On the large-time behavior of two-phase fluid models

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(based on collaboration with Prof. Y.-P. Choi)

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Outline

Our system of interest

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{T}^d,\ t > 0, \\
&\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = -\rho(u - v) - \rho \nabla K \star (\rho - \rho_c), \\
&\partial_t v + (v \cdot \nabla) v + \nabla p - \mu \Delta v = \rho(u - v), \\
&\nabla \cdot v = 0,
\end{aligned}
\tag{EPNS}
$$

subject to initial data:

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(\rho(x,0), u(x,0), v(x,0)) = (\rho_0(x), u_0(x), v_0(x)), \quad x \in \mathbb{T}^d.
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$$

- $\rho = \rho(x,t)$, $u = u(x,t)$: the density and the velocity of compressible fluid, resp.
- $v = v(x,t)$: the velocity of incompressible fluid.
- K: the fundamental solution to the Poisson equation, i.e. $-\Delta K = \delta_0$
- ρ_c: the background state given by $\rho_c := \int_{\mathbb{T}^d} \rho \, dx$. (set $\rho_c = 1$ without loss of generality)
- $\mu > 0$: viscosity coefficient (set $\mu = 1$ for simplicity)

Derivation of the system

(EPNS) can be derived from the kinetic-fluid system consisting of Vlasov-Poisson-Navier-Stokes system with strong local alignment forces:

$$\partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon - \nabla_\xi \cdot ((\nabla_x K \star (\rho^\varepsilon - \rho_c^\varepsilon) - (v^\varepsilon - \xi))f^\varepsilon) = -\frac{1}{\varepsilon}\nabla_\xi \cdot ((u^\varepsilon - \xi)f^\varepsilon),$$

$$\partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla_x)v^\varepsilon + \nabla_x p^\varepsilon - \Delta_x v^\varepsilon = -\int_{\mathbb{R}^d}(v^\varepsilon - \xi)f^\varepsilon d\xi,$$

$$\nabla_x \cdot v^\varepsilon = 0,$$

$$\rho^\varepsilon(x,t) := \int_{\mathbb{R}^d} f^\varepsilon(x,\xi,t)\,d\xi, \quad (\rho^\varepsilon u^\varepsilon)(x,t) := \int_{\mathbb{R}^d} \xi f^\varepsilon(x,\xi,t)\,d\xi.$$

When $\varepsilon \ll 1$, we get the monokinetic ansatz [Choi-J., '20]:

$$f^\varepsilon(x,\xi,t)dxd\xi \approx \rho^\varepsilon(x,t)dx \otimes \delta_{u^\varepsilon(x,t)}(d\xi)$$

- Define averaged momentum and velocity:

$$m_c = m_c(t) := \frac{\int_{\mathbb{T}^d} (\rho u)(x,t)\,dx}{\int_{\mathbb{T}^d} \rho(x,t)\,dx} \quad \text{and} \quad v_c = v_c(t) := \int_{\mathbb{T}^d} v(x,t)\,dx.$$

- A **modulated total energy** measuring the fluctuation of momentum and mass from the corresponding averages:

$$\mathscr{E}(t) := \frac{1}{2}\int_{\mathbb{T}^d} \rho|u-m_c|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^d} |v-v_c|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^d} |\nabla K \star (\rho-\rho_c)|^2\,dx + \frac{1}{4}|m_c - v_c|^2.$$

Theorem

Let $d \geq 2$ *and* (ρ, u, v) *be a global-in-time classical solution to* (EPNS)*. We assume the followings:*

$$\rho \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}_+) \quad \textit{and} \quad \rho(x,t) \geq \underline{\rho} > 0, \quad \forall (x,t) \in \mathbb{T}^d \times \mathbb{R}_+.$$

Then there exists $C > 0$ *which is independent of* t *such that*

$$\mathscr{E}(t) \leq C\mathscr{E}(0)e^{-Ct}, \quad \forall\, t > 0.$$

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- Compared to the previous works [Choi-Kwon, '16] and [Choi, '16], we do not require the smallness of the total energy:

$$\tilde{E}(t) := \frac{1}{2}\left(\int_{\mathbb{T}^d} \rho|u|^2\,dx + \int_{\mathbb{T}^d} |v|^2\,dx + \int_{\mathbb{T}^d} |\nabla K \star (\rho - 1)|^2\,dx\right) \leq \tilde{E}(0), \qquad (1)$$

- The existence of a global-in-time classical solution is given in [Choi-J. '20].

Outline

Lemma

The averages m_c and v_c satisfy the followings:

$$|m_c(t)|^2 + |v_c(t)|^2 \le C\tilde{E}(0) \quad \text{and} \quad |m_c'(t)|^2 + |v_c'(t)|^2 \le C \int_{\mathbb{T}^d} \rho |u - v|^2 \, dx$$

for all $t > 0$, where $C > 0$ is independent of t.

Lemma

Let (ρ, u, v) be a global-in-time solution to the system (EPNS) *with sufficient regualrity. Then we have*

$$\frac{d}{dt}\mathscr{E}(t) + \mathscr{D}(t) = 0,$$

for all $t > 0$, where the dissipation $\mathscr{D}$ is given by

$$\mathscr{D}(t) := \int_{\mathbb{T}^d} \rho |u - v|^2 \, dx + \int_{\mathbb{T}^d} |\nabla v|^2 \, dx.$$

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$\Longrightarrow$ $\mathscr{D}(t)$ does not have a dissipation for $\int_{\mathbb{T}^d} |\nabla K \star (\rho - 1)|^2\,dx$!

- To have a proper dissipation term, an analogue of [Bogovskii, '80] was employed in previous works.
- We use a slightly different version. Namely, we set

$$\mathscr{E}^{\lambda}(t) := \mathscr{E}(t) + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \cdot \nabla K \star (\rho - 1)\, dx, \tag{2}$$

  where $\lambda > 0$ will be determined later.
- Using Young's inequality and bounded assumption of ρ, we easily show that

$$\mathscr{E}^{\lambda}(t) \approx \mathscr{E}(t) \quad \text{for } \lambda > 0 \text{ small enough.} \tag{3}$$

- The following computation implies that the perturbation $\mathscr{E}^{\lambda}$ produces a proper dissipation for $\int_{\mathbb{T}^d} |\nabla K \star (\rho - 1)|^2\, dx$.

Lemma

The perturbed energy functional $\mathscr{E}^\lambda$ satisfies the following relation:

$$\frac{d}{dt}\mathscr{E}^\lambda(t) + \mathscr{D}^\lambda(t) = 0,$$

for all $t > 0$, where $\mathscr{D}^\lambda(t)$ is given by

$$\begin{aligned}
\mathscr{D}^\lambda(t) &:= \mathscr{D} - \lambda \frac{d}{dt}\int_{\mathbb{R}^d} \rho(u - m_c)\cdot \nabla K \star (\rho - 1)\, dx \\
&= \int_{\mathbb{T}^d} |\nabla v|^2\, dx + \int_{\mathbb{T}^d} \rho|u - v|^2\, dx - \lambda \int_{\mathbb{T}^d} \rho u \otimes u : \nabla^2(K \star (\rho - 1))\, dx \\
&\quad + \lambda \int_{\mathbb{T}^d} \rho|\nabla K \star (\rho - 1)|^2\, dx + \lambda \int_{\mathbb{T}^d} \rho(u - v)\cdot \nabla K \star (\rho - 1)\, dx \\
&\quad + \lambda \int_{\mathbb{T}^d} \rho(u - m_c)\cdot \nabla K \star (\nabla \cdot (\rho u))\, dx + \lambda \int_{\mathbb{T}^d} \partial_t(\rho m_c)\cdot \nabla K \star (\rho - 1)\, dx =: \sum_{i=1}^{7} \mathcal{J}_i.
\end{aligned}$$

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&\quad + \lambda\int_{\mathbb{T}^d} \rho|\nabla K \star (\rho - 1)|^2\,dx + \lambda\int_{\mathbb{T}^d} \rho(u - v)\cdot\nabla K \star (\rho - 1)\,dx \\
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\end{aligned}$$

For sufficiently small $\lambda > 0$, we claim

$$\mathscr{E}(t) \le C\mathscr{D}^\lambda(t)$$

for some $C > 0$ independent of t, which gives the proof of the main theorem.

- (Estimate for $\mathcal{J}_5$): We use Young's inequality to get

$$\mathcal{J}_5 \geq -\lambda^{1/2} \int_{\mathbb{T}^d} \rho |u - v|^2 \, dx - \lambda^{3/2} \int_{\mathbb{T}^d} \rho |\nabla K \star (\rho - 1)|^2 \, dx.$$

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- (Estimate for $\mathcal{J}_6$): We split $\mathcal{J}_6$ into two terms:

$$\mathcal{J}_6 = \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \cdot \nabla K \star (\nabla \cdot (\rho(u - m_c)) \, dx + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \cdot \nabla K \star (\nabla \cdot ((\rho - 1) m_c)) \, dx.$$

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The second term on the right hand side of the above can be rewritten as

$$\int_{\mathbb{T}^d} \rho(u - m_c) \cdot \nabla K \star (\nabla \cdot ((\rho - 1) m_c)) \, dx = \int_{\mathbb{T}^d} \rho(u - m_c) \otimes m_c : \nabla^2 K \star (\rho - 1) \, dx.$$

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Together with this, we estimate

$$\begin{aligned}\mathcal{J}_6 &\geq -\lambda\|\rho(u - m_c)\|_{L^2(\mathbb{T}^d)}\|\nabla K \star (\nabla \cdot (\rho(u - m_c))\|_{L^2(\mathbb{T}^d)} + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \otimes m_c : \nabla^2 K \star (\rho - 1)\, dx \\ &\geq -C\lambda\|\rho\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}_+)} \int_{\mathbb{T}^d} \rho|u - m_c|^2\, dx + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \otimes m_c : \nabla^2 K \star (\rho - 1)\, dx,\end{aligned}$$

where we used

$$\|\nabla K \star h\|_{L^2(\mathbb{T}^d)} \leq C\|h\|_{\dot{H}^{-1}(\mathbb{T}^d)} \quad \forall\, h \in L^2(\mathbb{T}^d) \quad \text{with} \quad \int_{\mathbb{T}^d} h\, dx = 0.$$

- (Estimate for $\mathcal{J}_7$): By using $\partial_t(\rho m_c) = -\nabla \cdot (\rho u) m_c + \rho m_c'$, we obtain

$$\begin{aligned}\mathcal{J}_7 &= -\lambda \int_{\mathbb{T}^d} \nabla \cdot (\rho u) m_c \cdot \nabla K \star (\rho - 1)\, dx + \lambda \int_{\mathbb{T}^d} \rho m_c' \cdot \nabla K \star (\rho - 1)\, dx \\ &\geq \lambda \int_{\mathbb{T}^d} \rho u \otimes m_c : \nabla^2 K \star (\rho - 1)\, dx - C\lambda^{1/2} \int_{\mathbb{T}^d} \rho |u - v|^2\, dx - C\lambda^{3/2} \int_{\mathbb{T}^d} \rho |\nabla K \star (\rho - 1)|^2\, dx,\end{aligned}$$

where $C = C(\|\rho\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}_+)})$ is a constant independent of t and λ.

- (Estimate for $\mathcal{J}_7$): By using $\partial_t(\rho m_c) = -\nabla\cdot(\rho u)m_c + \rho m_c'$, we obtain

$$\begin{aligned}\mathcal{J}_7 &= -\lambda\int_{\mathbb{T}^d}\nabla\cdot(\rho u)m_c\cdot\nabla K\star(\rho-1)\,dx + \lambda\int_{\mathbb{T}^d}\rho m_c'\cdot\nabla K\star(\rho-1)\,dx\\ &\geq \lambda\int_{\mathbb{T}^d}\rho u\otimes m_c : \nabla^2 K\star(\rho-1)\,dx - C\lambda^{1/2}\int_{\mathbb{T}^d}\rho|u-v|^2\,dx - C\lambda^{3/2}\int_{\mathbb{T}^d}\rho|\nabla K\star(\rho-1)|^2\,dx,\end{aligned}$$

where $C = C(\|\rho\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}_+)})$ is a constant independent of t and λ. Thus, we have

$$\begin{aligned}\mathcal{J}_3+\mathcal{J}_6+\mathcal{J}_7 \geq& -\lambda\int_{\mathbb{T}^d}\rho(u-m_c)\otimes(u-m_c) : \nabla^2 K\star(\rho-1)\,dx - C\lambda\int_{\mathbb{T}^d}\rho|u-m_c|^2\,dx\\ &- C\lambda^{1/2}\int_{\mathbb{T}^d}\rho|u-v|^2\,dx - C\lambda^{3/2}\int_{\mathbb{T}^d}\rho|\nabla K\star(\rho-1)|^2\,dx,\end{aligned} \tag{4}$$

where $C = C(\|\rho\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}_+)}) > 0$ is independent of t and λ and we used the symmetry of $\nabla^2 K\star(\rho-1)$ to get

$$(u-m_c)\otimes m_c : \nabla^2 K\star(\rho-1) = m_c\otimes(u-m_c) : \nabla^2 K\star(\rho-1).$$

We then estimate the first term on the right hand side of (4) as

$$
\begin{aligned}
&-\lambda \int_{\mathbb{T}^d} \rho(u - m_c) \otimes (u - m_c) : \nabla^2 K \star (\rho - 1)\, dx \\
&\quad \geq -\lambda \|\nabla^2 K \star (\rho - 1)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}_+)} \int_{\mathbb{T}^d} \rho |u - m_c|^2\, dx \\
&\quad \geq -\lambda \|\nabla K\|_{L^1(\mathbb{T}^d)} \|\nabla \rho\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}_+)} \int_{\mathbb{T}^d} \rho |u - m_c|^2\, dx \\
&\quad \geq -C\lambda \int_{\mathbb{T}^d} \rho |u - m_c|^2\, dx,
\end{aligned}
$$

where $C = C(\|\nabla K\|_{L^1(\mathbb{T}^d)}, \|\nabla \rho\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}_+)})$ is independent of t and λ.

Thus, we choose $\lambda > 0$ sufficiently small, use $\rho \geq \underline{\rho} > 0$ and combine all the above estimates to get

$$\begin{aligned} \mathscr{D}^\lambda(t) &\geq C_1 \left(\int_{\mathbb{T}^d} |\nabla v|^2 \, dx + \int_{\mathbb{T}^d} \rho |u - v|^2 \, dx \right) + C_2 \int_{\mathbb{T}^d} |\nabla K \star (\rho - 1)|^2 \, dx \\ &\geq C\mathscr{E}(t), \end{aligned}$$

Here C is a positive constant which depends on $\|\rho\|_{W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}_+)}$ and $\|\nabla K\|_{L^1(\mathbb{T}^d)}$.
This implies

$$\frac{d}{dt}\mathscr{E}^\lambda(t) + C\mathscr{E}^\lambda(t) \leq 0 \quad \forall\, t > 0$$

for some $C > 0$ independent of t. Applying Grönwall's lemma to the above concludes the proof of Theorem 1.

Outline

Euler-Riesz-Navier-Stokes system

Our main system can be recast in the following form:

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{T}^d,\ t > 0, \\
&\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = -\rho(u - v) - \rho \nabla \Lambda^{\alpha-d}(\rho - \rho_c), \\
&\partial_t v + (v \cdot \nabla) v + \nabla p - \mu \Delta v = \rho(u - v), \\
&\nabla \cdot v = 0,
\end{aligned}$$

where $\Lambda^s = (-\Delta)^s$ is the Riesz operator and $\alpha = d - 2$. If we let $\alpha \in (d-2, d)$, we can still have the same results under the same conditions.

However, we should choose a different total energy, modulated total energy and perturbation of the energy functional $\mathscr{E}$:

$$\tilde{E}(t) := \frac{1}{2}\left(\int_{\mathbb{T}^d} \rho|u|^2\,dx + \int_{\mathbb{T}^d} |v|^2\,dx + \int_{\mathbb{T}^d} (\rho-1)\Lambda^{\alpha-d}(\rho-1)\,dx\right),$$

$$\mathscr{E}(t) := \frac{1}{2}\int_{\mathbb{T}^d} \rho|u-m_c|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^d} |v-v_c|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^d} (\rho-1)\Lambda^{\alpha-d}(\rho-1)\,dx + \frac{1}{4}|m_c-v_c|^2,$$

$$\mathscr{E}^\lambda(t) := \mathscr{E}(t) + \lambda\int_{\mathbb{T}^d} (u-m_c)\cdot\nabla K\star(\rho-1)\,dx.$$

Note that $\mathscr{E}^\lambda \approx \mathscr{E}$ since

$$\|\nabla K\star(\rho-1)\|_{L^2(\mathbb{T}^d)} \approx \|\rho-1\|_{\dot{H}^{-1}(\mathbb{T}^d)} \le \|\rho-1\|_{\dot{H}^{-\frac{d-\alpha}{2}}(\mathbb{T}^d)}.$$

The reason for the removal of ρ in $\mathscr{E}^\lambda(t)$ comes from

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d}(u-m_c)\cdot\nabla K\star(\rho-1)\,dx &= \cdots - \int_{\mathbb{T}^d}\nabla\Lambda^{\alpha-d}(\rho-1)\cdot\nabla K\star(\rho-1)\,dx\\ &= \cdots - \int_{\mathbb{T}^d}(\rho-1)\Lambda^{\alpha-d}(\rho-1)\,dx.\end{aligned}$$

References

- M. E. Bogovskii, Solution of some vector analysis problems connected with operators div and grad (in Russian), Trudy Sem. S. L. Sobolev 80, (1980), 5–40.
- Y.-P. Choi, Global classical solutions and large-time behavior of the two-phase fluid model, SIAM J. Math. Anal., 48, (2016), 3090–3122.
- Y.-P. Choi and J., On the dynamics of charged particles in an incompressible flow: from kinetic-fluid to fluid-fluid models, arXiv: 2008.01964.
- Y.-P. Choi and J., On the large-time behavior of Euler–Poisson/Navier–Stokes equations, Appl. Math. Lett., 118, (2021), 107123.
- Y.-P. Choi and J., The damped Euler-Riesz euqations, arXiv: 2104.05153.
- Y.-P. Choi and B. Kwon, The Cauchy problem for the pressureless Euler/isentropic Navier–Stokes equations, J. Differential Equations, 261, (2016), 654–711.
- G. P. Galdi, An Introduction to the mathematical theory of the Navier-Stokes Equations I, Springer-Verlag, New York, 1994.

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Thank you for your attention.