From Kuramoto to Lohe Tensor II

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Outline

0-th story: From Lecture 1

Aggregation models for low-rank tensors

- The Lohe matrix model for complex-valued rank-2 tensors:

$$\mathrm{i}\dot{U}_i U_i^* = H_i + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N}\left(U_i U_j^* - U_j U_i^*\right).$$

- The swarm sphere model for real-valued rank-1 tensors:

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}(\langle x_i, x_i\rangle x_k - \langle x_k, x_i\rangle x_i).$$

- The Kuramoto model for real-valued rank-0 tensors:

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i).$$

Gradient flow formulation

- The Kuramoto model on $\mathbb{R}^N$: van Hemmen-Wreszinski (1993), Dong-Xue ('13), H-Kim-Ryoo ('16)

$$R_k := \left| \frac{1}{N} \sum_{j=1}^{N} e^{\mathrm{i}\theta_j} \right|, \quad V_k(\Theta) = -\nu \cdot \Theta - \kappa N R_k^2.$$

$$\text{The Kuramoto model} \quad \Longleftrightarrow \quad \dot{\Theta} = -\nabla_\Theta V_k(\Theta).$$

- The SS model on $\mathbb{S}^{dN}$: H-Ko-Ryoo ('18)

$$R_s := \left\| \frac{1}{N} \sum_{j=1}^{N} x_j \right\|, \quad V_s(X) = -\frac{\kappa}{2} N R_s^2.$$

$$\text{The SS model with } \Omega_i = \Omega \quad \Longleftrightarrow \quad \dot{x}_i = -\nabla_{x_i} V_s(X)\Big|_{T_{x_i}\mathbb{S}^d}.$$

cf. For a heterogeneous ensemble, the SS model is not a gradient flow on $\mathbb{S}^{dN}$.

- The LM model on $\mathbb{U}(d)^N$: H-Ko-Ryoo ('18)

$$R_m := \Big\| \frac{1}{N} \sum_{j=1}^{N} U_j \Big\|_F, \quad \mathcal{V}_m := -\frac{\kappa}{2} N R_m^2.$$

$$\text{The LM model with } H_i = O \quad \Longleftrightarrow \quad \dot{U}_i = -\nabla_{U_i} \mathcal{V}_{m\, T_{U_i}(d)}.$$

cf. For a heterogeneous ensemble, the LM model is not a gradient flow on $\mathbb{U}(d)^N$.

Lesson from previous models

Consider an ensemble $\{T_j\}_{j=1}^N$ of rank-m tensors over complex field $\mathbb{C}$, and for notational simplicity, we set

$$\alpha_* = (\alpha_1, \cdots, \alpha_m), \qquad \beta_* = (\beta_1, \cdots, \beta_m).$$

Then, we begin with following structure:

$$\frac{d}{dt}[T_j]_{\alpha_*} = \text{free flow} + \text{cubic interactions among components}.$$

- (Modeling of free flow)

Contraction of rank-$2m$ tensor A_j and rank-m tensor T_j:

$$\text{free flow part} = [A_j]_{\alpha_*\beta_*}[T_j]_{\beta_*}.$$

- (Modeling of cubic interactions): for a dummy variable β,

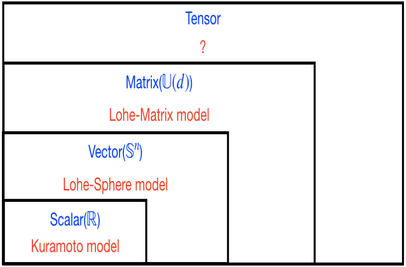
$$[T_c]_{i_1}[\bar{T}_j]_\beta [T_j]_{i_2} - [T_j]_{i_1}[\bar{T}_c]_\beta [T_j]_{i_2}.$$

- **Definition**:

We define the inner product of size $d_1 \times d_2 \times \cdots \times d_m$ as follows.

$$\langle T_i, T_j \rangle_F := [\bar{T}_i]_{\alpha_*}[T_j]_{\alpha_*}, \quad i,j = 1, \cdots, N.$$

1st story: The Lohe tensor model



The Lohe Tensor(LT) Model

- Handy notation:

$$\alpha_{*0} = \alpha_{10}\alpha_{20}\cdots\alpha_{m0}, \quad \alpha_{*1} = \alpha_{11}\alpha_{21}\cdots\alpha_{m1},$$
$$\alpha_{*i_*} = \alpha_{1i_1}\alpha_{2i_2}\cdots\alpha_{mi_m}, \quad \alpha_{*(1-i_*)} = \alpha_{1(1-i_1)}\alpha_{2(1-i_2)}\cdots\alpha_{m(1-i_m)},$$
$$\beta_* = \beta_1\beta_2\cdots\beta_m, \quad i_* = i_1 i_2\cdots i_m.$$

$$\frac{d}{dt}[T_i]_{\alpha_{*0}} = \underbrace{[A_i]_{\alpha_{*0}\beta_*}[T_i]_{\beta_*}}_{\text{Free Flow}}$$
$$+ \underbrace{\sum_{i_*\in\{0,1\}^m} \kappa_{i_*}\left([T_c]_{\alpha_{*i_*}}[\bar{T}_i]_{\alpha_{*1}}[T_i]_{\alpha_{*(1-i_*)}} - [T_i]_{\alpha_{*i_*}}[\bar{T}_c]_{\alpha_{*1}}[T_i]_{\alpha_{*(1-i_*)}}\right)}_{\text{Cubic coupling Terms}}$$

cf. 2^m cubic -coupling terms

We set

$$||T_i||_F := \sqrt{[\bar{T}_i]_{\alpha_*}[T_i]_{\alpha_*}}.$$

- **Lemma**: (Conservation law)

$$||T_i(t)||_F = ||T_i^{in}||_F, \quad t \geq 0.$$

Emergent dynamics

We set

$$\mathcal{D}(T) := \max_{i,j} ||T_i - T_j||_F, \quad \mathcal{D}(A) := \max_{i,j} ||A_i - A_j||_F, \quad \hat{\kappa}_0 := 2\sum_{i_* \neq 0} \kappa_{i_*}.$$

- **Theorem**: (Complete aggregation) H-Park '20, JSP

Suppose that the coupling strength and the initial data satisfy

$$A_j = 0, \quad \hat{\kappa}_0 < \frac{\kappa_0}{2||T_c^{in}||_F^2}, \quad ||T_j^{in}||_F = 1, \quad 0 < \mathcal{D}(T^{in}) < \frac{\kappa_0 - 2\hat{\kappa}_0||T_c^{in}||_F^2}{2\kappa_0}.$$

Then, there exist positive constants C_0 and C_1 depending on κ_{i_*} and T^{in} such that

$$C_0 e^{-\left(\kappa_0 + 2\hat{\kappa}_0||T_c^{in}||_F\right)t} \leq \mathcal{D}(T(t)) \leq C_1 e^{-\left(\kappa_0 - 2\hat{\kappa}_0||T_c^{in}||_F\right)t}, \quad t \geq 0.$$

Proof: By direct estimates, one has Gronwall differential inequality:

$$\left|\frac{d}{dt}\mathcal{D}(T) + \kappa_0\mathcal{D}(T)\right| \leq 2\kappa_0\mathcal{D}(T)^2 + \hat{\kappa}_0||T_c^{in}||_F\mathcal{D}(T), \quad \text{a.e. } t > 0.$$

Let η be the largest root of the quadratic equation:

$$2\kappa_0 x^2 + (\kappa_0 - 2\hat{\kappa}_0 ||T_c^{in}||_F^2)x = \mathcal{D}(A).$$

Then, the root η satisfies

$$0 < \eta < \frac{\kappa_0 - 2\hat{\kappa}_0 ||T_c^{in}||_F^2}{2\kappa_0}.$$

- **Theorem**: (Practical aggregation) H-Park '20, JSP

Suppose that coupling strength, initial data and frequency matrices satisfy

$$\kappa_0 > 0, \quad 0 \leq \mathcal{D}(T(0)) \ll 1 \quad \text{and} \quad \mathcal{D}(A) < \frac{|\kappa_0 - 2\hat{\kappa}_0 ||T_c^{in}||_F^2|^2}{8\kappa_0},$$

Then practical synchronization emerges:

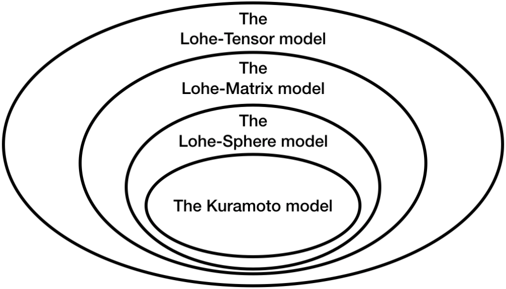
$$\lim_{D(A)/\kappa_0 \to 0+} \limsup_{t \to \infty} D(T(t)) = 0.$$

Proof: By direct estimates, one has Gronwall differential inequality:

$$\frac{d}{dt}\mathcal{D}(T) \leq 2\kappa_0 \mathcal{D}(T)^2 - (\kappa_0 - 2\hat{\kappa}_0 ||T_c^{in}||_F^2)\mathcal{D}(T) + \mathcal{D}(A), \quad \text{a.e. } t > 0.$$

Summary of 1st-story

We have proposed the Lohe tensor model for the set of tensors with the same rank and size:



Under suitable frameworks, we can also show that the above very complicated model exhibits emergent dynamics.

2nd story: Two reductions

- Can we propose an aggregation model on Hermitian unit sphere $\mathbb{HS}^{d-1}$?
- Are there aggregation models for non square matrices, for example $\mathbb{C}^{n \times m}$ with $n \neq m$?

The Lohe hermitian sphere(LHS) model

For rank-1 tensors with size d, the LT model becomes

$$\frac{d}{dt}[z_i]_{\alpha_{10}} = [\Omega_i]_{\alpha_{10}\beta_1}[z_i]_{\beta_1} + \kappa_0\Big([z_c]_{\alpha_{10}} \underbrace{[\bar{z}_i]_{\alpha_{11}}[z_i]_{\alpha_{11}}}_{\text{Contracted}} - [z_i]_{\alpha_{10}} \underbrace{[\bar{z}_c]_{\alpha_{11}}[z_i]_{\alpha_{11}}}_{\text{Contracted}}\Big)$$
$$+ \kappa_1\Big(\underbrace{[z_c]_{\alpha_{11}}[\bar{z}_i]_{\alpha_{11}}}_{\text{Contracted}}[z_i]_{\alpha_{10}} - \underbrace{[z_i]_{\alpha_{11}}[\bar{z}_c]_{\alpha_{11}}}_{\text{Contracted}}[z_i]_{\alpha_{10}}\Big).$$

After contractions, one derive the LHS model:

$$\dot{z}_i = \underbrace{\Omega_i z_i}_{\text{Free Flow}} + \underbrace{\kappa_0(\langle z_i, z_i\rangle z_c - \langle z_c, z_i\rangle z_i)}_{\text{swarm sphere coupling}} + \underbrace{\kappa_1(\langle z_i, z_c\rangle - \langle z_c, z_i\rangle) z_i}_{\text{new coupling}},$$

For $z_j = x_j \in \mathbb{R}^d$, one has the SS model:

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}(\langle x_i, x_i\rangle x_k - \langle x_k, x_i\rangle x_i).$$

Solution splitting property

Consider two Cauchy problems:

$$\begin{cases} \dot{z}_j = \Omega z_j + \kappa_0 \Big(\langle z_j, z_j \rangle z_c - \langle z_c, z_j \rangle z_j \Big) + \kappa_1 \Big(\langle z_j, z_c \rangle - \langle z_c, z_j \rangle \Big) z_j, \\ z_j(0) = z_j^{in}, \quad j = 1, \cdots, N, \end{cases}$$

and

$$\begin{cases} \dot{w}_j = \kappa_0 \Big(w_c \langle w_j, w_j \rangle - w_j \langle w_c . w_j \rangle \Big) + \kappa_1 \Big(\langle w_j, w_c \rangle - \langle w_c, w_j \rangle \Big) w_j, \quad t > 0, \\ w_j(0) = z_j^{in}, \quad j = 1, \cdots, N. \end{cases}$$

- **Proposition**: H-Park '19

$$z_j = e^{\Omega t} w_j, \quad j = 1, \cdots, N.$$

Emergent dynamics of Subsystem A

$$\dot{z}_j = \underbrace{\kappa_0\big(\langle z_j, z_j\rangle z_c - \langle z_c, z_j\rangle z_j\big)}_{\text{swarm sphere coupling}}.$$

- **Theorem**: H-Park '19

Suppose that the coupling strength and initial data satisfy

$$\kappa_0 > 0, \quad ||z_i^{in}|| = 1, \quad \max_{i \neq j} |1 - \langle z_i^{in}, z_j^{in}\rangle| < 1/2.$$

Then, $\exists \ \Lambda = \Lambda(Z^0) > 0$ such that

$$\mathcal{D}(Z(t)) \leq \mathcal{D}(Z^{in}) e^{-\kappa_0 \Lambda t}, \quad t \geq 0.$$

Emergent dynamics of Subsystem B

$$\dot{z}_j = \kappa_1(\langle z_j, z_c\rangle - \langle z_c, z_j\rangle)z_j, \quad t > 0.$$

- **Theorem**: H-Park '19

1. There exists a time-dependent phase θ_j such that

$$z_j(t) = e^{\mathrm{i}\theta_j(t)} z_j^{in}, \quad j = 1, \cdots, N.$$

2. If we set R_{jk}^{in} and α_{ji} such that

$$\langle z_j^{in}, z_k^{in}\rangle = R_{jk}^{in} e^{\mathrm{i}\alpha_{jk}},$$

then the phase θ_j in (1) is a solution to the following Cauchy problem:

$$\begin{cases} \dot{\theta}_j = \dfrac{2\kappa_1}{N}\displaystyle\sum_{k=1}^{N} R_{jk}^{in}\sin(\theta_k - \theta_j + \alpha_{jk}), \quad t > 0, \\ \theta_j(0) = 0, \end{cases}$$

where R_{jk}^{in} and α_{jk} satisfy symmetry and anti-symmetry properties:

$$R_{jk}^{in} = R_{kj}^{in}, \qquad \alpha_{jk} = -\alpha_{kj}, \quad \forall\ k, j = 1, \cdots, N.$$

Emergent dynamics of the full system

$$\dot{z}_j = \kappa_0\Big(z_c\langle z_j, z_j\rangle - z_j\langle z_c.z_j\rangle\Big) + \kappa_1\Big(\langle z_j, z_c\rangle - \langle z_c, z_j\rangle\Big)z_j.$$

Following the terminology in literature, we set

$$\rho = R_s = \Big|\frac{1}{N}\sum_{j=1}^{N} z_j\Big|.$$

- **Proposition**: H-Park '19

Let $\{z_j\}$ be a solution with initial condition $\rho^{in} > 0$. Then,

$$\exists\ \rho^{\infty} := \lim_{t\to\infty}\rho(t) > 0, \quad \lim_{t\to\infty}\langle z_i, z_c\rangle \in \{1, -1\}.$$

where $\rho = \|z_c\|$.

Proof: The above results are based on

$$\frac{d\rho^2}{dt} = \frac{2\kappa_0}{N}\sum_{i=1}^{N}\Big(\rho^2 - |\langle z_i, z_c\rangle|^2\Big) + \frac{4(\kappa_0+\kappa_1)}{N}\sum_{i=1}^{N}\Big|\mathrm{Im}(\langle z_i, z_c\rangle)\Big|^2.$$

- **Remark**:

1. Let $\{z_j\}$ be a solution with initial condition $\rho^{in} > \frac{N-2}{N}$. Then we have
$$\lim_{t\to\infty} \rho(t) = 1.$$
If each clusters contain l and $N-l$ particles, then we have $\rho = \frac{|N-2l|}{N}$. So if $\rho^{in} > \frac{N-2}{N}$, we can obtain $l = 0$ or $l = N$. That means there is only one cluster. i.e. complete aggregation.

2. Let $\{z_j\}$ be a solution with initial condition $\rho^{in} > 0$. Then, ρ is increasing along the flow. Thus, they will be no nontrivial periodic solution.

- **Theorem**: H-Park '19

Suppose that the coupling strengths and initial data satisfy

$$\Omega_j = 0, \quad 0 < \kappa_1 < \frac{1}{4}\kappa_0, \quad \rho^{in} > \frac{N-2}{N},$$

Then $\mathcal{D}(X)$ converges to zero exponentially fast.

Proof. We introduce a Lyapunov functional $\mathcal{L}(Z)$:

$$\mathcal{L}(Z) := \max_{1 \le i,j \le N} |1 - \langle z_i, z_j \rangle|^2,$$

Then, one has

$$\frac{d}{dt}\mathcal{L}(Z) \le -\kappa_0 \mathcal{L}(Z)\left(\mathrm{Re}(\langle x_{i_0} + x_{j_0}, z_c\rangle) - \frac{4\kappa_1}{\kappa_0}\right).$$

The generalized Lohe matrix(GLM) model

If we take

$$m = 2,$$

then the LT model becomes

$$\dot{T}_i = \underbrace{A_i T_i}_{\text{free flow}} + \underbrace{\kappa_{00}(\mathrm{tr}(T_i^* T_i) T_c - \mathrm{tr}(T_c^* T_i) T_i)}_{\text{LHS coupling}}$$
$$+ \underbrace{\kappa_{01}(T_c T_i^* T_i - T_i T_c^* T_i)}_{\text{Lohe matrix coupling}} + \underbrace{\kappa_{10}(T_i T_i^* T_c - T_i T_c^* T_i)}_{\text{Lohe matrix coupling}}$$
$$+ \underbrace{\kappa_{11}(\mathrm{tr}(T_i^* T_c) - \mathrm{tr}(T_c^* T_i)) T_i}_{\text{LHS coupling}}.$$

If we set

$$\kappa_{00} = 0, \quad \kappa_{01} = \kappa_1, \quad \kappa_{10} = \kappa_2, \quad \kappa_{11} = 0.$$

then, we can obtain the GLM model in a mean-field form H-Park '20 :

$$\begin{cases} \dot{T}_i = A_i T_i + \kappa_1 (T_c T_i^* T_i - T_i T_c^* T_i) + \kappa_2 (T_i T_i^* T_c - T_i T_c^* T_i), \\ T_i(0) = T_i^0, \quad \| T_i^0 \|_F = 1, \quad T_c := \frac{1}{N} \sum_{k=1}^{N} T_k. \end{cases}$$

cf. Emergent dynamics (DCDS-B (2021): Emergent behaviors of the generalized Lohe matrix model:

Exponential aggregation: homogeneous ensemble,

Practical aggregation: heterogeneous ensemble.

Two reductions from GLM

- From the GLM model to the LM model

Let $T_i \in \mathbb{U}(d)$, i.e.,

$$T_i T_i^* = T_i^* T_i = I_d.$$

Note that interaction terms are the same:

$$\begin{aligned} T_c T_i^* T_i - T_i T_c^* T_i &= T_i T_i^* T_c - T_i T_c^* T_i = T_c - T_i T_c^* T_i \\ &= T_c - \langle T_c, T_i \rangle_F T_i. \end{aligned}$$

Thus, the GLM model reduces to the LM model:

$$\dot{T}_i = A_i T_i + (\kappa_1 + \kappa_2)(T_c - \langle T_c, T_i \rangle_F T_i).$$

• From the GLM to the LHS

Let T_j be a rank-2 tensor with size $d \times 1$, i.e.,

$$d_1 = d, \quad d_2 = 1, \quad T_j = z_j.$$

Recall the GLM model:

$$\dot{T}_j = A_j T_j + \kappa_1 (T_c T_j^* T_j - T_j T_c^* T_j) + \kappa_2 (T_j T_j^* T_c - T_j T_c^* T_j).$$

Note that

$$T_j^* T_j = \langle T_j, T_j \rangle, \quad T_c^* T_j = \langle T_c, T_j \rangle,$$
$$T_j^* T_c = \langle T_j, T_c \rangle, \quad T_c^* T_j = \langle T_c, T_j \rangle.$$

Thus, one has the LHS model:

$$\dot{z}_j = \Omega_j z_j + \kappa_1 (z_c \langle z_j, z_j \rangle - z_j \langle z_c . z_j \rangle) + \kappa_2 (\langle z_j, z_c \rangle - \langle z_c z_j \rangle) z_j,$$

Summary of 2nd story

- We provided the LHS model on $\mathbb{HS}^{d-1}$ which generalize the swarm sphere model:

$$\dot{z}_j = \Omega_j z_j + \kappa_1 (z_c \langle z_j, z_j \rangle - z_j \langle z_c . z_j \rangle) + \kappa_2 (\langle z_j, z_c \rangle - \langle z_c z_j \rangle) z_j.$$

- We also provide the GLM model on $\mathbb{C}^{d_1 \times d_2}$ with $d_1 \neq d_2$.

$$\dot{T}_j = A_j T_j + \kappa_1 (T_c T_j^* T_j - T_j T_c^* T_j) + \kappa_2 (T_j T_j^* T_c - T_j T_c^* T_j).$$

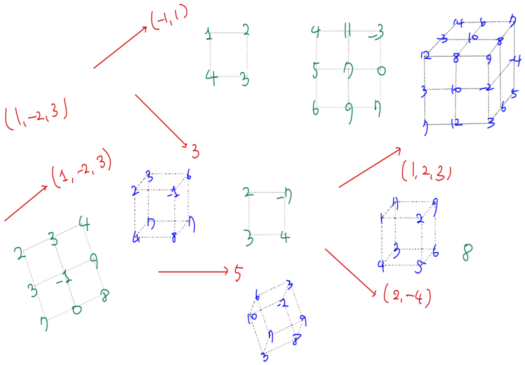
  This model reduces to the LM model and LHS model for special cases.

Emergent dynamics for heterogeneous ensemble is largely open except weak estimate (practical aggregation).

3rd story: From low-rank to high-rank

How to introduce a weak coupling between LT models with the same rank?

Mixture of tensors



"How to segregate a mixture of tensors into ensembles with the same rank and size?"

Weak coupling of two SS models with different sizes

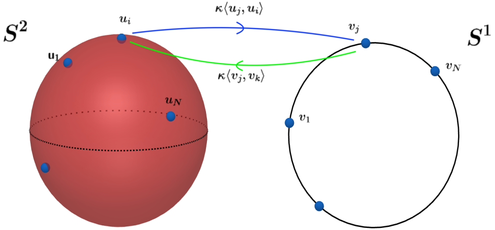
Consider two SS models on $\mathbb{S}^{d_1-1}$ and $\mathbb{S}^{d_2-1}$:

$$\dot{u}_i = \Omega_i u_i + \frac{\kappa}{N}\sum_{j=1}^{N}(u_j - \langle u_i, u_j\rangle u_i), \quad u_i \in \mathbb{S}^{d_1-1},$$

$$\dot{v}_i = \Lambda_i v_i + \frac{\kappa}{N}\sum_{j=1}^{N}(v_j - \langle v_i, v_j\rangle v_i), \quad v_i \in \mathbb{S}^{d_2-1}.$$

"How to couple the above SS models weakly?"

Lohe's idea on weak coupling



M. A. Lohe: On the double sphere model of synchronization, Physica D (2020).

The double sphere model

- Weakly coupled double sphere model:

$$\begin{cases} \dot{u}_i = \Omega_i u_i + \frac{\kappa}{N}\sum_{j=1}^{N}\langle v_i, v_j\rangle(u_j - \langle u_i, u_j\rangle u_i), \quad t > 0, \\ \dot{v}_i = \Lambda_i v_i + \frac{\kappa}{N}\sum_{j=1}^{N}\langle u_i, u_j\rangle(v_j - \langle v_i, v_j\rangle v_i), \\ (u_i, v_i)(0) = (u_i^0, v_i^0) \in \mathbb{S}^{d_1-1} \times \mathbb{S}^{d_2-1}, \end{cases}$$

where $\Omega_i \in \mathbb{R}^{d_1 \times d_1}$ and $\Lambda_i \in \mathbb{R}^{d_2 \times d_2}$ are real skew-symmetric matrices, respectively and $\kappa > 0$.

The double sphere on $\mathbb{S}^{d_1-1} \times \mathbb{S}^{d_2-1}$ is positively invariant under the DS flow.

A gradient flow formulation

Define a potential:

$$\mathcal{E}(U, V) := 1 - \frac{1}{N^2} \sum_{i,j=1}^{N} \langle u_i, u_j \rangle \langle v_i, v_j \rangle.$$

- **Theorem**: (H-Kim-Park '21, JSP)

The DS model with $\Omega_i = 0$ and $\Lambda_i = 0$ is a gradient flow on the compact state space $(\mathbb{S}^{d_1-1} \times \mathbb{S}^{d_2-1})^N$:

$$\dot{u}_i = -\frac{N\kappa}{2} \mathbb{P}_{T_{u_i}\mathbb{S}^{d_1-1}} \Big(\nabla_{u_i} \mathcal{E}(U, V) \Big),$$
$$\dot{v}_i = -\frac{N\kappa}{2} \mathbb{P}_{T_{v_i}\mathbb{S}^{d_2-1}} \Big(\nabla_{v_i} \mathcal{E}(U, V) \Big),$$

- **Corollary**: $\exists\ (U^\infty, V^\infty) \in (\mathbb{S}^{d_1-1})^N \times (\mathbb{S}^{d_2-1})^N$ such that

$$\lim_{t\to\infty} (U(t), V(t)) = (U^\infty, V^\infty).$$

- **Proposition** Suppose system parameters and initial data satisfy

$$\Omega_i = \Omega, \quad \Lambda_i = \Lambda, \quad \min_{1\le i,j\le N} \langle u_i^0, u_j^0 \rangle > 0, \quad \min_{1\le i,j\le N} \langle v_i^0, v_j^0 \rangle > 0.$$

Then, one has the complete segregation:

$$\lim_{t\to\infty} \max_{1\le i,j\le N} |u_i(t) - u_j(t)| = 0 \quad \text{and} \quad \lim_{t\to\infty} \max_{1\le i,j\le N} |v_i(t) - v_j(t)| = 0.$$

Connection between DS model and LT model

- **Proposition**: (H-Kim-Park '21, JSP)

1. Let $\{(u_i, v_i)\}$ be a solution to DS model with initial data $\{(u_i^0, v_i^0)\}$. Then, rank-2 tensor $T_i := u_i \otimes v_i \in \mathbb{R}^{d_1 \times d_2}$ is a completely separable solution to the GLM model with

$$A_i T_i := \Omega_i T_i + T_i \Lambda_i^\top, \quad \kappa_1 = \kappa_2 = \kappa, \quad T_i^0 =: u_i^0 \otimes v_i^0.$$

2. For a solution T_i to the LGM model with completely separable initial data $T_i^0 =: u_i^0 \otimes v_i^0$, there exist two unit vectors $u_i = u_i(t)$ and $v = v_i(t)$ such that

$$T_i(t) = u_i(t) \otimes v_i(t), \quad t > 0,$$

where (u_i, v_i) is a solution to the SDS model with $(u_i, v_i)(0) = (u_i^0, v_i^0)$.

The multi-sphere(MS) model

Using a gradient flow approach with the potential:

$$\mathcal{E} := 1 - \frac{1}{N^2} \sum_{i,j=1}^{N} \prod_{k=1}^{m} \langle u_i^k, u_j^k \rangle,$$

one can derive

$$\begin{cases} \dot{u}_i^k = \Omega_i^k u_i^k + \frac{\kappa}{N} \sum_{j=1}^{N} \left(\prod_{\substack{\ell \neq k \\ \ell=1}}^{N} \langle u_i^\ell, u_j^\ell \rangle \right) (u_j^k - \langle u_i^k, u_j^k \rangle u_i^k), \quad t > 0, \\ u_i^k(0) = u_i^{k,0} \in \mathbb{S}^{d_k - 1}, \quad 1 \leq i \leq N, \quad 1 \leq k \leq m. \end{cases}$$

cf. (H-Kim-Park '21, JSP)

- **Theorem** (H-Kim-Park '21, JSP) Suppose the initial data satisfy

$$\min_{1\le i,j\le N}\langle u_i^{k,0}, u_j^{k,0}\rangle > 0, \quad k = 1, \cdots, m,$$

then one has

$$\lim_{t\to\infty}\max_{1\le i,j\le N}|u_i^k(t) - u_j^k(t)| = 0, \quad k = 1, \cdots, m.$$

- **Theorem**: (H-Kim-Park '21, JSP) Under suitable assumptions on the coupling strengths and natural frequency tensors, if the initial data $\{T_i^0\}$ is completely separable

$$T_i^0 = u_i^{1,0} \otimes u_i^{2,0} \otimes \cdots \otimes u_i^{m,0},$$

then a solution $T_i = T_i(t)$ is uniquely determined by the following relation:

$$T_i(t) = u_i^1(t) \otimes u_i^2(t) \otimes \cdots \otimes u_i^m(t), \quad t > 0.$$

where $\{u_i\}$ is a solution to the MS model.

The double matrix model

- Weak coupling of two LM models (H-Kim-Park '21):

$$\begin{cases} \dot{U}_j = -\mathrm{i}H_jU_j + \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\left(\langle V_j, V_k\rangle_F\, U_kU_j^*U_j - \langle V_k, V_j\rangle_F\, U_jU_k^*U_j\right) \\ \dot{V}_j = -\mathrm{i}G_jV_j + \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\left(\langle U_j, U_k\rangle_F\, V_kV_j^*V_j - \langle U_k, U_j\rangle_F\, V_jV_k^*V_j\right). \end{cases}$$

Gradient flow formulation, emergent dynamics, extension to the multiple matrix model. Algebraic method for coupling of LT models is under way.

Weak coupling of SS model and LM model

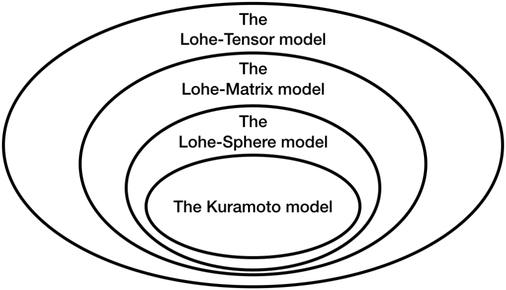
$$
\begin{cases}
\dot{x}_j = \Omega_j x_j + \frac{\kappa}{N} \sum_{k=1}^{N} \langle U_j, U_k \rangle_F (x_k - \langle x_k, x_j \rangle x_j), \\
\dot{U}_j = A_j U_j + \frac{\kappa}{2N} \sum_{k=1}^{N} \langle x_j, x_k \rangle (U_k - U_j U_k^* U_j), \\
(x_j, U_j)(0) = (x_j^0, U_j^0) \in \mathbb{S}^{d-1} \times \mathbb{U}(d).
\end{cases}
$$

Summary of 3rd story

We have discussed a part of story for the systematic weak couplings of multiple LT models. As byproducts of our generalized approach, we can derive Lohe hermitian sphere model and generalized Lohe matrix model.

Summary of two lectures

- In these two lectures, we have provided a picture on "Hierarchy of finite-dimensional counterpart of the Lohe type aggregation models"



- Via weak coupling of LT type models, we can derive a systematic algebraic methodology to combine multiple LT models weakly.

The Schrödinger-Lohe model

M. Lohe, J. Phys. A ('10)

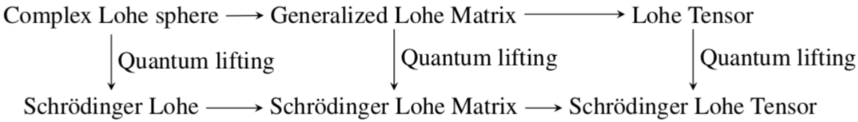
$$\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + V_j\psi_j + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N}\left(\psi_k - \frac{\langle \psi_k, \psi_j\rangle}{\langle \psi_j, \psi_j\rangle}\psi_j\right).$$

Here, $V_j = V_j(x)$ represents an external one-body potential acted on j-th node, and κ measures a coupling strength between oscillators.

- The S-L model enjoys L^2-conservation:

$$\|\psi_j(t)\|_{L^2(^d)} = 1, \quad t > 0.$$

Quantum lifting



"On the Schrodinger-Lohe hierarchy for aggregation and its emergent dynamics" by Ha, S.-Y and Park, H. appeared in JSP (2020).

The END

Questions and Comments

Thank you for your attention !!!