A relativistic generalization of the BGK model for gas mixtures

Byung-Hoon Hwang
(with Myeong-Su Lee and Seok-Bae Yun)

Department of Mathematics, Sungkyunkwan University

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Classical BGK model

- BGK model (Bhatnagar-Gross-Krook, 1954)

$$\partial_t f + v \cdot \nabla_x f = Q := \nu\,(\mathcal{M} - f)$$

  - $f \equiv f(t,x,v)$ is the velocity distribution function representing the number density of particles on the phase space $\mathbb{R}^3_x (or\ \mathbb{T}^3_x) \times \mathbb{R}^3_v$ at time $t \geq 0$.
  - ν is the collision frequency, and $\mathcal{M}$ is the Maxwellian distribution function describing a state of gases in equilibrium

$$\mathcal{M}(\rho, U, T; v) = \frac{\rho(t,x)}{(2\pi T(t,x))^{3/2}} e^{-\frac{|v-U(t,x)|^2}{2T(t,x)}}$$

  where ρ is the macroscopic density, U the bulk velocity, and T the temperature.

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- Classical particles
- Relativistic particles
- Quantum particles
- Polyatomic molecules
- Gas mixtures

The BGK model was basically proposed for a monatomic non-ionized gas, but with the development of the Boltzmann equation, it has been also developed for various types of a gaseous system;

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- Relativistic particles
- Quantum particles
- Polyatomic molecules
- Gas mixtures

In this talk, we introduce the BGK model for gas mixtures in a special relativistic setting.

Classical BGK model for gas mixtures

- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

$$\partial_t f^i + v \cdot \nabla_x f^i = Q^i := \nu^i \left(\mathcal{M}^i - f^i \right), \qquad i = 1, \cdots, N.$$

  - $f^i \equiv f^i(t,x,v)$ is the velocity distribution function for species i
  - $\mathcal{M}^i$ is the Maxwellian attractors

$$\mathcal{M}^i(\widetilde{n}^i, \widetilde{U}, \widetilde{T}; v) = \frac{\widetilde{n}^i(t,x)}{\left(\frac{2\pi \widetilde{T}(t,x)}{m^i}\right)^{\frac{3}{2}}} e^{-m^i \frac{|v-\widetilde{U}(t,x)|^2}{2\widetilde{T}(t,x)}}$$

where $\widetilde{n}^i$ is the auxiliary species number densities, $\widetilde{U}$ the common velocity, and $\widetilde{T}$ the common temperature

$$\int_{\mathbb{R}^3} Q^i \, dv = 0, \qquad \sum_{i=1}^{N} \int_{\mathbb{R}^3} (v, |v|^2) Q^i \, dv = 0.$$

- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

$$\partial_t f^i + v \cdot \nabla_x f^i = Q^i := \nu^i \left(\mathcal{M}^i - f^i \right), \qquad i = 1, \cdots, N.$$

  - Conservation laws for species number densities

$$\partial_t \int_{\mathbb{R}^3} f^i \, dv + \operatorname{div} \int_{\mathbb{R}^3} v f^i \, dv = 0,$$

  and global momentum and total energy.

$$\sum_{i=1}^N \partial_t \int_{\mathbb{R}^3} (v, |v|^2) f^i \, dv + \sum_{i=1}^N \operatorname{div} \int_{\mathbb{R}^3} v(v, |v|^2) f^i \, dv = 0.$$

  - Entropy inequality

$$\partial_t \sum_{i=1}^N \int_{\mathbb{R}^3} f^i \ln f^i \, dv + \sum_{i=1}^N \operatorname{div} \int_{\mathbb{R}^3} v f^i \ln f^i \, dv \leq 0.$$

Relativistic BGK model

- Relativistic generalizations of the BGK model (Marle, 1965)

$$\partial_t f + \frac{cp}{p^0} \cdot \nabla_x f = Q_{Marle} := \frac{cm}{\tau p^0} (\mathcal{J} - f).$$

  - $f \equiv f(x^\alpha, p^\alpha)$ is the momentum distribution function representing the number density of particles in the phase space spanned by the spacetime coordinates x^α and four-momentum p^α

$$x^\alpha = (ct, x), \qquad p^\alpha = \left(\sqrt{(mc)^2 + |p|^2}, p\right).$$

  - τ is the characteristic time of order of the mean free time.
  - $\mathcal{J} \equiv \mathcal{J}(\mu_E, u, T; p)$ is the Jüttner distribution function (Jüttner, 1911)

Relativistic BGK model for gas mixtures

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0}\cdot\nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i) \qquad (i = 1, ..., N)$$

  - $f_i \equiv f_i(x^\alpha, p_i^\alpha)$ is the momentum distribution function for species i with

$$p_i^\alpha = \left(\sqrt{(m_i c)^2 + |p_i|^2}, p_i\right)$$

  - $\mathcal{J}_i$ is the Jüttner distribution attractors

$$\mathcal{J}_i = \frac{\int_{\mathbb{R}^3} f_i \, \frac{dp_i}{p_i^0}}{\int_{\mathbb{R}^3} e^{-c\beta p_i^0} \, \frac{dp_i}{p_i^0}} e^{-\beta u^\mu p_{i\mu}}.$$

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i) \qquad (i = 1, ..., N)$$

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Here u^μ $(u^\mu u_\mu = c^2)$ is the common four-velocity

$$u^\mu = c \frac{\sum_{i=1}^N \frac{m_i}{\tau_i} N_i^\mu}{\sqrt{\left(\sum_{i=1}^N \frac{m_i}{\tau_i} N_i^\mu\right)\left(\sum_{j=1}^N \frac{m_j}{\tau_j} N_{j\mu}\right)}} \qquad \text{with} \qquad N_i^\mu = c\int_{\mathbb{R}^3} p_i^\mu f_i \frac{dp_i}{p_i^0},$$

and β $(= 1/kT)$ denotes the inverse of common temperature which is determined by the nonlinear relation

$$\sum_{i=1}^N \frac{m_i}{\tau_i} \frac{\int_{\mathbb{R}^3} e^{-c\beta p_i^0} \, dp_i}{\int_{\mathbb{R}^3} e^{-c\beta p_i^0} \, \frac{dp_i}{p_i^0}} \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c}\left\{\left(\sum_{i=1}^N \frac{m_i}{\tau_i} N_i^\mu\right)\left(\sum_{j=1}^N \frac{m_j}{\tau_j} N_{j\mu}\right)\right\}^{\frac{1}{2}}.$$

Determination problem of the common temperature

Note that β is determined through the nonlinear relation

$$\sum_{i=1}^{N} \frac{m_i}{\tau_i} \frac{\int_{\mathbb{R}^3} e^{-c\beta p_i^0}\, dp_i}{\int_{\mathbb{R}^3} e^{-c\beta p_i^0} \frac{dp_i}{p_i^0}} \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c} \left\{ \sum_{i,j}^{N} (2-\delta_{ij}) \frac{m_i m_j}{\tau_i \tau_j} n_i n_j u_i^{\mu} u_{j\mu} \right\}^{\frac{1}{2}}.$$

Here the integrals for β are not explicitly computed due to the factor p_i^0

$$\int_{\mathbb{R}^3} e^{-c\beta p_i^0}\, dp_i = \int_{\mathbb{R}^3} e^{-c\beta\sqrt{(mc)^2+|p_i|^2}}\, dp_i$$

$$\int_{\mathbb{R}^3} e^{-c\beta p_i^0} \frac{dp_i}{p_0^i} = \int_{\mathbb{R}^3} \frac{1}{\sqrt{(mc)^2+|p_i|^2}} e^{-c\beta\sqrt{(mc)^2+|p_i|^2}}\, dp_i$$

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In order to study the existence theorem of the this model, we need to show whether or not β can be uniquely determined as the moments of f_i throught the above relation.

Note that β is determined through the nonlinear relation

$$\sum_{i=1}^{N} \frac{m_i}{\tau_i} \frac{\int_{\mathbb{R}^3} e^{-c\beta p_i^0}\, dp_i}{\int_{\mathbb{R}^3} e^{-c\beta p_i^0}\, \frac{dp_i}{p_i^0}} \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c} \left\{ \sum_{i,j}^{N} (2-\delta_{ij}) \frac{m_i m_j}{\tau_i \tau_j} n_i n_j u_i^{\mu} u_{j\mu} \right\}^{\frac{1}{2}}.$$

- Key idea of proof
  1. Strict monotonicity

$$\frac{d}{d\beta} \left\{ \frac{\int_{\mathbb{R}^3} e^{-c\beta p_i^0}\, dp_i}{\int_{\mathbb{R}^3} e^{-c\beta p_i^0}\, \frac{dp_i}{p_i^0}} \right\} < 0$$

  2. Comparison of ranges on the left and right sides

$$\text{range of l.h.s} \supseteq \text{range of r.h.s}$$

As a result, we can conclude that there exists a 1-1 correspondence which guarantees the unique existence of β.

Properties

(1) Conservation laws for each species, and total energy-momentum tensor

$$\partial_\mu N_i^\mu = 0, \qquad \sum_{i=1}^N \partial_\nu T_i^{\mu\nu} = 0.$$

(2) The momentum distribution functions are non-negative

$$f_i \geq 0 \qquad i = 1, \cdots, N$$

(3) Entropy inequality

$$\sum_{i=1}^N \partial_t \int_{\mathbb{R}^3} f_i \ln f_i \, dp_i + \sum_{i=1}^N \mathrm{div}_x \int_{\mathbb{R}^3} \frac{cp_i}{p_i^0} f_i \ln f_i \, dp_i \leq 0$$

(4) Indifferentiability principle
: When all m_i and τ_i are identical, the total distribution $f = \sum f_i$ obeys the single species relativistic BGK model.

Conservation laws

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i) \qquad (i = 1, ..., N)$$

Since $\mathcal{J}_i$ is defined in a way to satisfy the following constraints

$$\int_{\mathbb{R}^3} Q_i \, dp_i = 0, \qquad \sum_{i=1}^{N} \int_{\mathbb{R}^3} p_i^\mu Q_i \, dp_i = 0,$$

we get

$$\int_{\mathbb{R}^3} \partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i \, dp_i = 0, \qquad \sum_{i=1}^{N} \int_{\mathbb{R}^3} p_i^\mu \left(\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i \right) dp_i = 0,$$

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i) \qquad (i = 1, ..., N)$$

Since $\mathcal{J}_i$ is defined in a way to satisfy the following constraints

$$\int_{\mathbb{R}^3} Q_i \, dp_i = 0, \qquad \sum_{i=1}^{N} \int_{\mathbb{R}^3} p_i^\mu Q_i \, dp_i = 0,$$

we get

$$\frac{\partial N_i^\mu}{\partial x^\mu} = 0, \qquad \sum_{i=1}^{N} \frac{\partial T_i^{\mu\nu}}{\partial x^\nu} = 0.$$

Non-negativity

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i) \qquad (i = 1, ..., N)$$

- Mild form of the model equation

$$f_i = e^{-\frac{cm_i}{\tau_i p_i^0} t} f_0 + \int_0^t e^{-\frac{cm_i}{\tau_i p_i^0}(t-s)} \frac{cm_i}{\tau_i p_i^0} \mathcal{J}_i(f_i)\, ds$$

Entropy inequality

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0}\cdot\nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i) \qquad (i = 1, ..., N)$$

Since Q_i satisfies

$$\sum_{i=1}^N \int_{\mathbb{R}^3} (1, p_i^\mu) Q_i \, dp_i = 0,$$

we obtain

$$\begin{aligned}\sum_{i=1}^N \int_{\mathbb{R}^3} Q_i \ln f_i \, dp_i &= \sum_{i=1}^N \int_{\mathbb{R}^3} Q_i (\ln f_i - \ln \mathcal{J}_i)\, dp_i \\ &= \sum_{i=1}^N \frac{cm_i}{\tau_i} \int_{\mathbb{R}^3} \mathcal{J}_i \Big(1 - \frac{f_i}{\mathcal{J}_i}\Big) \ln \frac{f_i}{\mathcal{J}_i} \frac{dp_i}{p_i^0} \\ &\le 0\end{aligned}$$

Thus,

$$\sum_{i=1}^N \partial_t \int_{\mathbb{R}^3} f_i \ln f_i \, dp_i + \sum_{i=1}^N \mathrm{div}_x \int_{\mathbb{R}^3} \frac{cp_i}{p_i^0} f_i \ln f_i \, dp_i \le 0$$

Indifferentiability principle

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i) \qquad (i = 1, ..., N)$$

Letting

$$m_1 = m_2 = \cdots = m_N \ (= m), \qquad \tau_1 = \tau_2 = \cdots = \tau_N \ (= \tau),$$

and summing the model equations over i, we get

$$\partial_t f + \frac{cp}{p^0} \cdot \nabla_x f = \frac{cm}{\tau p^0}\left(\sum_{i=1}^N \mathcal{J}_i - f\right)$$

where $\sum_{i=1}^N \mathcal{J}_i$ is given by

$$\sum_{i=1}^N \mathcal{J}_i = \frac{\sum_{i=1}^N \int_{\mathbb{R}^3} f_i \frac{dp}{p^0}}{\int_{\mathbb{R}^3} e^{-c\beta p^0} \frac{dp}{p^0}} e^{-\beta u^\mu p_\mu} = \frac{\int_{\mathbb{R}^3} f \frac{dp}{p^0}}{\int_{\mathbb{R}^3} e^{-c\beta p^0} \frac{dp}{p^0}} e^{-\beta u^\mu p_\mu}.$$

It only remains to show that the total distribution $f = \sum f_i$ obeys the common four-velocity u^μ and the inverse of common temperature β.

- common four-velocity u^μ

$$u^\mu = c\frac{\frac{m}{\tau}\sum_{i=1}^N N_i^\mu}{\sqrt{\left(\frac{m}{\tau}\sum_{i=1}^N N_i^\mu\right)\left(\frac{m}{\tau}\sum_{j=1}^N N_{j\mu}\right)}}.$$

- β, the inverse of common temperature

$$\frac{m}{\tau}\frac{\int_{\mathbb{R}^3} e^{-c\beta p^0}\,dp}{\int_{\mathbb{R}^3} e^{-c\beta p^0}\,\frac{dp}{p^0}}\sum_{i=1}^N\int_{\mathbb{R}^3} f_i\,\frac{dp}{p^0} = \frac{1}{c}\left\{\left(\frac{m}{\tau}\sum_{i=1}^N N_i^\mu\right)\left(\frac{m}{\tau}\sum_{j=1}^N N_{j\mu}\right)\right\}^{\frac{1}{2}}.$$

Since

$$\sum_{i=1}^N N_i^\mu = \sum_{i=1}^N c\int_{\mathbb{R}^3} f_i\,\frac{dp}{p^0} \equiv c\int_{\mathbb{R}^3} f\,\frac{dp}{p^0}$$

we obtain

$$\partial_t f + \frac{cp}{p^0}\cdot\nabla_x f = \frac{cm}{\tau p^0}\left(\sum_{i=1}^N \mathcal{J}_i - f\right) \equiv \frac{cm}{\tau p^0}\left(\mathcal{J} - f\right).$$

Newtonian limit

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i), \qquad i = 1, ..., N.$$

- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

$$\partial_t f^i + v \cdot \nabla_x f^i = Q^i := \nu^i \left(\mathcal{M}^i - f^i\right), \qquad i = 1, \cdots, N.$$

Question) Does the relativistic BGK model approach the classical BGK model in the Newtonian limit?

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0}\cdot\nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0}(\mathcal{J}_i - f_i), \qquad i = 1, ..., N.$$

- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

$$\partial_t f^i + v\cdot\nabla_x f^i = Q^i := \nu^i\left(\mathcal{M}^i - f^i\right), \qquad i = 1, \cdots, N.$$

  - Dimensionless number

$$t = \bar{t}s, \quad x = \bar{x}L, \quad p_i = \bar{v}\mu_i, \quad f_i(t, x, p_i) = \frac{\mathcal{N}_i}{\mu_i^3}\bar{f}_i(\bar{t}, \bar{x}, \bar{v}) \quad \Rightarrow \quad \mu_i = m_i L/s.$$

  - Notations

$$\nu_i := \frac{s}{\tau_i}, \qquad \varepsilon := \frac{\mu_i}{cm_i}.$$

Substituting the dimensionless numbers into the relativistic BGK model, we get

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\frac{\partial}{\partial \bar{t}}\bar{f}_i + \frac{1}{\sqrt{1+|\varepsilon \bar{v}|^2}}\bar{v}\cdot\nabla_{\bar{x}}\bar{f}_i = \frac{\nu_i}{\sqrt{1+|\varepsilon \bar{v}|^2}}\left(\bar{\mathcal{J}}_i - \bar{f}_i\right), \qquad i = 1, \cdots, N.$$

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- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

$$\partial_t f^i + v\cdot\nabla_x f^i = Q^i := \nu^i\left(\mathcal{M}^i - f^i\right), \qquad i = 1, \cdots, N.$$

- In the Newtonian limit ($\varepsilon \to 0$), we obtain

$$\bar{\mathcal{J}}_i \to \mathcal{M}_i$$

which completes the proof.

Future works

1. Chapman-Enskog method
2. Existence theory
3. Extension to (polyatomic molecules/chemically reactive mixtures)
4. Anderson-Witting's formulation

Thank you very much