The rigorous derivation of the Boltzmann equation:

how to generalize Lanford's theorem in various domains?

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Plan of the talk

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1. Introducing the Boltzmann equation and Lanford's theorem
   - The Boltzmann equation
   - From the dynamics of the particles to a statistical description of the system
   - The observation of Grad: a way to obtain a rigorous derivation
   - The convergence of the solutions
2. The extensions of Lanford's theorem to domains with boundary
   - Prescribing the boundary conditions
   - The case of the half-space
   - The case of a general convex obstacle

Statistical mechanics: the description of the matter at a mesoscopic level

Goal:

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The fluid will be described by the quantity $f(t,x,v)$, the density of particles lying at time t at point x and moving with velocity v.

f is called the *one-particle density function in the phase space*.

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$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d_{v_*}} \int_{\mathbb{S}^{d-1}_\omega} B(v - v_*, \omega) \Big[f(t,x,v') f(t,x,v'_*) - f(t,x,v) f(t,x,v_*) \Big] \, \mathrm{d}\omega \, \mathrm{d}v_*,$$

For a solution f of the Boltzmann equation, if one considers the *entropy*:

$$H(f)(t) = \int_x \int_v f(t,x,v) \ln f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x,$$

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one can prove that, if f is not an equilibrium (i.e. a Maxwellian), then:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f)(t) = -\frac{1}{4} \int_x \int_v \int_{v_*} \int_\omega B(v - v_*, \omega) \big(f(v') f(v'_*) - f(v) f(v_*) \big) \times \ln \left(\frac{f(v') f(v'_*)}{f(v) f(v_*)} \right) \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x < 0.$$

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This is the H-theorem (1872).

Defining the dynamics of the particles: the hard sphere model

One assumes that the gas is monoatomic and electrically neutral. The gas is composed of spherical particles of diameter ε, evolving in a domain without boundary: the Euclidean space $\mathbb{R}^d$ ($d \geq 2$), or the torus $\mathbb{T}^d$. The position of the particle i at time t will be denoted $x_i(t)$, and its velocity at time t $v_i(t)$.

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$$\text{(so that locally } x_i(t) = x_i(t_0) + (t - t_0) v_i(t_0)\text{).}$$

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If $d(x_i(t), x_j(t)) = \varepsilon$, we define

$$\begin{cases} v_1' = v_1 - (\omega \cdot (v_1 - v_2))\omega, \\ v_2' = v_2 + (\omega \cdot (v_1 - v_2))\omega \end{cases}$$

$$\text{with } \omega = \frac{x_2 - x_1}{|x_2 - x_1|}.$$

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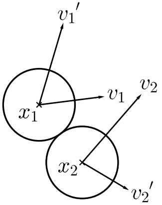

Figure: Collision between two particles

Introducing the BBGKY hierarchy

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One denotes:

$$Z_N = (x_1, v_1, \dots, x_N, v_N) = (z_1, \dots, z_N) \in \mathbb{R}^{2dN},$$

with $z_i = (x_i, v_i) \in \mathbb{R}^{2d}$, and

$$\mathcal{D}_N^\varepsilon = \Big\{ Z_N \in \mathbb{R}^{2dN} / \ \forall i \neq j,\ |x_i - x_j| > \varepsilon \Big\}.$$

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$$f_N(t, x_1, v_1, \dots, x_i, v_i, \dots, x_j, v_j, \dots, x_N, v_N) \\ = f_N(t, x_1, v_1, \dots, x_i, v_i', \dots, x_j, v_j', \dots, x_N, v_N)$$

when $|x_i - x_j| = \varepsilon$ and $(x_i - x_j) \cdot (v_i - v_j) > 0$.

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Introducing the marginals $f^{(s)}_N$ of the distribution function:

$$f^{(s)}_N(Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N) \mathbb{1}_{\mathcal{D}^\varepsilon_N} \, \mathrm{d}z_{s+1} \dots \mathrm{d}z_N,$$

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one can show that each marginal satisfies the equation (for $1 \leq s \leq N-1$):

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where $\mathcal{C}^{N,\varepsilon}_{s,s+1}$ is the collision term, which writes:

$$\mathcal{C}^{N,\varepsilon}_{s,s+1} f^{(s+1)} = \sum_{i=1}^{s} (N-s)\varepsilon^{d-1} \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^d_{v_{s+1}}} \omega \cdot (v_{s+1} - v_i) \times f^{(s+1)}_N(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \, \mathrm{d}\omega \, \mathrm{d}v_{s+1}.$$

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Those N equations constitute the BBGKY hierarchy.

The Boltzmann-Grad limit, and the Boltzmann hierarchy

So far, no link was given between the number N of particles of the system, and the radius $\varepsilon/2$ of those particles.

The Boltzmann-Grad limit, and the Boltzmann hierarchy

One will consider the *Boltzmann-Grad* limit:

$$N\varepsilon^{d-1} = 1.$$

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This means that the mean free path does not depend on the number N of particles of the system.

The Boltzmann-Grad limit, and the Boltzmann hierarchy

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Taking the limit $\varepsilon \to 0$, $N\varepsilon^{d-1} = 1$, the collision term becomes (formally):

$$\sum_{i=1}^{s} \int_{\mathbb{S}_\omega^{d-1}} \int_{\mathbb{R}^d_{v_{s+1}}} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ f_N^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i', \ldots, x_i, v_{s+1}') \,\mathrm{d}\omega \,\mathrm{d}v_{s+1}$$
$$- \sum_{i=1}^{s} \int_{\mathbb{S}_\omega^{d-1}} \int_{\mathbb{R}^d_{v_{s+1}}} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ f_N^{(s+1)}(t, Z_s, x_i, v_{s+1}) \,\mathrm{d}\omega \,\mathrm{d}v_{s+1}.$$

The Boltzmann-Grad limit, and the Boltzmann hierarchy

One will consider the *Boltzmann-Grad* limit:

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One defines the Boltzmann hierarchy as the *infinite* sequence of equations:

$$\forall s \geq 1,\ \partial_t f^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f^{(s)} = \mathcal{C}^0_{s,s+1} f^{(s+1)},$$

with $\mathcal{C}^0_{s,s+1} f^{(s+1)}$ denoting

$$\sum_{i=1}^{s} \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^d_{v_{s+1}}} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ \big(f^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i', \ldots, x_i, v_{s+1}') \\ - f^{(s+1)}(t, Z_s, x_i, v_{s+1})\big) \,\mathrm{d}v_{s+1}\,\mathrm{d}\omega.$$

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if one assumes in addition that the second marginal is tensorized:

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<u>Goal</u>: proving the convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy.

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For the Boltzmann hierarchy, the hard sphere transport is replaced by the free transport with boundary conditions $T^{s,0}$.

An explicit expression of the solutions to the hierarchies

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Notations for the integrated in time transport-collision operator for the BBGKY hierarchy. For any positive integers N and s, and any sequence of functions $\big(f_N^{(s)}\big)_{1\leq s\leq N}$ belonging to the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$, we will denote the function

$$t \mapsto \int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}f_N^{(s+1)}(u,\cdot)\,\mathrm{d}u$$

as $\mathcal{I}_s^{N,\varepsilon}f_N^{(s+1)}$.

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as $\mathcal{I}_s^{N,\varepsilon} f_N^{(s+1)}$.

Similarly, for the Boltzmann hierarchy we will denote

$$t \mapsto \int_0^t \mathcal{T}_{t-u}^{s,0}\mathcal{C}_{s,s+1}^{0} f^{(s+1)}(u,\cdot)\,\mathrm{d}u$$

as $\mathcal{I}_s^0 f^{(s+1)}$ for any sequence of functions $\left(f^{(s)}\right)_{s\geq 1}$ belonging to $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$.

An explicit expression of the solutions to the hierarchies

Few notations (II)

The iterations of those operators

$$t \mapsto \int_0^t \mathcal{T}^{s,\varepsilon}_{-t_1} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_{t_1} \int_0^{t_1} \mathcal{T}^{s+1,\varepsilon}_{-t_2} \mathcal{C}^{N,\varepsilon}_{s+1,s+2} \mathcal{T}^{s+2,\varepsilon}_{t_2} \dots \int_0^{t_{k-1}} \mathcal{T}^{s+k-1,\varepsilon}_{-t_k} \mathcal{C}^{N,\varepsilon}_{s+k-1,s+k} \mathcal{T}^{s+k,\varepsilon}_{t_k} f_N^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k \dots \mathrm{d}t_2\,\mathrm{d}t_1$$

and

$$t \mapsto \int_0^t \mathcal{T}^{s,0}_{-t_1} \mathcal{C}^{0}_{s,s+1} \mathcal{T}^{s+1,0}_{t_1} \int_0^{t_1} \mathcal{T}^{s+1,0}_{-t_2} \mathcal{C}^{0}_{s+1,s+2} \mathcal{T}^{s+2,0}_{t_2} \dots \int_0^{t_{k-1}} \mathcal{T}^{s+k-1,0}_{-t_k} \mathcal{C}^{0}_{s+k-1,s+k} \mathcal{T}^{s+k,0}_{t_k} f^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k \dots \mathrm{d}t_2\,\mathrm{d}t_1$$

An explicit expression of the solutions to the hierarchies

Few notations (II)

The iterations of those operators

$$t \mapsto \int_0^t \mathcal{T}^{s,\varepsilon}_{-t_1} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_{t_1} \int_0^{t_1} \mathcal{T}^{s+1,\varepsilon}_{-t_2} \mathcal{C}^{N,\varepsilon}_{s+1,s+2} \mathcal{T}^{s+2,\varepsilon}_{t_2} \dots \int_0^{t_{k-1}} \mathcal{T}^{s+k-1,\varepsilon}_{-t_k} \mathcal{C}^{N,\varepsilon}_{s+k-1,s+k} \mathcal{T}^{s+k,\varepsilon}_{t_k} f_N^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k \dots \mathrm{d}t_2\,\mathrm{d}t_1$$

and

$$t \mapsto \int_0^t \mathcal{T}^{s,0}_{-t_1} \mathcal{C}^{0}_{s,s+1} \mathcal{T}^{s+1,0}_{t_1} \int_0^{t_1} \mathcal{T}^{s+1,0}_{-t_2} \mathcal{C}^{0}_{s+1,s+2} \mathcal{T}^{s+2,0}_{t_2} \dots \int_0^{t_{k-1}} \mathcal{T}^{s+k-1,0}_{-t_k} \mathcal{C}^{0}_{s+k-1,s+k} \mathcal{T}^{s+k,0}_{t_k} f^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k \dots \mathrm{d}t_2\,\mathrm{d}t_1$$

will be respectively denoted as

$$\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f_N^{(s+k)} \text{ and } \mathcal{I}^{0}_{s,s+k-1} f^{(s+k)}.$$

The detailed expression of the elementary terms

The Duhamel formula (BBGKY version)

It is then possible to prove the following result, giving an explicit expression of the solutions to the hierarchy in terms of the initial data.

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Iterated Duhamel formula for the solution of the BBGKY hierarchy

Let N be a positive integer and $\varepsilon > 0$. In the Boltzmann-Grad limit $N\varepsilon^{d-1} = 1$, for any strictly real numbers $\beta_0 > 0$, μ_0, and any sequence of initial data

$$F_{N,0} = \left(f_{N,0}^{(s)}\right)_{1 \leq s \leq N} \in \mathbf{X}_{N,\varepsilon,\beta_0,\mu_0},$$

the unique solution of the integrated form of the conjugated BBGKY hierarchy with initial datum $F_{N,0}$ is

$$H_N = t \mapsto \left(f_{N,0}^{(s)} + \sum_{k=1}^{N-s} \left(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)}\right)(t,\cdot)\right)_{1 \leq s \leq N}.$$

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Iterated Duhamel formula for the solution of the Boltzmann hierarchy

For any strictly real numbers $\beta_0 > 0$, μ_0, and any sequence of initial data

$$F_0 = \left(f_0^{(s)} \right)_{s \geq 1} \in \mathbf{X}_{0,\beta_0,\mu_0},$$

the unique solution of the integrated form of the Boltzmann hierarchy with initial datum F_0 is

$$F = t \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{+\infty} \mathcal{I}^0_{s,s+k-1} \left(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)} \right)(t,\cdot) \right)_{s \geq 1}.$$

The detailed expression of the elementary terms

Another step forward into the decomposition

Keeping in mind that the collision operator was defined as

$$\mathcal{C}^0_{s,s+1} f^{(s+1)} = \sum_{i=1}^{s} \Bigg[\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega\cdot(v_{s+1}-v_i)\big]_+ f^{(s+1)}(t,\dots,x_i,v_i',\dots,x_i,v_{s+1}') - \big[\omega\cdot(v_{s+1}-v_i)\big]_- f_N^{(s+1)}(t,Z_s,x_i,v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1} \Bigg],$$

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this suggests the decomposition:

$$\int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1} \mathcal{T}^{s+1,0}_{t_1} f_0^{(s+1)}(t_1, \cdot)\,\mathrm{d}t_1 = \sum_{j_1=1}^{s} \Big(\mathcal{I}^0_{\substack{s\\+,j_1}} - \mathcal{I}^0_{\substack{s\\-,j_1}} \Big) \big(t_1 \mapsto \mathcal{T}^{s+1,0}_{t_1} f_0^{(s+1)} \big),$$

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and then

$$\mathcal{I}^0_{s,s+k-1} = \Big(\sum_{\substack{1\leq j_1\leq s\\ \pm_1}} (\pm_1)\mathcal{I}^0_{\substack{s\\ \pm_1,j_1}}\Big)\circ\cdots\circ\Big(\sum_{\substack{1\leq j_k\leq s+k-1\\ \pm_k}} (\pm_k)\mathcal{I}^0_{\substack{s+k-1\\ \pm_k,j_k}}\Big).$$

The detailed expression of the elementary terms

The final definition of the elementary terms

The final decomposition of the solution into *elementary terms* (for example, of the Boltzmann hierarchy) will be:

$$F = \Big(\mathcal{T}_t^{s,0} f_0^{(s)} + \sum_{k=1}^{+\infty} \sum_{J_k, M_k} (\pm_1) \dots (\pm_k) \mathcal{I}^0_{\substack{s,s+k-1 \\ J_k, M_k}} \big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)} \big) \Big)_{s \geq 1},$$

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with $J_k = (j_1, \dots, j_k)$ and $M_k = (\pm_1, \dots, \pm_k)$, and $s \leq j_l \leq s + l - 1$.

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What can be said about a generic elementary term?

From the operators to the pseudo-trajectories

Let us take $k = 1$, $m_1 = -$ (and s and j_1 being generic), that is we consider $\mathcal{I}^0_{\substack{s,s \\ (j_1,-)}}\left(u \mapsto \mathcal{T}^{s+1,0}_u f^{(s+1)}_0\right)$.

$$\int_0^t \int_\omega \int_{v_{s+1}} \Big[\omega \cdot \Big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j}\Big)\Big]_- \\ \times f^{(s+1)}_0\Big(T^{s+1,0}_{-t_1}\Big(T^{s,0}_{t_1-t}(Z_s), \Big(T^{s,0}_{t_1-t}(Z_s)\Big)^{X,j}, v_{s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$

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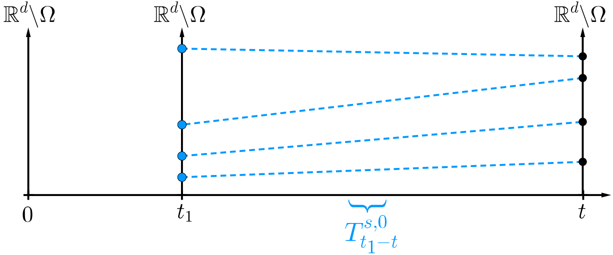
$$\int_0^t \int_\omega \int_{v_{s+1}} \Big[\omega \cdot \Big(v_{s+1} - \big(\overbrace{T^{s,0}_{t_1-t}(Z_s)}^{②}\big)^{V,j}\Big)\Big]_- \\ \times f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\big(\underbrace{T^{s,0}_{t_1-t}(Z_s)}_{②}, \big(\underbrace{T^{s,0}_{t_1-t}(Z_s)}_{②}\big)^{X,j}, v_{s+1}\big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$



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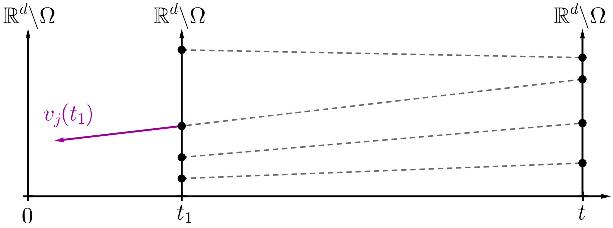

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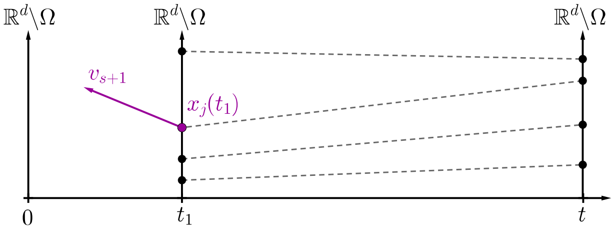

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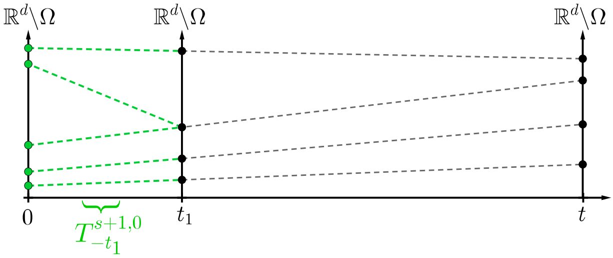

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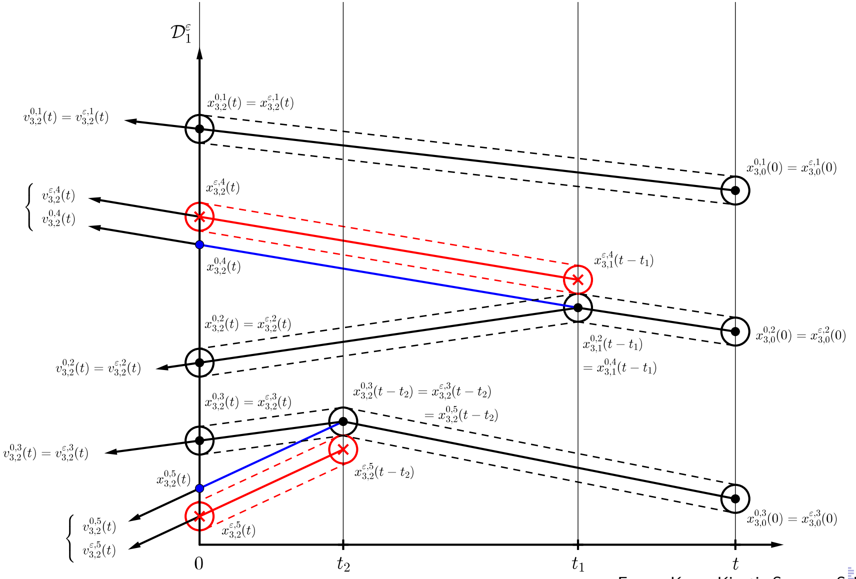
We are then naturally led to consider pseudo-trajectories.

The plan for the convergence of the solutions

The behaviour of the pseudo-trajectories



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The main idea of the proof is now to proceed as follows:

The plan for the convergence of the solutions

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For the same adjunction parameters J_k, $(t_1, \ldots, t_k)$, $(\omega_1, \ldots, \omega_k)$ and $(v_1, \ldots, v_k)$:

Pseudo-trajectory of the BBGKY hierarchy

$\downarrow$ as $\varepsilon \to 0$

Pseudo-trajectory of the Boltzmann hierarchy

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$\Rightarrow$

For any number of adjunction k, any parameters of adjunction $1 \leq j_k \leq s + k - 1$ and $m_k = \pm_k$:

$$\mathcal{I}^{N,\varepsilon}_{\substack{s,s+k-1\\J_k,M_k}}$$

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$\Rightarrow$

Since the solutions are sums of the elementary terms, the ones of the BBGKY hierarchy converging towards those of the Boltzmann hierarchy:

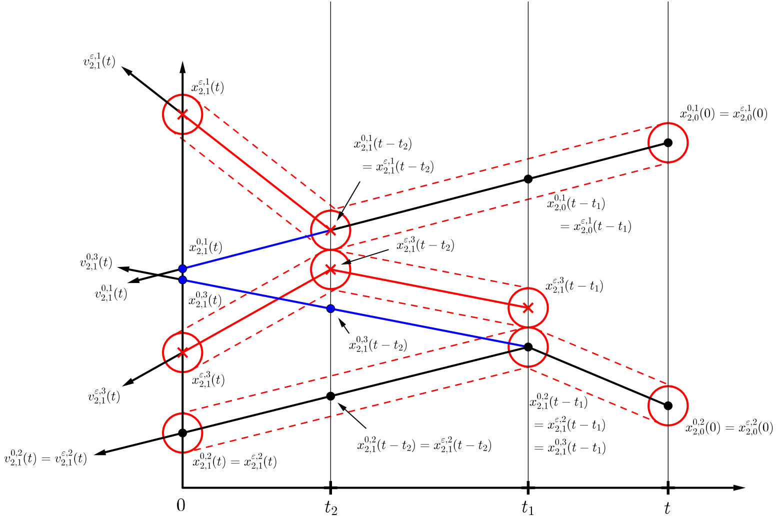
F_N

$\downarrow$ as $\varepsilon \to 0$

F.

A serious obstacle to the convergence of the pseudo-trajectories:

The recollisions



The elimination of the recollisions: the geometrical lemma

The decisive contribution of Gallagher, Saint-Raymond and Texier (2014) (I)

We are now at the heart of the control of the recollisions. It relies on the following lemma of geometry:

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Shooting lemma [Gallagher, Saint-Raymond, Texier 2014]

Let ε, a and ε_0 be three positive numbers such that $\varepsilon \ll a \ll \varepsilon_0$. Let $\overline{x}_1$ and $\overline{x}_2$ be two vectors of $\mathbb{R}^d$ such that $|\overline{x}_2 - \overline{x}_1| \geq \varepsilon_0$, and v_1 a vector of $B(0,R)$ such that $|v_1| \leq R$. Then, for any $x_1 \in B(\overline{x}_1, a)$, $x_2 \in B(\overline{x}_2, a)$ and $v_2 \in B(0,R)$:

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- if v_2 is not in the cylinder of radius $6Ra/\varepsilon_0$ and of axis $v_1 + \text{Vect}(\overline{x}_2 - \overline{x}_1)$, then:

$$\forall t \geq 0,\ |(x_1 - tv_1) - (x_2 - tv_2)| > \varepsilon,$$

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- if v_2 is not in the cylinder of radius $6\varepsilon_0/\delta$ and of axis $v_1 + \text{Vect}(\overline{x}_2 - \overline{x}_1)$, then:

$$\forall t \geq \delta,\ |(\overline{x}_1 - tv_1) - (\overline{x}_2 - tv_2)| > \varepsilon_0.$$

The elimination of the recollisions: the stability of the good configurations by adjunction

The decisive contribution of Gallagher, Saint-Raymond and Texier (2014) (II)

With the shooting lemma, we can now prove that:

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If $\overline{Z}_k$ is a good configuration of type ε_0, there exists a subset $\mathcal{B}_k(\overline{Z}_k) \subset \mathbb{S}^{d-1} \times B(0,R)$ of small measure:

$$|\mathcal{B}_k(\overline{Z}_k)| \leq Ck\Big(R\eta^{d-1} + R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\Big)$$

The elimination of the recollisions: the stability of the good configurations by adjunction

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With the shooting lemma, we can now prove that:

Stability of the good configurations [Gallagher, Saint-Raymond, Texier 2014]

If $\overline{Z}_k$ is a good configuration of type ε_0, there exists a subset $\mathcal{B}_k(\overline{Z}_k) \subset \mathbb{S}^{d-1} \times B(0,R)$ of small measure:

$$|\mathcal{B}_k(\overline{Z}_k)| \leq Ck\Big(R\eta^{d-1} + R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\Big)$$

such that if $(\omega, v_{k+1}) \notin \mathcal{B}_k(\overline{Z}_k)$, then:

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- the configuration $(\overline{Z}_k, \overline{x}_k, v_{k+1})$ is a good configuration of type ε_0 after at most δ,

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such that if $(\omega, v_{k+1}) \notin \mathcal{B}_k(\overline{Z}_k)$, then:

- the configuration $(\overline{Z}_k, \overline{x}_k, v_{k+1})$ is a good configuration of type ε_0 after at most δ,
- for all $X_k \in B(\overline{X}_k, a)$, the configuration $(X_k, \overline{V}_k, \overline{x}_k + \varepsilon\omega, v_{k+1})$ is a good configuration of type ε.

The elimination of the recollisions: final comments

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The stability of the good configurations shows that, except for a small amount of adjunction parameters, the pseudo-trajectories that are built are without recollision.

The elimination of the recollisions: final comments

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The stability of the good configurations shows that, except for a small amount of adjunction parameters, the pseudo-trajectories that are built are without recollision.

As a consequence, the difference between the positions of the pseudo-trajectories of the BBGKY and the Boltzmann hierarchies are only due to the size of the particles, and is then given by:

$$k\varepsilon$$

after the k-th adjunction.

We completed our program!

The Lanford's theorem in the Euclidean space

Theorem [Lanford 1975], [Gallagher, Saint-Raymond, Texier 2014]

Let $f_0 : \mathbb{R}^{2d} \to \mathbb{R}_+$ be a continuous density of probability such that

$$\left\| f_0(x,v) \exp\left(\frac{\beta}{2}|v|^2\right) \right\|_{L^\infty(\mathbb{R}^{2d})} < +\infty$$

for some $\beta > 0$.

Then, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, $f_N^{(1)}$ converges towards the solution f of the Boltzmann equation with the cross section $b(v,\omega) = (v \cdot \omega)_+$ with f_0 as initial data, in the following sense. For all compact set $K \subset \mathbb{R}^d$:

$$\left\| \mathbb{1}_K(x) \int_{\mathbb{R}^d_v} \varphi(v) \big(f_N^{(1)} - f \big)(x,v) \, \mathrm{d}v \right\|_{L^\infty([0,T]\times\mathbb{R}^d_x)} \underset{N\to+\infty}{\longrightarrow} 0.$$

If in addition f_0 is Lipschitz-continuous, the rate of convergence is of order $O(\varepsilon^a)$ with

$$a < \frac{d-1}{d+1}.$$

Plan of the talk

1. Introducing the Boltzmann equation and Lanford's theorem
   - The Boltzmann equation
   - From the dynamics of the particles to a statistical description of the system
   - The observation of Grad: a way to obtain a rigorous derivation
   - The convergence of the solutions

2. The extensions of Lanford's theorem to domains with boundary
   - Prescribing the boundary conditions
   - The case of the half-space
   - The case of a general convex obstacle

Prescribing the boundary conditions

Defining the dynamics of the particles (bis): the hard spheres, with specular reflection

We now assume that the particles evolve outside an obstacle $\Omega \in \mathbb{R}^d$. Far enough from this obstacle, we keep the same assumptions concerning the dynamics of the particles.

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Far enough from the obstacle (i.e. when $d(\Omega, x_i(t)) > \varepsilon/2$) and from the other particles (i.e. when $d(x_i(t), x_j(t)) > \varepsilon$ for $j \neq i$), each particle i moves in straight line, with constant velocity.

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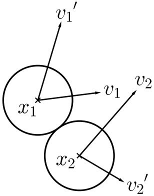

$$\begin{cases} v_1' &= v_1 - (\omega \cdot (v_1 - v_2))\omega, \\ v_2' &= v_2 + (\omega \cdot (v_1 - v_2))\omega, \end{cases}$$

with $\omega = \dfrac{x_2 - x_1}{|x_2 - x_1|}$

Figure: Collision between two particles: $|x_1 - x_2| = \varepsilon$

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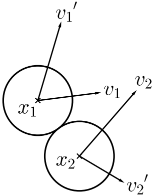

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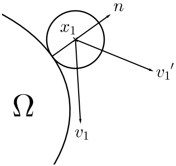

$$v_1' = v_1 - 2(v_1 \cdot n)n$$

Figure: Bouncing against the obstacle : $d(x_1, \Omega) = \varepsilon/2$

A first case of a domain with boundary: the case of the half-space

What does it change?

We assume here that $\Omega = \{(x_1, \dots, x_d) \in \mathbb{R}^d \ / \ x_1 \leq 0\}$.

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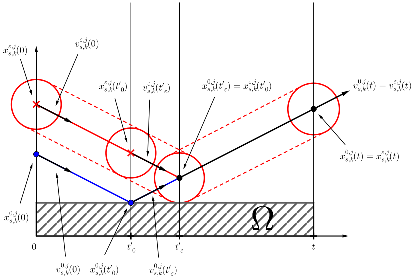

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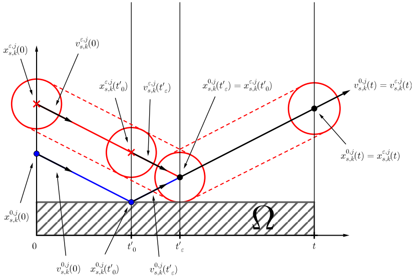

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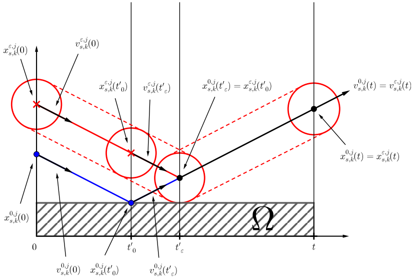

→ The bouncings increase the distance between the particles,

→ There are time intervals during which the velocities are *very* different.

A first case of a domain with boundary: the case of the half-space

A more complicated (but solvable) shooting lemma

In the case of the half-space, there are more possibilities for two particles to collide:

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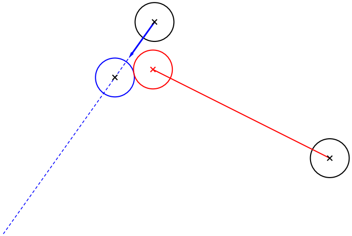

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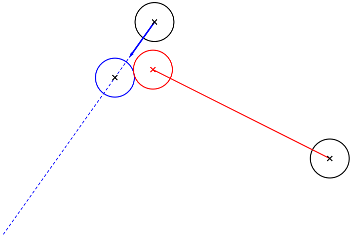

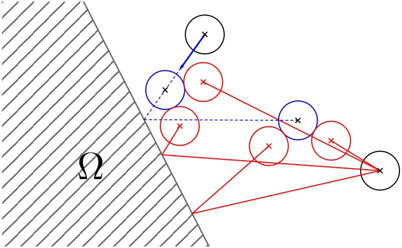

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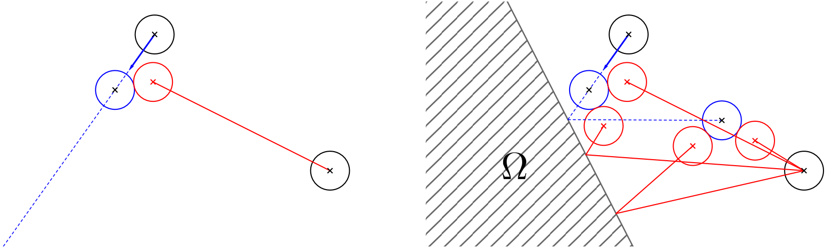

There are more cases to consider to obtain an analogous shooting lemma.

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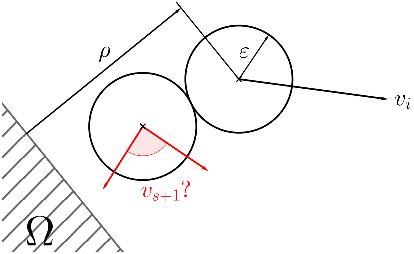
An important obstruction in the stability of the good configurations



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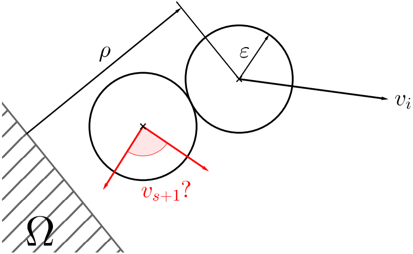

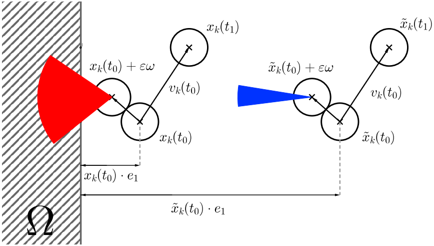

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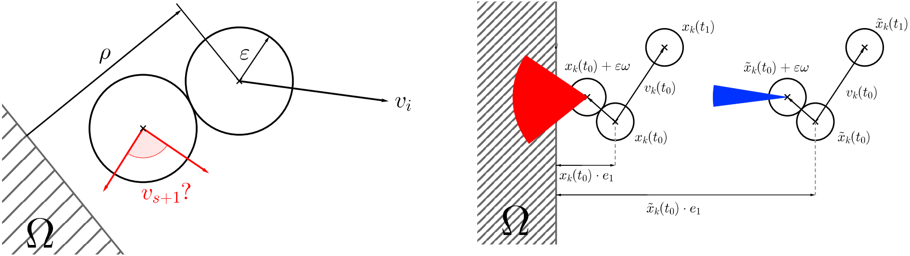

In the case when there is an obstacle, one has to introduce a cut-off on the proximity between the obstacle and the particle undergoing an adjunction.

A first case of a domain with boundary: the case of the half-space

An extended proof to take into account the cut-off in proximity with obstacle

In the case when there is an obstacle, the particle that undergoes the adjunction has to be at a distance at least ρ from this obstacle.

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An extended proof to take into account the cut-off in proximity with obstacle

In the case when there is an obstacle, the particle that undergoes the adjunction has to be at a distance at least ρ from this obstacle.

How could we take this condition into account, since the positions are not part of the adjunction parameters?

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It is not possible to prevent the particles to be too close to the obstacle in general. But actually, it is sufficient that the particle experiencing the adjunction is far from the obstacle *at the time of adjunction*.

Therefore, we can exclude the times such that the chosen particle is too close to the obstacle. But this amount of times can be huge, if the particle is grazing the obstacle!

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One has to make sure in addition that no particle of the system has a grazing velocity, which implies another cut-off.

A first case of a domain with boundary: the case of the half-space

Lanford's theorem in the half-space with specular reflexion, [D. 2019]

Let $f_0 : \{x \in \mathbb{R}^d\} \times \mathbb{R}^d \to \mathbb{R}_+$ be a continuous density of probability such that

$$f(x,v) \underset{|(x,v)|\to+\infty}{\longrightarrow} 0 \text{ and } \left\| f_0(x,v) \exp\left(\frac{\beta}{2}|v|^2\right)\right\|_{L^\infty(\mathbb{R}^{2d})} < +\infty$$

for some $\beta > 0$. Consider the system of N hard spheres of diameter ε inside the half-space with specular reflexion, initially distributed according to f_0 and independent. Then, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, its distribution function $f_N^{(1)}$ converges to the solution of the Boltzmann equation f with the cross section $b(v,\omega) = (v\cdot\omega)_+$, with specular reflexion and with initial data f_0, in the following sense:

$$\left\| \mathbb{1}_K(x,v)\big(f_N^{(1)} - f\big)(x,v)\right\|_{L^\infty([0,T]\times\{x\cdot e_1>0\}\times\{v\cdot e_1\neq 0\})} \underset{N\to+\infty}{\longrightarrow} 0.$$

If in addition $\sqrt{f_0}$ is Lipschitz with respect to the position variable uniformly in the velocity variable, the rate of convergence is $O(\varepsilon^a)$ with $a < 13/128$.

Towards more general domains: outside a convex obstacle

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But in that general case, the trajectories are *not explicit anymore*!
$\Rightarrow$ It is much more complicated to obtain the shooting lemma in that case.

Solving the shooting lemma outside a general convex obstacle

Characterizing the velocities solving the shooting problem

In the case of the whole Euclidean space, or of the half-space, the set of velocities of a particle travelling from a disk to another one are easily pictured: this set is a cone.

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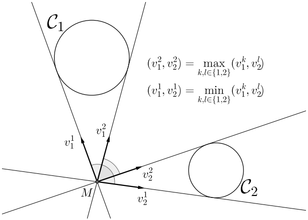
We introduce the notion of *interior and exterior pairs of tangent lines* to two disks.



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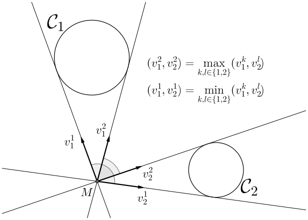

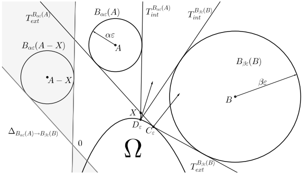

Such pairs exist, are unique, and the velocities solving the shooting lemma are contained between those lines.

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It remains to control the angle between the interior and the exterior tangent lines to the disk containing the starting point.

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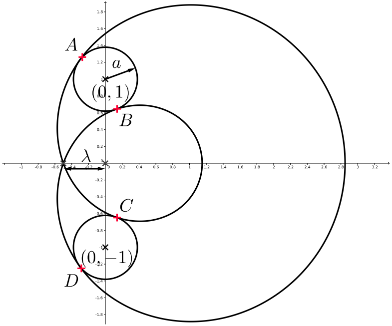
We can reach that goal using the family of isoptics of the two disks.



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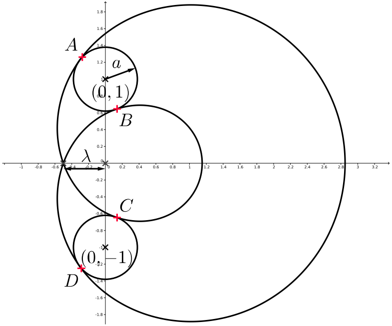

Shooting lemma [D., to be published]

Let $a \neq 1$, $0 < \beta < 1$, and $u \geq a^\beta$. We assume that there is a trajectory, starting from $x_1 = (0, 1)$ and reaching $x_2 = (0, -1)$, with a bouncing at (u, v). The set of velocities of the trajectories starting from $B(x_1, a)$ and reaching $B(x_2, a)$ after a bouncing has a size bounded by:

$$C(\beta)a^{1-\beta} + \frac{2M}{\cos\theta}a + o(a/d).$$

Works in progress and open questions

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The case of the general convex obstacle is performed only in the two-dimensional case.
Besides even more general obstacles, one can study other boundary conditions. For example, the case of the diffusive boundary condition turns out to be very difficult. Indeed, the very first steps (well-posedness of the dynamics of the particles, analog of the BBGKY hierarchy?) of Lanford's program seem hard to tackle.