From Kuramoto to Lohe Tensor I

Seung Yeal Ha

Department of Mathematical Sciences
Seoul National University

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Outline

A master aggregation model

A hierarchy of finite-dimensional aggregation models

Aggregation of tensors

Lecture plan

- Lecture 1: Aggregation of numbers, vectors and matrices
- Lecture 2: Aggregation of tensors

Some jargons to be used in this lecture:

Consensus in position: aggregation,

Consensus in velocity: flocking,

Consensus in frequency: synchronization

As long as there is no confusion, we still use "aggregation" to denote consensus of state.

Outline of Lecture 1

- A master aggregation model
- A hierarchy of (finite-dimensional) aggregation models
- Aggregation of tensors

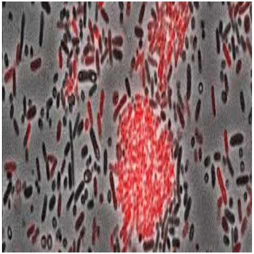
The first story: A master aggregation model

$$\dot{q}_i = \nu_i + \frac{1}{N}\sum_{k=1}^{N} \Psi(q_k - q_i).$$

$q_i \in \mathcal{M}$: state for the i-th particle, Ψ : coupling function.

Collective behaviors of biological systems

- Aggregation of bacteria Flocking of birds, Synchronization of fireflies



PDE models for collective dynamics

- The Keller-Segel model :Patlak (1953), Keller-Segel (1970s)

$$\partial_t \rho + \nabla \cdot (\rho \nabla c) = \sigma \Delta \rho, \quad -\Delta c = \rho,$$

- The hydrodynamic Cucker-Smale model H-Tadmor '08

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) = -\kappa \int_{R^d} \psi(|x-y|)(u(y)-u(x))\rho(x)\rho(y)dy.$$

- The kinetic Kuramoto model Kuramoto '75

$$\partial_t F + \partial_\theta(\omega[F]F) = 0,$$
$$\omega[F](\theta,\nu,t) := \nu - \kappa \int_0^{2\pi} \int_R \sin(\theta_* - \theta) F(\theta_*, \nu_*, t) d\nu_* d\theta.$$

At PDE level, three PDE models look different.

Particle models

- The deterministic Keller-Segel model in $\mathbb{R}^3$

$$\dot{x}_i = \frac{\kappa}{N} \sum_{k \neq i} \frac{x_k - x_i}{|x_k - x_i|^3}.$$

- The Cucker-Smale model: Cucker-Smale '07

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{\kappa}{N} \sum_{k=1}^{N} \psi_{cs}(x_k - x_i)(v_k - v_i).$$

- The Kuramoto model: Kuramoto '75

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i).$$

First-order formulation of the C-S model on a line

- The C-S model in $1D$: H-Kim-Park-Zhang '19 ARMA

$$\dot{x}_i = v_i, \qquad \dot{v}_i = \frac{\kappa}{N}\sum_{k=1}^{N}\psi(x_k - x_i)(v_k - v_i).$$

Idea

$$\psi(x_k - x_i)(v_k - v_i) = \frac{d}{dt}\int_0^{x_k - x_i}\psi(s)ds =: \frac{d}{dt}\Psi_{cs}(x_k - x_i).$$

Then, C-S flocking becomes a first-order aggregation model:

$$\dot{x}_i = \nu_i(X^0, V^0) + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi_{cs}(x_k - x_i),$$

$$\nu_i(X^0, V^0) := v_i^0 - \frac{\kappa}{N}\sum_{j=1}^{N}\psi(x_k^0 - x_i^0).$$

Particle Pictures

q_i : generalized position of the i-th particle.

- The deterministic Keller-Segel model in 3d

$$\dot{q}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi_a(q_k - q_i), \quad \Psi_a(q) = \frac{q}{|q|^3}.$$

- The Cucker-Smale model in 1d

$$\dot{q}_i = \nu_i(q^0, p^0) + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi_{cs}(q_j - q_i), \quad \Psi_{cs}(q) = \int^q \psi_{cs}(y)dy.$$

- The Kuramoto model

$$\dot{q}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi_k(q_k - q_i), \quad \Psi_k(q) = \sin q.$$

Summary of the first story

Many collective behaviors of many-body systems can be described by the first-order master aggregation model:

$$\dot{q}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i), \quad q_i \in \mathcal{M}.$$

In other words, there exists a kind of triality relation:

Keller-Segel aggregation $\Longleftrightarrow$ 1*d CS flocking*

$\Longleftrightarrow$ Kuramoto synchronization.

A three-minute tour with the Kuramoto model

Kuramoto's seminal paper (1975)

Lecture Notes in Physics

Edited by J. Ehlers, München, K. Hepp, Zürich, and H. A. Weidenmüller, Heidelberg
Managing Editor: W. Beiglböck, Heidelberg

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SELF-ENTRAINMENT OF A POPULATION OF COUPLED NON-LINEAR OSCILLATORS

Yoshiki Kuramoto
Department of Physics, Kyushu University, Fukuoka, Japan

Temporal organization of matter is a widespread phenomenon over a macroscopic world in far from thermodynamic equilibrium. A previous study on chemical instability[1)] implies that a simplest nontrivial model for a temporally organized system may be represented by a macroscopic self-sustained oscillator Q obeying the equation of motion

$$\dot{Q} = (i\omega + \alpha)Q - \beta|Q|^2 Q, \qquad \alpha,\beta > 0. \tag{1}$$

Consider a population of such oscillators $Q_1, Q_2, \cdots Q_N$ with various frequencies, and introduce interactions between every pair as follows.

$$\dot{Q}_s = (i\omega_s + \alpha)Q_s + \sum_{r \neq s} V_{rs} Q_r - \beta|Q_s|^2 Q_s, \qquad r,s = 1, 2, \cdots N. \tag{2}$$

- Kuramoto's approach: Stewart-Landau oscillator

$$\dot{z} = (1 - |z|^2 + \mathrm{i}\nu)z \quad \Longleftrightarrow \quad \dot{r} = r(1 - r^2), \quad \dot{\theta} = \nu.$$

where $z = re^{\mathrm{i}\theta} \in \mathbb{C}$: location of oscillator, ν: natural frequency or intrinsic phase velocity

◇ Linearly coupled Stewart-Landau oscillators:

$$\dot{z}_j = (1 - |z_j|^2 + \mathrm{i}\nu_j)z_j + \frac{\kappa}{N}\sum_{i=1}^{N}(z_i - z_j).$$

We set

$$z_j = e^{\mathrm{i}\theta_j}$$

and compare the imaginary part of the resulting relation

Kuramoto's mean-field analysis

$$\dot{\theta}_i = \underbrace{\nu_i}_{\text{random}} + \underbrace{\frac{\kappa}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i)}_{\text{nonlinear coupling}} .$$

Introduce order parameters R and ϕ:

$$Re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j}, \quad R \in [0,1].$$

This yields

$$Re^{\mathrm{i}(\phi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}, \quad \text{i.e.,} \quad R\sin(\phi-\theta_i) = \frac{1}{N}\sum_{i=1}^{N}\sin(\theta_j-\theta_i).$$

Thus

$$\text{KM} \quad \Leftrightarrow \quad \dot{\theta}_i = \nu_i + KR \sin(\phi - \theta_i).$$

If $|\nu_i| > KR$, then i-th oscillator will drift over the circle.

If $|\nu_i| \leq KR$, then i-th oscillator will approach to some equilibrium.

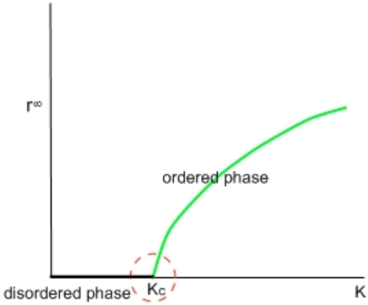
- Asymptotic order parameter

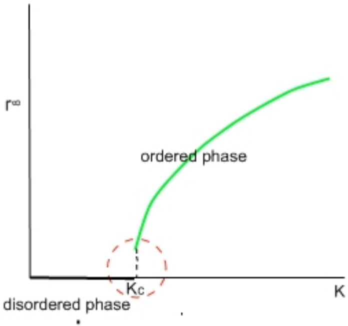
$$R^{\infty}(\kappa) := \lim_{t \to \infty} \lim_{N \to \infty} R^N(\kappa, t).$$

Phase transitions at the critical coupling strength κ_c

- Self-consistent analysis

$$\partial_t f + \partial_\theta(\omega[f]f) = 0,$$
$$\omega[f](x, \Omega, t) := \Omega - K \int_0^{2\pi} \int_R \sin(\theta_* - \theta) f(\theta_*, \Omega_*, t) g(\Omega_*) d\Omega_* d\theta.$$





Why aggregation model?

- Decentralized control algorithms from control theory

- Consensus-based optimization(CBO) algorithm

$$dX_t^i = \kappa \sum_k \omega_k (X_t^k - X_t^i) dt + \sigma \sum_k \omega_k (X_t^k - X_t^i) \odot dW_t,$$

$$\omega_k = \frac{e^{-\beta L(X_t^k)}}{\sum_{l=1}^N e^{-\beta L(X_t^l)}}.$$

Askari-Sichani–Jalili '13, Pinnau–Totzeck–Tse–Martin '17, Carrillo–Choi–Totzeck–Tse '18, Carrillo–Jin–Li–Zhu '19, · · ·

- Systemic risk in financial market Garnier-Papanicolaou-Yang '13

$$dx_t^i = -hV'(x_t^i)dt + \frac{\kappa}{N}\sum_k (x_t^k - x_t^i)dt + \sigma dw_t^i.$$

The second story: A hierarchy of (finite-dimensional) aggregation models

$$a, \quad \mathbf{A} = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}, \quad U = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \vdots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}.$$

Well-known Lohe type aggregation models

- The Lohe matrix(LM) model: Lohe ('09, '10)

U_i : $d \times d$ unitary matrix, H_i : $d \times d$ Hermitian matrix.

$$\mathrm{i}\dot{U}_i U_i^* = H_i + \frac{\mathrm{i}\kappa}{2N} \sum_{k=1}^{N} \left(U_i U_j^* - U_j U_i^* \right).$$

- The swarm sphere(SS) model: Olfati-Saber '06, Lohe '09

x_i : a real vector in $\mathbb{R}^d$, Ω_i : $d \times d$ skew-symmetric matrix.

$$||x_i||^2 \dot{x}_i = \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} \Big(\langle x_i, x_i \rangle x_k - \langle x_k, x_i \rangle x_i \Big),$$

- The Kuramoto model: Kuramoto '75

θ_i : real number, ν_i : real number.

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i).$$

A hierarchy of aggregation models

- From LM model to SS model

For $d = 2$, we set

$$U_i := \mathrm{i}\sum_{k=1}^{3} x_i^k \sigma_k + x_i^4 I_2 = \begin{pmatrix} x_i^4 + \mathrm{i}x_i^1 & x_i^2 + \mathrm{i}x_i^3 \\ -x_i^2 + \mathrm{i}x_i^3 & x_i^4 - \mathrm{i}x_i^1 \end{pmatrix},$$

$$H_i = \sum_{k=1}^{3} \omega_i^k \sigma_k + \nu_i I_2,$$

where

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the SS model:

$$||x_i||^2 \dot{x}_i = \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (||x_i||^2 x_k - \langle x_i, x_k \rangle x_i),$$

where Ω_i is a real 4×4 skew-symmetric matrix:

$$\Omega_i := \begin{pmatrix} 0 & -\omega_i^3 & \omega_i^2 & -\omega_i^1 \\ \omega_i^3 & 0 & -\omega_i^1 & -\omega_i^2 \\ -\omega_i^2 & \omega_i^1 & 0 & -\omega_i^3 \\ \omega_i^1 & \omega_i^2 & \omega_i^3 & 0 \end{pmatrix}.$$

cf. Special skew-symmetric matrix.

- From SS to Kuramoto We set

$$d = 2, \quad x_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}, \quad \Omega_i = \begin{bmatrix} 0 & -\nu_i \\ \nu_i & 0 \end{bmatrix},$$

Then, the SS model

$$\|x_i\|^2 \dot{x}_i = \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (\langle x_i, x_i \rangle x_k - \langle x_k, x_i \rangle x_i),$$

reduces to the Kuramoto model:

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i).$$

Remark: The SS model and the Lohe matrix model can be regarded as high-dimensional Kuramoto model.

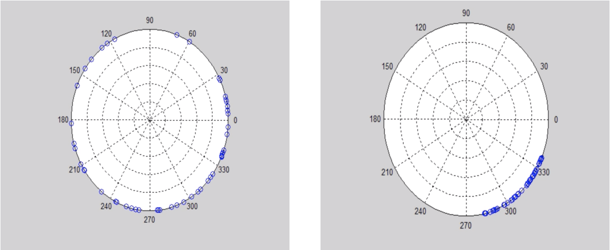
Emergent dynamics

Synchronization of the Kuramoto model

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N.$$

◇ Desync. and Sync.:

$$N = 50, \quad \nu_i \in [-1, 1], \quad \kappa = 0.8, \quad \kappa = 2.2$$



Remarks on the complete synchronization

For a phase configuration $\Theta = (\theta_1, \cdots, \theta_N)$, we set a phase diameter:

$$D(\Theta) := \max_{i,j} |\theta_i - \theta_j|.$$

- Lyapunov functional approach:
  1. Chopra-Spong (2009), H-Ha-Kim (2010): $D(\Theta) < \frac{\pi}{2}$
  2. Dorfler-Bullo (2011), Choi-H-Jung-Kim (2012),$\cdots$: $D(\Theta) < \pi$.

Next, we introduce Kuramoto order parameters (R, ϕ):

$$Re^{\mathrm{i}\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j}, \quad R\sin(\phi - \theta_i) = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i).$$

Hence

$$\text{Kuramoto model} \quad \Longleftrightarrow \quad \dot\theta_i = \nu_i + \kappa R\sin(\phi - \theta_i).$$

Non-synchronizing Kuramoto flow

Exact Kuramoto flow:

$$N = 4, \quad \nu_1 = \nu_2 \neq \nu_3 = \nu_4,$$
$$\theta_1 = \nu_1 t, \quad \theta_2 = \nu_1 t + \pi, \quad \theta_3 = \nu_3 t, \quad \theta_4 = \nu_3 t + \pi.$$

In this case, R is identically zero:

$$R = \frac{1}{N}\left(e^{\mathrm{i}\nu_1 t} + e^{\mathrm{i}\nu_1 t + \pi \mathrm{i}} + e^{\mathrm{i}\nu_3 t} + e^{\mathrm{i}\nu_3 t + \pi \mathrm{i}}\right) = 0, \quad t \geq 0.$$

So no matter how large the coupling κ is, Kuramoto flow cannot achieve complete synchronization

Complete Synchronization for generic initial data

- **Theorem** H-Kim-Ryoo '16, H-Ryoo '18

Suppose initial configuration and natural frequencies satisfy

$$R^0 := \Big|\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(0)}\Big| > 0 \quad \sum_j \nu_j = 0.$$

Then, $\exists\ \kappa_\infty = 1.6\frac{D(\nu)}{R_0^2} > 0$ and phase-locked state Θ^∞ such that

$$\kappa \ge \kappa_\infty \quad \Longrightarrow \quad \lim_{t\to\infty} ||\Theta(t) - \Theta^\infty||_\infty = 0.$$

cf. Sufficient conditions, a gradient flow formulation and uniform boundedness of fluctuations for $\kappa \gg 1$.

Some references on synchronization

- J. A. Acebron, L. L. Bonilla, C. J. P. Pérez Vicente, F. Ritort and R. Spigler: *The Kuramoto model: A simple paradigm for synchronization phenomena.* Rev. Mod. Phys. **77** (2005), 137-185.
- A. Pikovsky, M. Rosenblum and J. Kurths: *Synchronization: A universal concept in nonlinear sciences*. Cambridge University Press, Cambridge, 2001.
- F. Dörfler, and F. Bullo: *Synchronization in complex networks of phase oscillators: A survey.* Automatica **50** (2014), 1539-1564.
- S.-Y. Ha and D. Kim: *Collective dynamics of Lohe type aggregation models*. To appear in arxiv.

Aggregation estimate for the SS model

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle x_i).$$

- For identical ensemble with:

$$\Omega_i = \Omega, \quad i = 1, \cdots, N,$$

  aggregation estimates have extensively studied by H-Choi, J. Markdahl, J. Thunberg, J. Goncalves, V. Jaclmovic, A. Crnkic, J. Zhu and their collaborators.

- For non-identical ensemble with

$$\Omega_i \neq \Omega_j,$$

  aggregation estimates are largely open except practical aggregation

cf. Phase-transition like phenomena: Michelle Girvan and Edward Ott.

An aggregation model for the LM model

- Aggregation on $\mathbb{U}(d)$:

Recall the unitary group $\mathbb{U}(d)$:

$$\mathbb{U}(d) := \{A \in \mathbb{C}^{d\times d} : \; UU^* = U^*U = I_d\}$$

and the Kuramoto model on $\mathbb{S}^1$:

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i).$$

cf. $\mathbb{U}(1)$: circle group of complex numbers with absolute value 1 under multiplication.

- The Lohe matrix model (2009):

$$\underbrace{\mathrm{i}\dot{U}_i U_i^*}_{\text{generalized frequency}} = H_i + \underbrace{\frac{\mathrm{i}\kappa}{2N}\sum_{j=1}^{N}\Big(U_j U_i^* - U_i U_j^*\Big)}_{\text{nonlinear coupling}},$$

cf. 1. Lohe, M. A.: Non-abelian Kuramoto model and synchronization. J. Phys. A: Math. Theor. 42, 395101-395126 (2009).

$$U_i(t) : d \times d \text{ unitary matrix}, \quad H_i : d \times d \text{ Hermitian matrix}.$$

2. DeVille ('19), Bronski-Carty-Simpson ('20).

3. H-Golse ('19): Mean-field limit

From Lohe to Kuramoto and Schrödinger

- From Lohe to Kuramoto: For $d = 1$, we set

$$U_i = e^{-i\theta_i}, \quad H_i = \nu_i.$$

Thus, the Lohe matrix model reduces the Kuramoto model:

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i).$$

- From Lohe to Schrödinger: For $\kappa = 0$,

$$i\dot{U}_i U_i^* = H_i, \quad \text{or} \quad \dot{U}_i = -iH_iU_i.$$

This yields

$$U_i(t) = e^{-iH_it}U_i(0), \quad t > 0.$$

Mathematical beauties

$$\mathrm{i}\dot{U}_i U_i^* = H_i + \frac{i\kappa}{2N} \sum_{j=1}^{N} \Big(U_j U_i^* - U_i U_j^* \Big),$$

- Invariance of $U_i U_i^*$:

$$\frac{d}{dt}(U_i U_i^*) = 0, \quad t > 0.$$

Note that for $U \in \mathbb{U}(d)$, the LM model can rewritten as

$$\begin{aligned} \dot{U}_i &= -\mathrm{i}H_i U_i + \frac{\kappa}{2N} \sum_{j=1}^{N} (U_j - U_j U_j^* U_i) \\ &= -\mathrm{i}H_i U_i + \frac{\kappa}{2}(U_c - \langle U_c, U_i \rangle_F U_i). \end{aligned}$$

- Invariance under the right-multiplication by a unitary matrix: if $L \in U(d)$ and $V_i = U_i L$, then V_i satisfies

$$\mathrm{i}\dot{V}_i V_i^* = H_i + \frac{\mathrm{i}\kappa}{2N}\sum_{j=1}^{N}\Big(V_j V_i^* - V_i V_j^*\Big), \quad V_i(0) = U_{i0}L.$$

- Solution splitting property: For identical hamiltonians $H_i = H$, the solution operator of the Lohe matrix model can be split as a composition of two solution operators of the following two systems:

$$\mathrm{i}\dot{U}_i U_i^* = H \quad \text{and} \quad \mathrm{i}\dot{U}_i U_i^* = \frac{\mathrm{i}\kappa}{2N}\sum_{j=1}^{N}\Big(U_j U_i^* - U_i U_j^*\Big).$$

A Lyapunov functional approach

1. Step A: Introduce an ensemble diameters:

$$D(U) := \max_{1\leq i,j\leq N} \|U_i - U_j\|_F, \qquad D(H) := \max_{1\leq i,j\leq N} \|H_i - H_j\|_F.$$

2. Step B: Derive a Gronwall type differential inequality for $D(U)$:

$$\left| \frac{d}{dt} D(U)^2 + 2\kappa D(U)^2 \right| \leq 2D(H)D(U) + \kappa D(U)^4 \quad \text{a.e. } t \in (0, \infty).$$

3. Step C: Establish the existence of PLSs. For example, for identical oscillators with $D(H) = 0$,

$$\lim_{t\to\infty} D(U(t)) = 0.$$

- **Theorem**: (H-Ryoo '16, J. Stat. Phys.) Suppose that κ and U^0 satisfy

$$\kappa > \frac{54}{17} D(H), \quad \max_{i,j} \|U_i^0 - U_j^0\|_F < \alpha \quad \text{for some } \alpha > 0$$

Then, we have

1. $\{U_i\}$ achieves asymptotic phase-locking:

$$\lim_{t \to \infty} U_i U_j^*$$

converges exponentially fast, with exponential rate bounded above by $-\kappa(1 - 3\alpha_1)$.

2. There exists a phase-locked state $\{V_i\}$ and $L \in \mathcal{U}(d)$ such that

$$\lim_{t \to \infty} \|U_i - V_i L\|_F = 0$$

converges exponentially fast.

Summary of the second story

- The Kuramoto model, the swarm sphere model and the Lohe matrix model for real numbers, real vectors and unitary matrices.
- Sufficient frameworks for the complete aggregation of the Lohe matrix model for restricted initial data.

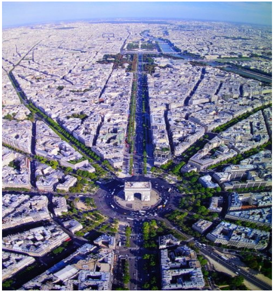
Some open problems

- The Kuramoto model: Complete synchronization for generic data has been verified using the gradient flow formulation for $\kappa \gg 1$, so the remaining open issue is to identify a minimal coupling strength for generic data.
- The SS and LM models: Complete aggregation for restricted initial data has been verified using a Lyapunov functional approach for $\kappa \gg 1$. As in the Kuramoto model, is the complete aggregation true for generic data?

Third story: An aggregation model for tensors

- Can we construct an aggregation model on the space of non square matrices, for example $\mathbb{R}^{n\times m}$ with $n \neq m$?
- Can we propose an aggregation model on Hermitian unit sphere $\mathbb{HS}^{d-1}$?

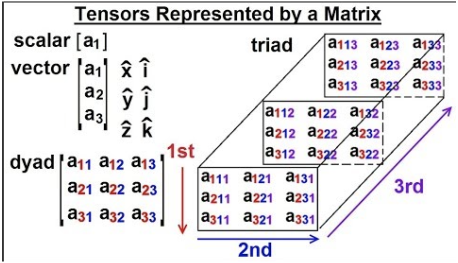
General strategy: A bird view approach



We first design a master consensus model incorporating aforementioned aggregation models with emergent dynamics, possibly on the space of tensors, and then derive aggregation models on $\mathbb{R}^{n\times m}$ and $\mathbb{HS}^{d-1}$.

What is a tensor?

◇ Physicist's Definition: Tensor is a multi-dimensional array of complex numbers, and the rank of a tensor is the number of indices.



complex number: rank-0 tensor, $\mathbb{C}^d$-vector: rank-1tensor, $m \times n$ complex matrix: rank-2 tensor

⋄ Remark: 1. We denote the set of all rank-m tensors with size $d_1 \times \cdots \times d_m$ by $\mathcal{T}_m(\mathbb{C}; d_1 \times \cdots d_m)$. Then, the set $\mathcal{T}_m(\mathbb{C})$ is a vector space. For a given tensor $T \in \mathcal{T}_m(\mathbb{C}; d_1 \times \cdots d_m)$ and $\alpha \in \Pi_{i=1}^m \{1, \cdots, d_i\}$, we denote $[T]_\alpha$ to be the α-th component of T.

2. Einstein summation convention: For rank-1 tensor u and v,

$$\langle u, v \rangle = \sum_\alpha [\bar{u}]_\alpha [v]_\alpha =: [\bar{u}]_\alpha [v]_\alpha.$$

Question

Design a generalized aggregation model which can include previous models

Learning from the SS and LM models

- The swarm sphere model in vector form

$$\dot{x}_i = \underbrace{\Omega_i x_i}_{\text{free flow}} + \underbrace{\frac{\kappa}{N}\sum_{k=1}^{N}(\langle x_i, x_i\rangle x_k - \langle x_k, x_i\rangle x_i)}_{\text{cubic interactions}}$$
$$= \Omega_i x_i + \kappa(\langle x_i, x_i\rangle x_c - \langle x_c, x_i\rangle x_i).$$

◇ The swarm sphere model in component form

$$\frac{d}{dt}[x_i]_\alpha = [\Omega_i x_i]_\alpha + \kappa([x_i]_\beta [x_i]_\beta [x_c]_\alpha - [x_i]_\beta [x_c]_\beta [x_i]_\alpha)$$
$$= [\Omega_i]_{\alpha\beta}[x_i]_\beta + \kappa([x_i]_\beta [x_i]_\beta [x_c]_\alpha - [x_i]_\beta [x_c]_\beta [x_i]_\alpha)$$

where $x_c = \frac{1}{N}\sum_{k=1}^{N} x_k$.

- The Lohe matrix model in matrix form

$$\mathrm{i}\dot{U}_j U_j^* = H_j + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N}(U_k U_j^* - U_j U_k^*).$$

or equivalently

$$\dot{U}_j = -\mathrm{i}H_j U_j + \frac{\kappa}{2}(U_c U_j^* U_j - U_j U_c^* U_j).$$

or

$$\dot{U}_j = \underbrace{-\mathrm{i}H_j U_j}_{\text{free flow}} + \underbrace{\frac{\kappa}{2}(U_j U_j^* U_c - U_j U_c^* U_j)}_{\text{cubic couplings}}.$$

◇ The Lohe matrix model in component form

$$\frac{d}{dt}[U_j]_{\alpha\beta} = [-\mathrm{i}H_jU_j]_{\alpha\beta} + \frac{\kappa}{2}\big([U_j]_{\alpha\gamma}[\bar{U}_j]_{\delta\gamma}[U_c]_{\delta\beta} - [U_j]_{\alpha\gamma}[\bar{U}_c]_{\delta\gamma}[U_j]_{\delta\beta}\big)$$

Next, we interpret the free flow term $[-\mathrm{i}H_jU_j]_{\alpha\beta}$ as a contraction of rank-4 tensor A_j and rank-2 tensor U_j. For this, we define rank-4 tensor A_j as follows:

$$[A_j]_{\alpha\beta\gamma\delta} := [-\mathrm{i}H_j]_{\alpha\gamma}\delta_{\beta\delta} \quad \text{and} \quad \delta_{\beta\delta} := \begin{cases} 1, & \beta = \delta, \\ 0, & \beta \neq \delta. \end{cases}$$

Then, one can see

$$[\bar{A}_j]_{\gamma\delta\alpha\beta} = -[A_j]_{\alpha\beta\gamma\delta} \quad \text{and} \quad [A_j]_{\alpha\beta\gamma\delta}[U_j]_{\gamma\delta} = [-\mathrm{i}H_jU_j]_{\alpha\beta}.$$

Lessons from the SS and LM models

Consider an ensemble $\{T_j\}_{j=1}^N$ of complex rank-m tensors, and for notational simplicity, we set

$$\alpha_* = (\alpha_1, \cdots, \alpha_m), \qquad \beta_* = (\beta_1, \cdots, \beta_m).$$

Then, we begin with following structure:

$$\frac{d}{dt}[T_j]_{\alpha_*} = \text{free flow} + \text{cubic interactions}.$$

- (Modeling of free flow)

Contraction of rank-$2m$ tensor A_j and rank-m tensor T_j:

$$\text{free flow part} = [A_j]_{\alpha_*\beta_*}[T_j]_{\beta_*}.$$

- (Modeling of cubic interactions): for a dummy variable β,

$$[T_c]_{i_1}[\overline{T_j}]_\beta[T_j]_{i_2} - [T_j]_{i_1}[\overline{T_c}]_\beta[T_j]_{i_2}.$$

- **Definition**:

We define the Frobenius inner product on $T_m(\mathbb{C}; d_1 \times d_2 \times \cdots \times d_m)$ as follows.

$$\langle T_i, T_j \rangle_F := [\overline{T_i}]_{\alpha_*}[T_j]_{\alpha_*}, \quad i, j = 1, \cdots, N.$$

Summary of the third Story

- The SS model in component form:

$$\frac{d}{dt}[x_i]_\alpha = [\Omega_i]_{\alpha\beta}[x_i]_\beta + \kappa([x_i]_\beta[x_i]_\beta[x_c]_\alpha - [x_i]_\beta[x_c]_\beta[x_i]_\alpha)$$

- The LM model in component form:

$$\frac{d}{dt}[U_j]_{\alpha\beta} = [A_j]_{\alpha\beta\gamma\delta}[U_j]_{\gamma\delta} + \frac{\kappa}{2}\left([U_c]_{\alpha\gamma}[U_j^*]_{\gamma\delta}[U_j]_{\delta\beta} - [U_j]_{\alpha\gamma}[U_c^*]_{\gamma\delta}[U_j]_{\delta\beta}\right).$$

- Lesson from above models

$$\frac{d}{dt}[T_j]_{\alpha_*} = \text{free flow} + \text{cubic interactions.}$$

Summary of Lecture 1

We have reviewed the emergent dynamics of a hierarchy of Lohe type aggregation models:

$$\dot{U}_i = -\mathrm{i}H_i U_i + \frac{\kappa}{2N}\sum_{k=1}^{N}\Big(U_k - \langle U_k, U_i\rangle_F U_i\Big),$$

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Big(x_k - \langle x_k, x_i\rangle x_i\Big),$$

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i).$$

Gradient flow formulation

• The Kuramoto model on $\mathbb{R}^N$: van Hemmen-Wreszinski (1993), Dong-Xue ('13), H-Kim-Ryoo ('16)

$$R_k := \left| \frac{1}{N} \sum_{j=1}^{N} e^{\mathrm{i}\theta_j} \right|, \quad V_k(\Theta) = -\nu \cdot \Theta - \kappa N R_k^2.$$

$$\text{The Kuramoto model} \quad \Longleftrightarrow \quad \dot{\Theta} = -\nabla_\Theta V_k(\Theta).$$

• The SS model on $\mathbb{S}^{dN}$: H-Ko-Ryoo ('18)

$$R_s := \left\| \frac{1}{N} \sum_{j=1}^{N} x_j \right\|, \quad V_s(X) = -\frac{\kappa}{2} N R_s^2.$$

$$\text{The SS model with } \Omega_i = \Omega \quad \Longleftrightarrow \quad \dot{x}_i = -\nabla_{x_i} V_s(X) \Big|_{T_{x_i}\mathbb{S}^d}.$$

cf. For a heterogeneous ensemble, the SS model is not a gradient flow on $\mathbb{S}^{dN}$.

- The LM model on $\mathbb{U}(d)^N$: H-Ko-Ryoo ('18)

$$R_m := \Big\| \frac{1}{N} \sum_{j=1}^{N} U_j \Big\|_F, \quad \mathcal{V}_m := -\frac{\kappa}{2} N R_m^2.$$

$$\text{The LM model with } H_i = O \quad \Longleftrightarrow \quad \dot{U}_i = -\nabla_{U_i} \mathcal{V}_{m\, T_{U_i}(d)}.$$

cf. For a heterogeneous ensemble, the LM model is not a gradient flow on $\mathbb{U}(d)^N$.

Preview for Lecture 2

Tomorrow, we will continue the derivation of "the Lohe tensor model" and study its emergent dynamics.

Questions and Comments

Thank you for your attention !!!