A relativistic generalization of the BGK model for gas mixtures

Byung-Hoon Hwang
(with Myeong-Su Lee and Seok-Bae Yun)

Department of Mathematics, Sungkyunkwan University

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Classical BGK model

• BGK model (Bhatnagar-Gross-Krook, 1954)

\[ \partial_t f + v \cdot \nabla_x f = Q := \nu (M - f) \]

- \( f \equiv f(t, x, \nu) \) is the velocity distribution function representing the number density of particles on the phase space \( \mathbb{R}_x^3 (or \, T_x^3) \times \mathbb{R}_v^3 \) at time \( t \geq 0 \).
- \( \nu \) is the collision frequency, and \( M \) is the Maxwellian distribution function describing a state of gases in equilibrium

\[
M(\rho, U, T; \nu) = \frac{\rho(t, x)}{(2\pi T(t, x))^{3/2}} e^{-\frac{|v-U(t,x)|^2}{2T(t,x)}}
\]

where \( \rho \) is the macroscopic density, \( U \) the bulk velocity, and \( T \) the temperature.
The BGK model was basically proposed for a monatomic non-ionized gas, but with the development of the Boltzmann equation, it has been also developed for various types of a gaseous system.
The BGK model was basically proposed for a monatomic non-ionized gas, but with the development of the Boltzmann equation, it has been also developed for various types of a gaseous system;

- Classical particles
- Relativistic particles
- Quantum particles
- Polyatomic molecules
- Gas mixtures
The BGK model was basically proposed for a monatomic non-ionized gas, but with the development of the Boltzmann equation, it has been also developed for various types of a gaseous system;

- Classical particles
- Relativistic particles
- Quantum particles
- Polyatomic molecules
- Gas mixtures

In this talk, we introduce the BGK model for gas mixtures in a special relativistic setting.
Classical BGK model for gas mixtures

- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

\[ \partial_t f^i + v \cdot \nabla_x f^i = Q^i := \nu^i (M^i - f^i), \quad i = 1, \cdots, N. \]

- \( f^i \equiv f^i(t, x, v) \) is the velocity distribution function for species \( i \)
- \( M^i \) is the Maxwellian attractors

\[ M^i(\tilde{n}^i, \tilde{U}, \tilde{T}; v) = \frac{\tilde{n}^i(t, x)}{\left( \frac{2\pi \tilde{T}(t, x)}{m^i} \right)^{\frac{3}{2}}} e^{-m^i |v - \tilde{U}(t, x)|^2 / 2\tilde{T}(t, x)} \]

where \( \tilde{n}^i \) is the auxiliary species number densities, \( \tilde{U} \) the common velocity, and \( \tilde{T} \) the common temperature

\[ \int_{\mathbb{R}^3} Q^i \ dv = 0, \quad \sum_{i=1}^{N} \int_{\mathbb{R}^3} (\nu, |\nu|^2)Q^i \ dv = 0. \]
• Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

\[ \partial_t f^i + \nu \cdot \nabla_x f^i = Q^i := \nu^i \left( M^i - f^i \right), \quad i = 1, \cdots, N. \]

- Conservation laws for species number densities

\[ \partial_t \int_{\mathbb{R}^3} f^i \, dv + \text{div} \int_{\mathbb{R}^3} vf^i \, dv = 0, \]

and global momentum and total energy.

\[ \sum_{i=1}^{N} \partial_t \int_{\mathbb{R}^3} (\nu, |\nu|^2) f^i \, dv + \sum_{i=1}^{N} \text{div} \int_{\mathbb{R}^3} \nu (\nu, |\nu|^2) f^i \, dv = 0. \]

- Entropy inequality

\[ \partial_t \sum_{i=1}^{N} \int_{\mathbb{R}^3} f^i \ln f^i \, dv + \sum_{i=1}^{N} \text{div} \int_{\mathbb{R}^3} vf^i \ln f^i \, dv \leq 0. \]
Relativistic generalizations of the BGK model (Marle, 1965)

\[ \partial_t f + \frac{cp}{p_0} \cdot \nabla_x f = Q_{Marle} := \frac{cm}{\tau p_0} (\mathcal{J} - f). \]

- \( f \equiv f(x^\alpha, p^\alpha) \) is the momentum distribution function representing the number density of particles in the phase space spanned by the spacetime coordinates \( x^\alpha \) and four-momentum \( p^\alpha \)

\[ x^\alpha = (ct, x), \quad p^\alpha = \left( \sqrt{(mc)^2 + |p|^2}, p \right). \]

- \( \tau \) is the characteristic time of order of the mean free time.
- \( \mathcal{J} \equiv \mathcal{J}(\mu_E, u, T; p) \) is the Jüttner distribution function (Jüttner, 1911)
Relativistic BGK model for gas mixtures

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0} (J_i - f_i) \quad (i = 1, \ldots, N)$$

- $f_i \equiv f_i(x^\alpha, p_i^\alpha)$ is the momentum distribution function for species $i$ with

$$p_i^\alpha = \left(\sqrt{(m_i c)^2 + |p_i|^2}, p_i \right)$$

- $J_i$ is the Jüttner distribution attractors

$$J_i = \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} \frac{e^{-c\beta p_i^0}}{e^{-\beta u^\mu p_{i\mu}}}.$$
• Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[ \partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{c m_i}{\tau_i p_i^0} (\mathcal{J}_i - f_i) \quad (i = 1, \ldots, N) \]

- \( \mathcal{J}_i \) is the Jüttner distribution attractors

\[ \mathcal{J}_i = \frac{\int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} e^{-\beta u^\mu p_{i\mu}}}{\int_{\mathbb{R}^3} e^{-c \beta p_i^0} \frac{dp_i}{p_i^0}}. \]

Here \( u^\mu (u^\mu u_{\mu} = c^2) \) is the common four-velocity

\[ u^\mu = c \frac{\sum_{i=1}^N \frac{m_i}{\tau_i} N_i^\mu}{\sqrt{\left( \sum_{i=1}^N \frac{m_i}{\tau_i} N_i^\mu \right) \left( \sum_{j=1}^N \frac{m_j}{\tau_j} N_j^\mu \right)}} \quad \text{with} \quad N_i^\mu = c \int_{\mathbb{R}^3} p_i^\mu f_i \frac{dp_i}{p_i^0}, \]

and \( \beta \left( = 1/kT \right) \) denotes the inverse of common temperature which is determined by the nonlinear relation

\[ \sum_{i=1}^N \frac{m_i}{\tau_i} \left( \int_{\mathbb{R}^3} e^{-c \beta p_i^0} \frac{dp_i}{p_i^0} \right) \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} e^{-c \beta p_i^0} \frac{dp_i}{p_i^0} \right) f_i \frac{dp_i}{p_i^0} = \frac{1}{c} \left\{ \left( \sum_{i=1}^N \frac{m_i}{\tau_i} N_i^\mu \right) \left( \sum_{j=1}^N \frac{m_j}{\tau_j} N_j^\mu \right) \right\}^{1/2}. \]
Determination problem of the common temperature

Note that $\beta$ is determined through the nonlinear relation

$$
\sum_{i=1}^{N} \frac{m_i}{\tau_i} \int_{\mathbb{R}^3} e^{-c\beta p_i^0} \frac{dp_i}{p_i^0} \int_{\mathbb{R}^3} \frac{dp_i}{p_i^0} \frac{f_i}{p_i^0} = \frac{1}{c} \left\{ \sum_{i,j}^{N} (2 - \delta_{ij}) \frac{m_i m_j}{\tau_i \tau_j} n_i n_j u_i^\mu u_j^\mu \right\}^{\frac{1}{2}}.
$$

Here the integrals for $\beta$ are not explicitly computed due to the factor $p_i^0$

$$
\int_{\mathbb{R}^3} e^{-c\beta p_i^0} dp_i = \int_{\mathbb{R}^3} e^{-c\beta \sqrt{(mc)^2 + |p_i|^2}} dp_i
$$

$$
\int_{\mathbb{R}^3} \frac{e^{-c\beta p_i^0}}{p_i^0} dp_i = \int_{\mathbb{R}^3} \frac{1}{\sqrt{(mc)^2 + |p_i|^2}} e^{-c\beta \sqrt{(mc)^2 + |p_i|^2}} dp_i
$$
Note that $\beta$ is determined through the nonlinear relation

$$
\sum_{i=1}^{N} \frac{m_i}{\tau_i} \int_{\mathbb{R}^3} e^{-c^{\beta} p_i^0} \frac{dp_i}{p_i^0} \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c} \left\{ \sum_{i,j}^{N} (2 - \delta_{ij}) \frac{m_i m_j}{\tau_i \tau_j} n_i n_j u_i^\mu u_j^\mu \right\}^{\frac{1}{2}}.
$$

In order to study the existence theorem of the this model, we need to show whether or not $\beta$ can be uniquely determined as the moments of $f_i$ throught the above relation.
Note that $\beta$ is determined through the nonlinear relation

$$
\sum_{i=1}^{N} \frac{m_i}{\tau_i} \int_{\mathbb{R}^3} e^{-c\beta p_i^0} \frac{dp_i}{p_i^0} \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c} \left\{ \sum_{i,j}^{N} (2 - \delta_{ij}) \frac{m_i m_j}{\tau_i \tau_j} n_i n_j u_i^{\mu} u_{j\mu} \right\}^{\frac{1}{2}}.
$$

- Key idea of proof

1. Strict monotonicity

\[
\frac{d}{d\beta} \left\{ \int_{\mathbb{R}^3} e^{-c\beta p_i^0} \frac{dp_i}{p_i^0} \right\} < 0
\]

2. Comparison of ranges on the left and right sides

range of l.h.s $\supseteq$ range of r.h.s

As a result, we can conclude that there exists a 1-1 correspondence which guarantees the unique existence of $\beta$. 
Properties

(1) Conservation laws for each species, and total energy-momentum tensor

$$\partial_\mu N_i^{\mu} = 0, \quad \sum_{i=1}^{N} \partial_\nu T_i^{\mu\nu} = 0.$$ 

(2) The momentum distribution functions are non-negative

$$f_i \geq 0 \quad i = 1, \ldots, N$$

(3) Entropy inequality

$$\sum_{i=1}^{N} \partial_t \int_{\mathbb{R}^3} f_i \ln f_i \, dp_i + \sum_{i=1}^{N} \text{div} \int_{\mathbb{R}^3} \frac{cp_i}{p_i^0} f_i \ln f_i \, dp_i \leq 0$$

(4) Indifferentiability principle

: When all $m_i$ and $\tau_i$ are identical, the total distribution $f = \sum f_i$ obeys the single species relativistic BGK model.
Conservation laws

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[
\frac{\partial_t f_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{c m_i}{\tau_i p_i^0} (J_i - f_i) \quad (i = 1, ..., N)
\]

Since $J_i$ is defined in a way to satisfy the following constraints

\[
\int_{\mathbb{R}^3} Q_i \, dp_i = 0, \quad \sum_{i=1}^N \int_{\mathbb{R}^3} p_i^{\mu} Q_i \, dp_i = 0,
\]

we get

\[
\int_{\mathbb{R}^3} \partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i \, dp_i = 0, \quad \sum_{i=1}^N \int_{\mathbb{R}^3} p_i^{\mu} \left( \partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i \right) dp_i = 0,
\]
• Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[
\partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{c m_i}{\tau_i p_i^0} (J_i - f_i) \quad (i = 1, \ldots, N)
\]

Since \( J_i \) is defined in a way to satisfy the following constraints

\[
\int_{\mathbb{R}^3} Q_i \, dp_i = 0, \quad \sum_{i=1}^{N} \int_{\mathbb{R}^3} p_i^\mu Q_i \, dp_i = 0,
\]

we get

\[
\frac{\partial N_i^{\mu}}{\partial x^\mu} = 0, \quad \sum_{i=1}^{N} \frac{\partial T_i^{\mu\nu}}{\partial x^\nu} = 0.
\]
Non-negativity

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[
\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{cm_i}{\tau_i p_i^0} (J_i - f_i) \quad (i = 1, \ldots, N)
\]

- Mild form of the model equation

\[
f_i = e^{-\frac{cm_i}{\tau_i p_i^0} t} f_0 + \int_0^t e^{-\frac{cm_i}{\tau_i p_i^0} (t-s)} \frac{cm_i}{\tau_i p_i^0} J_i(f_i) \, ds
\]
Entropy inequality

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[ \partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{c m_i}{\tau_i p_i^0} (J_i - f_i) \quad (i = 1, \ldots, N) \]

Since \( Q_i \) satisfies

\[ \sum_{i=1}^{N} \int_{\mathbb{R}^3} (1, p_i^\mu) Q_i \, dp_i = 0, \]

we obtain

\[ \sum_{i=1}^{N} \int_{\mathbb{R}^3} Q_i \ln f_i \, dp_i = \sum_{i=1}^{N} \int_{\mathbb{R}^3} Q_i (\ln f_i - \ln J_i) \, dp_i \]

\[ = \sum_{i=1}^{N} \frac{c m_i}{\tau_i} \int_{\mathbb{R}^3} J_i (1 - \frac{f_i}{J_i}) \ln \frac{f_i}{J_i} \frac{dp_i}{p_i^0} \leq 0 \]

Thus,

\[ \sum_{i=1}^{N} \partial_t \int_{\mathbb{R}^3} f_i \ln f_i \, dp_i + \sum_{i=1}^{N} \text{div}_x \int_{\mathbb{R}^3} \frac{c p_i}{p_i^0} f_i \ln f_i \, dp_i \leq 0 \]
• Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[
\partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{c m_i}{\tau_i p_i^0} (\mathcal{J}_i - f_i) \quad (i = 1, \ldots, N)
\]

Letting

\[m_1 = m_2 = \cdots = m_N \ (= m), \quad \tau_1 = \tau_2 = \cdots = \tau_N \ (= \tau),\]

and summing the model equations over \(i\), we get

\[
\partial_t f + \frac{c p}{p^0} \cdot \nabla_x f = \frac{c m}{\tau p^0} \left( \sum_{i=1}^N \mathcal{J}_i - f \right)
\]

where \(\sum_{i=1}^N \mathcal{J}_i\) is given by

\[
\sum_{i=1}^N \mathcal{J}_i = \frac{\sum_{i=1}^N \int_{\mathbb{R}^3} f_i \, \frac{dp}{p^0}}{\int_{\mathbb{R}^3} e^{-c\beta p^0} \, \frac{dp}{p^0}} e^{-\beta u^\mu p_\mu} = \frac{\int_{\mathbb{R}^3} f \, \frac{dp}{p^0}}{\int_{\mathbb{R}^3} e^{-c\beta p^0} \, \frac{dp}{p^0}} e^{-\beta u^\mu p_\mu}.
\]

It only remains to show that the total distribution \(f = \sum f_i\) obeys the common four-velocity \(u^\mu\) and the inverse of common temperature \(\beta\).
- common four-velocity \( u^\mu \)

\[
u^\mu = c \frac{\frac{m}{\tau} \sum_{i=1}^{N} N_i^\mu}{\sqrt{\left( \frac{m}{\tau} \sum_{i=1}^{N} N_i^\mu \right) \left( \frac{m}{\tau} \sum_{j=1}^{N} N_j^\mu \right)}}.
\]

- \( \beta \), the inverse of common temperature

\[
\frac{m}{\tau} \int_{\mathbb{R}^3} e^{-c\beta p_0} \frac{dp}{p_0} \sum_{i=1}^{N} \int_{\mathbb{R}^3} f_i \frac{dp}{p_0} = \frac{1}{c} \left\{ \left( \frac{m}{\tau} \sum_{i=1}^{N} N_i^\mu \right) \left( \frac{m}{\tau} \sum_{j=1}^{N} N_j^\mu \right) \right\}^{\frac{1}{2}}.
\]

Since

\[
\sum_{i=1}^{N} N_i^\mu = \sum_{i=1}^{N} c \int_{\mathbb{R}^3} f_i \frac{dp}{p_0} \equiv c \int_{\mathbb{R}^3} f \frac{dp}{p_0}
\]

we obtain

\[
\partial_t f + \frac{cp}{p_0} \cdot \nabla_x f = \frac{cm}{\tau p_0} \left( \sum_{i=1}^{N} J_i - f \right) \equiv \frac{cm}{\tau p_0} \left( J - f \right).
\]
Newtonian limit

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[
\partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{c m_i}{\tau_i p_i^0} (J_i - f_i), \quad i = 1, \ldots, N.
\]

- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

\[
\partial_t f_i^i + v \cdot \nabla_x f_i^i = Q_i^i := \nu^i \left( \mathcal{M}^i - f_i^i \right), \quad i = 1, \ldots, N.
\]

Question) Does the relativistic BGK model approach the classical BGK model in the Newtonian limit?
• Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[
\partial_t f_i + \frac{c p_i}{p_i^0} \cdot \nabla_x f_i = Q_i := \frac{c m_i}{\tau_i p_i^0} (J_i - f_i), \quad i = 1, \ldots, N.
\]

• Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

\[
\partial_t f^i + v \cdot \nabla_x f^i = Q^i := \nu^i \left( M^i - f^i \right), \quad i = 1, \ldots, N.
\]

- Dimensionless number

\[
t = \bar{t}s, \quad x = \bar{x}L, \quad p_i = \bar{v}\mu_i, \quad f_i(t, x, p_i) = \frac{N_i}{\mu_i^3} \bar{f}_i(\bar{t}, \bar{x}, \bar{v}) \Rightarrow \mu_i = m_iL/s.
\]

- Notations

\[
\nu_i := \frac{s}{\tau_i}, \quad \varepsilon := \frac{\mu_i}{c m_i}.
\]
Substituting the dimensionless numbers into the relativistic BGK model, we get

- Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)
  \[
  \frac{\partial}{\partial \tilde{t}} \tilde{f}_i + \frac{1}{\sqrt{1 + |\varepsilon \tilde{v}|^2}} \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{f}_i = \frac{\nu_i}{\sqrt{1 + |\varepsilon \tilde{v}|^2}} (\tilde{J}_i - \tilde{f}_i), \quad i = 1, \ldots, N.
  \]

- Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)
  \[
  \partial_t f^i + v \cdot \nabla_x f^i = Q^i := \nu^i \left( M^i - f^i \right), \quad i = 1, \ldots, N.
  \]
• Relativistic BGK model for inert gas mixtures (Hwang-Lee-Yun, 2021)

\[
\frac{\partial}{\partial t} \bar{f}_i + \frac{1}{\sqrt{1 + |\varepsilon \bar{v}|^2}} \bar{v} \cdot \nabla \bar{x}_i \bar{f}_i = \frac{\nu_i}{\sqrt{1 + |\varepsilon \bar{v}|^2}} \left( \bar{J}_i - \bar{f}_i \right), \quad i = 1, \cdots, N.
\]

• Classical BGK model for inert gas mixtures (Bisi-Monaco-Soares, 2018)

\[
\partial_t f^i + \nu \cdot \nabla_x f^i = Q^i := \nu^i \left( M^i - f^i \right), \quad i = 1, \cdots, N.
\]

• In the Newtonian limit ($\varepsilon \to 0$), we obtain

\[
\bar{J}_i \to M_i
\]

which completes the proof.
Future works

1. Chapman-Enskog method
2. Existence theory
3. Extension to (polyatomic molecules/chemically reactive mixtures)
4. Anderson-Witting’s formulation
Thank you very much