal{C}_b(\mathbb{R}^m)$

$$\iint_{\mathbb{R}^m \times \mathbb{R}^m} (\varphi(x) + \psi(y)) \, \pi(dxdy) = \int_{\mathbb{R}^m} \varphi(x) \, \mu(dx) + \int_{\mathbb{R}^m} \psi(y) \, \nu(dy).$$

Main result

Theorem C. Under suitable assumptions on solutions to **VFP** and **AD** equations, if W satisfies

$$\nabla W \in L^q(B_{2R}) \cap \mathcal{W}^{1,\infty}(\mathbb{R}^d \setminus B_R) \quad \text{for some } R > 0 \text{ and } q \in (1,\infty].$$

and one of the following conditions:

(i) (Weakly singular) $\nabla W \in \mathcal{W}^{1,1}(B_{2R})$,

(ii) (Purely repulsive) W is positive definite, or

(iii) (Attractive Newtonian) W is given by the Newtonian potential, i.e. $\Delta W = \delta_0$,

then

$$\sup_{0 \le t \le T_*} \mathrm{d}_2^2(\rho^\varepsilon, \rho) \le C \left(\mathrm{d}_2^2(\rho_0^\varepsilon, \rho_0) + \varepsilon^2 \right),$$

for some constants $C > 0$, $T_* > 0$ independent of $\varepsilon \le 1$.

Examples.

$$\text{repulsive cases:} \quad W(x) = \frac{c_{\alpha,d}}{|x|^\alpha} \quad \text{with } -1 \le \alpha < d-1$$

$$\text{attractive cases:} \quad W(x) = -\frac{c_{\alpha,d}}{|x|^\alpha} \quad \text{with } -1 \le \alpha \le d-2$$

$$\text{repulsive/attractive cases:} \quad |W(x)| \le \frac{C}{|x|^\alpha} \quad \text{with } -1 \le \alpha < d-2$$

- Vlasov–Poisson–Fokker–Planck system $\longrightarrow$ Keller–Segel equations

Idea of Proof

(Step 1) Intermediate equation via a *coarse-graining map*:

$$\Gamma^\varepsilon : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d, \qquad \Gamma^\varepsilon(x, v) = (x + \varepsilon v, v).$$

Set

$$\bar{\rho}^\varepsilon := \int f^\varepsilon(x - \varepsilon v, v)\, dv.$$

Then $\bar{\rho}^\varepsilon$ satisfies

$$\partial_t \bar{\rho}^\varepsilon + \nabla_x \cdot \bar{\jmath}^\varepsilon = \Delta_x \bar{\rho}^\varepsilon\,,$$
$$\bar{\jmath}^\varepsilon(x, t) = \int \mathrm{F}(x - \varepsilon v, \rho^\varepsilon) f^\varepsilon(x - \varepsilon v, v, t)\, dv.$$

(Step 2) Error estimate between **VFP** and the intermediate equations:

$$\mathrm{d}_2^2(\bar{\rho}^\varepsilon, \rho^\varepsilon) \le \varepsilon^2 \iint |v|^2 f^\varepsilon\, dxdv\,.$$

We recall

$$\rho^\varepsilon := \int f^\varepsilon(x, v)\, dv.$$

Idea of Proof

(Step 3) Weighted L^p-norm by the exponential of Hamiltonian $H(x,v) = V(x) + |v|^2/2$:

$$\|f\|_{L^p_H} := \left(\iint f^p e^{(p-1)H}\, dxdv \right)^{1/p}$$

$$\|f\|_{W^{k,p}_{x,H}} := \left(\sum_{|\alpha|\le k} \iint |\nabla_x^\alpha f|^p e^{(p-1)H}\, dxdv \right)^{1/p}$$

Uniform-in-ε estimates:

$$\sup_{0\le t\le T_*} \|f^\varepsilon\|_{W^{1,p}_{x,H}} \le C$$

for some $C > 0$, $T_* > 0$ independent of $\varepsilon \le 1$. This yields

$$\rho^\varepsilon,\ \bar\rho^\varepsilon \in \mathcal{W}^{1,p}(\mathbb{R}^d),$$

and by Morrey's inequality, for $p > d$

$$\rho^\varepsilon,\ \bar\rho^\varepsilon \in L^\infty(\mathbb{R}^d).$$

Idea of Proof

(Step 4) Error estimate between the intermediate and **AD** equations (Evolution-Variational Inequality):

$$\frac{1}{2}\frac{d}{dt}\,\mathrm{d}_2^2(\bar{\rho}^{\varepsilon},\rho) \leq \lambda\,\mathrm{d}_2^2(\bar{\rho}^{\varepsilon},\rho) - 2\mathscr{D}_W(\bar{\rho}^{\varepsilon},\rho) + \frac{1}{2}\|\mathrm{e}^{\varepsilon}\|_{L^2(\bar{\rho}^{\varepsilon})}^2$$

Remark. gradient-flow structure of **AD** equation

- $\mathscr{D}_W$: modulated interaction energy given by

$$\mathscr{D}_W(\mu,\nu) := \iint W(x-y)(\mu-\nu)(dy)(\mu-\nu)(dx) \qquad \text{for } \mu,\nu \in \mathcal{P}(\mathbb{R}^d)$$

- e^{ε}: error term given by

$$\mathrm{e}^{\varepsilon}(x) := \frac{d\bar{\jmath}^{\varepsilon}}{d\bar{\rho}^{\varepsilon}}(x) - \mathrm{F}(x,\bar{\rho}^{\varepsilon}) \qquad \text{for } \bar{\rho}^{\varepsilon}\text{-almost every } x \in \mathbb{R}^d$$

On the estimate of e^{ε}:

$$\|\mathrm{e}^{\varepsilon}\|_{L^2(\bar{\rho}^{\varepsilon})}^2 \leq C\varepsilon^2\,(1+M(f^{\varepsilon}))^2,$$

where

$$M(f^{\varepsilon}) := \|f^{\varepsilon}\|_{W^{1,q'}_{x,H}} + \iint |v|\, f^{\varepsilon}\, dxdv.$$

Idea of Proof

On the estimates of $\mathscr{D}_W$:

1. Smooth interaction: If ∇W is globally Lipschitz, then

$$|\mathscr{D}_W(\mu,\nu)| \leq \|\nabla W\|_{\mathrm{Lip}} \, \mathrm{d}_2^2(\mu,\nu)\,.$$

2. Weakly singular interaction: If $\nabla^2 W \in L^1(\mathbb{R}^d)$, then

$$\|\nabla W \star (\mu-\nu)\|_{L^2} \leq \|\nabla^2 W\|_{L^1} \, \mathrm{d}_2(\mu,\nu)\,.$$

   In particular, we obtain

$$|\mathscr{D}_W(\mu,\nu)| \leq \|\nabla W \star (\mu-\nu)\|_{L^2(\mathbb{R}^d)} \|\mu-\nu\|_{\dot{H}^{-1}} \leq \sqrt{c_\infty}\, \|\nabla^2 W\|_{L^1} \, \mathrm{d}_2^2(\mu,\nu),$$

   where $c_\infty := \max\{\|\mu\|_{L^\infty}, \|\nu\|_{L^\infty}\}$.
3. Repulsive case: $\mathscr{D}_W \geq 0$.
4. Newtonian attractive: When W is the fundamental solution of the Laplacian, i.e. $\Delta W = \delta_0$, $\mathscr{D}_W$ takes the alternative form

$$\mathscr{D}_W(\mu,\nu) = -\int |\nabla W \star (\mu-\nu)|^2 \, d\mathscr{L}^d = -\|\mu-\nu\|_{\dot{H}^{-1}}^2\,,$$

   from which we obtain

$$\mathscr{D}_W(\mu,\nu) \geq -c_\infty \, \mathrm{d}_2^2(\mu,\nu)\,.$$

Remarks

- Regular case: $\nabla W \in L^\infty \cap \mathrm{Lip}(\mathbb{R}^d)$

  $(f^\varepsilon)_{\varepsilon \le 1} \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$, $\rho \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$ + entropy inequality $\to$ error estimate in 2-Wasserstein distance

- Irregular case: $\nabla W \in L^\infty(\mathbb{R}^d)$

  $e^{-V} \in L^1(\mathbb{R}^d)$ $\to$ error estimate in bounded and Lipschitz distance

Conclusion

Summary:

- Part I: Quantified hydrodynamic limit from kinetic to isothermal/pressureless Euler-type equations

- Part II: Quantified overdamped limit from kinetic to aggregation-diffusion equations

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Thank you for your attention.