Mean-field limits and emergent dynamics of the classical and quantum many-body systems Lecture 1

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What is a Mean-Field Kinetic Model?

System of N identical particles, with pairwise interactions; $N \gg 1$ (e.g. $N =$ Avogadro number $\simeq 6.02 \cdot 10^{23}$...)

Dynamics described either

(a) by Newton's 2nd law of motion for each particle, or
(b) by the motion equation for the "typical particle" driven by the collective interaction with all the other particles

Approach (b) is usually referred to as a mean-field kinetic model

System of N identical point particles of mass m, spatial domain $\mathbf{R}^d$

Newton's second law for the motion of the kth particle:

$$m\dot{x}_j = \xi_j\,, \qquad \dot{\xi}_j = \sum_{\substack{k=1\\k\neq j}}^{N} -\nabla \underbrace{V(x_j - x_k)}_{\text{interaction potential}}\,, \qquad 1 \le j \le N$$

Assumptions on V

(H1) $\quad V(z) = V(-z) \quad$ for all $z \in \mathbf{R}^d$

(H2) $\quad V \in C^1(\mathbf{R}^d)$ with $\nabla V \in L^\infty(\mathbf{R}^d) \cap \mathrm{Lip}(\mathbf{R}^d)$

Notation set $X_N := (x_1, \dots, x_N)$ and $\Xi_N := (\xi_1, \dots, \xi_N)$ in $\mathbf{R}^{dN}$

Solution of the differential system with initial data (X_N^{in}, Ξ_N^{in})

$$(X_N, \Xi_N)(t) = \Phi_N(t; X_N^{in}, \Xi_N^{in})$$

Rescaled time, position and momentum:

$$\hat{t} = t/N\,, \quad \hat{x}_j(\hat{t}) = x_j(t)\,, \quad \hat{\xi}_j(\hat{t}) = \xi_j(t)$$

Motion equations

$$mN\frac{d\hat{x}_j}{d\hat{t}} = \hat{\xi}_j\,, \qquad N\frac{d\hat{\xi}_j}{d\hat{t}} = \sum_{\substack{k=1\\k\neq j}}^{N} -\nabla V(\hat{x}_j - \hat{x}_k)$$

Finite total mass assumption

$$Nm = 1$$

Henceforth drop hats on all variables; our starting point is

$$\dot{x}_j = \xi_j\,, \qquad \dot{\xi}_j = -\frac{1}{N}\sum_{\substack{k=1\\k\neq j}}^{N} \nabla V(x_j - x_k)$$

Vlasov Equation

Unknown $f(t, dxd\xi)$ = single-particle phase-space number density

$$(\partial_t + \xi \cdot \nabla_x) f - \nabla_x V_f \cdot \nabla_\xi f = 0\,, \qquad x, \xi \in \mathbf{R}^d$$

where $V_f \equiv V_f(t,x)$ is the **mean-field potential**

$$V_f(t,x) := \iint_{\mathbf{R}^d \times \mathbf{R}^d} V(x-y) f(t, dyd\eta) = (V \star f(t))(x,\xi)$$

Notation set of Borel probability measures on $\mathbf{R}^n$ denoted $\mathcal{P}(\mathbf{R}^n)$

$$\mu \in \mathcal{P}_k(\mathbf{R}^n) \iff \int_{\mathbf{R}^n} |x|^k \mu(dx) < \infty$$

Existence/Uniqueness For each $f^{in} \in \mathcal{P}_1(\mathbf{R}^{2d})$, there exists a unique weak solution $f \in C([0,+\infty); w - \mathcal{P}(\mathbf{R}^{2d}))$ of the Vlasov equation such that $f\big|_{t=0} = f^{in}$

Liouville Equation

For a.e. $Y_N := (y_1, \ldots, y_N)$ and $H_N := (\eta_1, \ldots, \eta_N)$, consider

$$F_N(t, Y_N, H_N) := (f^{in})^{\otimes N}(\Phi_N(-\tfrac{t}{N}, Y_N, H_N))$$

By the method of characteristics, F_N is the solution of the N-body Liouville equation

$$\begin{cases} \partial_t F_N + \sum_{j=1}^{N} \left(\eta_j \cdot \nabla_{y_j} F_N - \frac{1}{N} \sum_{k=1}^{N} \nabla V(y_j - y_k) \cdot \nabla_{\eta_j} F_N \right) = 0 \\ F_N\big|_{t=0} = (f^{in})^{\otimes N} \end{cases}$$

Problem compare $f(t)^{\otimes n}$, where f is the Vlasov solution, with

$$F_N^{\mathbf{n}}(t, Y_n, H_n) := \int_{\mathbf{R}^{2d(N-n)}} F_N(t, Y_N, H_N) dy_{n+1} d\eta_{n+1} \ldots dy_N d\eta_N$$

Wasserstein distance of exponent p between μ and $\nu \in \mathcal{P}_p(\mathbf{R}^m)$

$$\mathbf{W_p}(\mu,\nu) := \inf_{\sigma \in \mathcal{C}(\mu,\nu)} \left(\iint_{\mathbf{R}^{2m}} |x-y|^p \sigma(dxdy) \right)^{1/p}$$

where $\mathcal{C}(\mu,\nu)$=set of $\sigma \in \mathcal{P}(\mathbf{R}^{2m})$ (**couplings** of μ and ν) s.t.

$$\iint_{\mathbf{R}^{2m}} (\phi(x) + \psi(y)\sigma(dxdy) = \int_{\mathbf{R}^m} \phi(x)\mu(dx) + \int_{\mathbf{R}^m} \psi(y)\nu(dy)$$

•Observe that $\mu \otimes \nu \in \mathcal{C}(\mu,\nu) \neq \varnothing$

•With $\sigma(dxdy) = \mu(dx)\delta(y-x) \in \mathcal{C}(\mu,\mu)$, one has $\mathbf{W_p}(\mu,\mu) = 0$

Theorem A [Dobrushin FA1979]
Assume that the potential V satisfies (H1)-(H2). Let $f(t)$ be the solution of the Vlasov equation with initial data f^{in} and F_N be the solution of the Liouville equation with initial data F_N^{in}. Then

$$\frac{1}{n}\mathbf{W}_2(f(t)^{\otimes n}, F_N^{\mathbf{n}}(t))^2 \leq \frac{(2\|\nabla V\|_{L^\infty})^2}{N}\frac{e^{\Lambda t}-1}{\Lambda}$$

for all $t \geq 0$ and $n = 1, \ldots, N$, with

$$\Lambda = 2 + \max(1, 2\,\mathrm{Lip}(\nabla V)^2)$$

Notation Henceforth, we denote

$$\rho_f(t,x) := \int_{\mathbf{R}^d} f(t,x,\xi)d\xi$$

Lemma 1 Let $t \mapsto P(t, dX_N d\Xi_N dY_N dH_N) \in \mathcal{P}(\mathbf{R}^{4dN})$ satisfy

$$\begin{cases} (\partial_t + \Xi_N \cdot \nabla_{X_N} + H_N \cdot \nabla_{Y_N})P \\ \qquad = \sum_{j=1}^{N} \left(\nabla V \star_x \rho_f(t, x_j) \cdot \nabla_{\xi_j} + \frac{1}{N} \sum_{k=1}^{N} \nabla V(y_j - y_k) \cdot \nabla_{\eta_j} \right) P \\ P\big|_{t=0} = (f^{in})^{\otimes N}(X_N, \Xi_N)\delta(Y_N - X_N)\delta(H_N - \Xi_N) \end{cases}$$

Then $P(t) \in \mathcal{C}(f(t)^{\otimes N}, F_N(t))$ for each $t \geq 0$, i.e.

$$\int P(t) dY_N dH_N = f(t)^{\otimes N}, \qquad \int P(t) dX_N d\Xi_N = F_N(t)$$

Proof: Integrate both sides of the equation for P in (Y_N, H_N) and in (X_N, Ξ_N), and use the uniqueness property for the Vlasov and the Liouville equations

The Functional $D_N(t)$

For $P(t, dX_N d\Xi_N dY_N dH_N)$ defined above, consider the quantity

$$D_N(t) := \int_{\mathbf{R}^{4dN}} \frac{1}{N} \sum_{j=1}^{N} (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t, dX_N d\Xi_N dY_N dH_N)$$

Lemma 2

$$D_N(t) \geq \frac{1}{n} \mathbf{W}_2(f(t)^{\otimes n}, F_N^{\mathbf{n}}(t))^2$$

Proof: By symmetry of $P(t)$, for all $j = 1, \ldots, N$, one has

$$D_N(t) := \int_{\mathbf{R}^{4dN}} (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t, dX_N d\Xi_N dY_N dH_N)$$

$$\geq \int_{\mathbf{R}^{4dN}} \frac{1}{n} \sum_{j=1}^{n} (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t, dX_N d\Xi_N dY_N dH_N)$$

Averaging out the last $d(N-n)$ position and momentum variables in $P(t)$ defines a coupling of $f(t)^{\otimes n}$ with $F_N^{\mathbf{n}}(t)$

Notation for $Y_N = (y_1, \ldots, y_N)$, we set

$$\mu_{Y_N} := \frac{1}{N}\sum_{j=1}^{N}\delta_{y_j}$$

•Multiplying each side of the equation for P by

$$\frac{1}{N}(|X_N - Y_N|^2 + |\Xi_N - H_N|^2)$$

and integrating in all variables shows that

$$\dot{D}_N(t) = \int \frac{1}{N}\sum_{j=1}^{N}(\xi_j \cdot \nabla_{x_j} + \eta_j \cdot \nabla_{y_j})|x_j - y_j|^2 P(t)$$

$$+\int \frac{1}{N}\sum_{j=1}^{N}\left(\nabla V \star_x \rho_f(t, x_j) \cdot \nabla_{\xi_j} + \nabla V \star \mu_{Y_N}(y_j)\nabla_{\eta_j}\right)|\xi_j - \eta_j|^2 P(t)$$

Thus

$$\dot{D}_N(t) = \int \frac{2}{N} \sum_{j=1}^{N} (\xi_j - \eta_j) \cdot (x_j - y_j) P(t)$$

$$+ \int \frac{2}{N} \sum_{j=1}^{N} (\nabla V \star_x \rho_f(t, x_j) - \nabla V \star \mu_{Y_N}(y_j)) \cdot (\xi_j - \eta_j) P(t)$$

so that

$$\dot{D}_N(t) \leq D_N(t)$$

$$+ \int \frac{2}{N} \sum_{j=1}^{N} (\nabla V \star_x \rho_f(t, x_j) - \nabla V \star \mu_{X_N}(x_j)) \cdot (\xi_j - \eta_j) P(t)$$

$$+ \int \frac{2}{N} \sum_{j=1}^{N} (\nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)) \cdot (\xi_j - \eta_j) P(t)$$

$$=: D_N(t) + I_N(t) + J_N(t)$$

Controling I_N and J_N

Since ∇V is Lipschitz continuous

$$
\begin{aligned}
J_N(t) \le \int \frac{1}{N}\sum_{j=1}^{N}(|\nabla V \star (\mu_{X_N}(x_j) - \mu_{Y_N}(y_j))|^2 + |\xi_j - \eta_j|^2)P(t) \\
\le \frac{1}{N}\int \sum_{j=1}^{N}(2\,\mathrm{Lip}(\nabla V)^2|x_j - y_j|^2 + |\xi_j - \eta_j|^2)P(t) \\
\le \max(1, 2\,\mathrm{Lip}(\nabla V)^2)D_N(t)
\end{aligned}
$$

Likewise

$$
\begin{aligned}
I_N(t) \le \int \frac{1}{N}\sum_{j=1}^{N}(|\nabla V \star (\rho_f(t,x_j) - \mu_{X_N}(x_j))|^2 + |\xi_j - \eta_j|^2)P(t) \\
\le \int \frac{1}{N}\sum_{j=1}^{N}|\nabla V \star (\rho_f(t,\cdot) - \mu_{X_N})(x_j))|^2 \rho_f(t)^{\otimes N} + D_N(t)
\end{aligned}
$$

Lemma 3

$$\int |\nabla V \star (\rho_f - \mu_{X_N})(x_1)|^2 \rho_f(t)^{\otimes N} \leq \frac{(2\|\nabla V\|_{L^\infty})^2}{N}$$

Proof Setting $\mathcal{V}(z) := \nabla V \star \rho_f(x_1) - \nabla V(x_1 - z)$, one has

$$|\nabla V \star (\rho - \mu_{X_N})(x_1))|^2 = \frac{1}{N^2} \sum_{k,l=1}^{N} \mathcal{V}(x_j) \cdot \mathcal{V}(x_k)$$

and

$$j \neq k \Rightarrow \int \mathcal{V}(x_j) \cdot \mathcal{V}(x_k) \rho^{\otimes N} = 0$$

Apply Gronwall's lemma to the differential inequality

$$\dot{D}_N(t) \leq \Lambda D_N(t) + \frac{(2\|\nabla V\|_{L^\infty})^2}{N}$$

(KL) Dynamics of N weakly coupled oscillators

$$\dot\theta_j = \nu_j + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k-\theta_j)\,,\qquad j=1,\ldots,N$$

(LMM) Motion of weakly coupled $U_j(t)\in\mathbf{U}(d)$, for $H_j=H_j^*\in M_d(\mathbf{C})$

$$i\dot U_j U_j^* = H_j + \frac{i\kappa}{2N}\sum_{k=1}^{N}(U_kU_j^*-U_jU_k^*)\,,\qquad 1\le j\le N$$

where

$$\mathbf{U}(d)=\{V\in M_d(\mathbf{C})\text{ s.t. } VV^*=V^*V=I\}$$

For $d=1$, writing $U_j(t)=e^{i\theta_j(t)}$ and $H_j=\nu_j\in\mathbf{R}$ shows that Lohe's model reduces to Kuramoto's

Tangent space at U of $\mathbf{U}(d)$:

$$T_U\mathbf{U}(d) := \mathfrak{su}(d)U\,,\quad \mathfrak{su}(d) := \{A \in M_d(\mathbf{C}) \text{ s.t. } A = -A^*\}$$

Single-oscillator phase-space

$$T\mathbf{U}(d) := \{(U, AU) \text{ s.t. } (U, A) \in \mathbf{U}(d) \times \mathfrak{su}(d)\}$$

Continuous vector field on $\mathbf{U}(d)$

$$\mathbf{U}(d) \ni U \mapsto X_U = A(U)U \in T_U\mathbf{U}(d)\,,\quad A(U) \in \mathfrak{su}(d)$$

Divergence of fX with $f \in \mathcal{P}(\mathbf{U}(d))$ and X continuous vector field on $\mathbf{U}(d)$: linear functional

$$C^1(\mathbf{U}(d)) \ni \phi \mapsto \langle \text{div}(fX), \phi\rangle := \int_{\mathbf{U}(d)} (d_U\phi, X_U) f(dU)$$

Liouville Equation for the LMM

Distribution function in the N-particle phase space $(\mathbf{U}(d)\times\mathfrak{su}(d))^N$:

$$F_N \equiv F_N(t, dU_1 dA_1 \dots dU_N dA_N)$$

Liouville equation N-particle system (with $A_j := -iH_j \in \mathfrak{su}(d)$)

$$\partial_t F_N + \sum_{j=1}^{N} \operatorname{div}_{U_j}(F_N(A_j + X_j(U_1, \dots, U_N))U_j) = 0$$

where

$$X_j := \frac{\kappa}{2N} \sum_{k=1}^{N} K(U_k, U_j)U_j\,, \quad K(U,V) := UV^* - VU^* \in \mathfrak{su}(d)$$

Remark: the variables $A_1, \dots, A_N$ are not dynamical, but only simple parameters — yet the mean-field limit requires considering the joint distribution of U and A

Lohe Kinetic Equation and Mean-Field Limit

Mean-field equation: 1-particle distribution $f \equiv f(t, dUdA)$

$$\partial_t f(t) + \mathrm{div}_U \left(f(t) \left(A + \frac{\kappa}{2} \int_{\mathbf{U}(d)\times\mathfrak{su}(d)} K(V,U) f(t, dVdB) \right) U \right) = 0$$

Theorem B [FG-S.-Y. Ha ARMA2019]
Let $f^{in} \in \mathcal{P}_2(\mathbf{U}(d) \times \mathfrak{su}(d))$ and let f be the solution of the Lohe kinetic equation with initial data f^{in}. Let F_N be the solution of the Liouville equation with initial data $(f^{in})^{\otimes N}$. Then

$$\sup_{\mathrm{Lip}(\phi)\le 1} \left| \int \phi(U,A)(F_N^{\mathbf{1}}(t, dUdA) - f(t, dUdA)) \right|^2$$
$$\le \mathbf{W}_2(F_N^{\mathbf{1}}(t), f(t))^2 \le \frac{8\kappa d}{5N}(e^{10\kappa t} - 1)$$

where $F_N^{\mathbf{1}}$ is the 1st marginal of F_N, defined by

$$\int \phi(U,A) F_N^{\mathbf{1}}(dUdA) = \int \phi(U_1, A_1) F_N(dU_1 dA_1 \ldots dU_N dA_N)$$

Bibliography

•References used in the talk

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•Reference for numerical methods (vortex blob method for 2D Euler)

6. C. Marchioro, M. Pulvirenti: "Mathematical Theory of Incompressible Nonviscous Fluids", Springer Verlag, New York 1994
(see chapter 5, and the appendix on K-R distance, which is $\mathbf{W}_1$)