On a generalized Kuramoto model and uniform stability

(based on the joint work with H. Ahn, H. Cho, Prof. S.-Y. Ha, Prof. C. Min)

Myeongju Kang
Seoul National University

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Generalized Kuramoto model

Kuramoto Model (1975).

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad \forall\ t > 0, \quad \forall\ i = 1, \cdots, N,$$

where ν_i and κ are natural frequency of the i-th oscillator and nonnegative coupling strength, respectively.

Generalized Kuramoto Model ([2] Min, Ahn, Ha, K.).

$$F(\dot{\theta}_i) = \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad \forall\ t > 0, \quad \forall\ i = 1, \cdots, N,$$

where F is an odd and continuously differentiable monotone increasing in the interval $(-L, L)$:

$$F'(\omega) > 0, \quad F(-\omega) = -F(\omega), \quad \forall\ \omega \in (-L, L).$$

Cucker-Smale model

Cucker-Smale Model (2007).

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad \forall\ t > 0, \quad i = 1, \cdots, N, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{B=1}^{N} \psi_{ij} \left(v_j - v_i \right), \\ (x_i(0), v_i(0)) = (x_i^{in}, v_i^{in}) \in T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d, \end{cases}$$

where ψ_{ij} is communication weight between i-th and j-th particle.

Relativistic Cucker-Smale Model

Relativistic Cucker-Smale Model ([3] Ha, Kim, Ruggeri, 2020).

$$
\begin{cases}
\dfrac{dx_i}{dt} = v_i, \quad \forall\ t > 0, \quad i = 1, \cdots, N, \\
\dfrac{d}{dt}\left[\Gamma_i v_i \left(1 + \dfrac{\Gamma_i}{c^2}\right)\right] = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi_{ij}\left(v_j - v_i\right), \\
\Gamma_i := \dfrac{1}{\sqrt{1 - \frac{\|v_i\|^2}{c^2}}} \\
(x_i(0), v_i(0)) = (x_i^{in}, v_i^{in}) \in T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d.
\end{cases}
$$

Remark. $\dot{v}_i$ is uniquely determined.

Relativistic Cucker-Smale Model ([1] Ahn, Ha, K., Shim).

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad \forall\ t > 0, \quad i = 1, \cdots, N, \\ \nabla_{\dot{x}_i}\left[\Gamma_i v_i \left(1 + \dfrac{\Gamma_i}{c^2}\right)\right] = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi_{ij}\left(P_{ij} v_j - v_i\right), \\ \Gamma_i := \dfrac{1}{\sqrt{1 - \frac{\|v_i\|^2}{c^2}}} \\ (x_i(0), v_i(0)) = (x_i^{in}, v_i^{in}) \in T\mathcal{M}, \end{cases}$$

where $\mathcal{M}$ is a a connected and smooth Riemannian manifold without boundary, ∇ is a compatible Levi-Civita connection, and P_{ij} is the parallel transport along the length-minimizing geodesic from x_j to x_i

Example ($\mathcal{M} = \mathbb{S}^d \subset \mathbb{R}^{d+1}$).

$$\begin{cases} \dot{x}_i = v_i, \quad \forall\ t > 0, \quad \forall\ i = 1, \cdots, N, \\ (\dot{v}_i + \|v_i\|^2 x_i)\left(\Gamma_i\left(1 + \frac{\Gamma_i}{c^2}\right)\right) + v_i \frac{d}{dt}\left(\Gamma_i\left(1 + \frac{\Gamma_i}{c^2}\right)\right) \\ \qquad = \frac{\kappa}{N}\sum_{j=1}^{N} \psi(x_i, x_j)\left(v_j - v_i - \frac{\langle x_i, v_j\rangle}{(1 + \langle x_i, x_j\rangle)}(x_i + x_j)\right), \\ (x_i(0), v_i(0)) = (x_i^{in}, v_i^{in}) \in T\mathbb{S}^d \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}. \end{cases}$$

Consider the case of $d = 1$: $x_i = (\cos\theta_i, \sin\theta_i)$ and $\psi(x_i, x_j) = \langle x_i, x_j\rangle$.

Generalized Kuramoto model

We can obtain the Kuramoto type model:

$$\dot{\theta}_i \Gamma_i \left(1 + \frac{\Gamma_i}{c^2}\right) = C_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad \forall\ t \geq 0, \quad \forall\ i = 1, \cdots, N,$$

where C_i is a constant depending on $\theta_i(0)$ and $\dot{\theta}_i(0)$. Note that

$$x \ \mapsto \ \frac{cx}{\sqrt{c^2 - x^2}} \left(1 + \frac{1}{c\sqrt{c^2 - x^2}}\right)$$

is monotone increasing odd function on $(-c, c)$ whose image is $\mathbb{R}$.

$$\implies F(\dot{\theta}_i) = \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad \forall\ t > 0, \quad \forall\ i = 1, \cdots, N.$$

Estimate of kinetic energy

$$\text{Generalized kinetic energy } \mathcal{E}_F(t) := \sum_{i=1}^{N} \int_1^{\gamma_i(t)} \frac{L^2}{x^3} \mathcal{F}(x) dx > 0,$$

$$\text{where} \quad \gamma_i(t) := \frac{1}{\sqrt{1 - \frac{\dot{\theta}_i(t)^2}{L^2}}}, \quad \mathcal{F}(x) := F'\left(L\sqrt{1 - \frac{1}{x^2}}\right).$$

$$\implies \frac{d\mathcal{E}_F}{dt} = -\frac{\kappa}{2N} \sum_{i,j=1}^{N} \cos(\theta_j - \theta_i)\left(\dot{\theta}_i - \dot{\theta}_j\right)^2.$$

1. If $F(x) = x$, then $\lim_{L\to\infty} \mathcal{E}_F(t) = \sum_{i=1}^{N} \frac{\dot{\theta}_i^2}{2}$.
2. If $L = c$ and $F(x) = \frac{cx}{\sqrt{c^2-x^2}}\left(1 + \frac{1}{c\sqrt{c^2-x^2}}\right)$, then

$$\mathcal{E}_F(t) = \sum_{i=1}^{N} c^2(\Gamma_i - 1) + \Gamma_i^2 - \log \Gamma_i - 1 \xrightarrow{c\to\infty} \sum_{i=1}^{N} \frac{\dot{\theta}_i^2}{2}.$$

Emergent collective dynamics

Notation. $\mathcal{D}(\Omega) = \max_{i,j} |\nu_i - \nu_j|$, $\mathcal{D}(\Theta^{in}) := \max_{i,j} \left|\theta_i^{in} - \theta_j^{in}\right|$

Theorem

Suppose that initial data, natural frequency, and coupling strength satisfy

$$\kappa > \frac{\mathcal{D}(\Omega)}{\mathcal{D}(\Theta^{in})} > 0, \quad \mathcal{D}(\Theta^{in}) < \pi - \theta_*, \quad \theta_* := \sin^{-1}\left(\frac{\mathcal{D}(\Omega)}{\kappa}\right) \in \left(0, \frac{\pi}{2}\right).$$

Then, complete-frequency synchronization occurs asymptotically:

$$\lim_{t\to\infty} \left|\dot{\theta}_i(t) - \dot{\theta}_j(t)\right| = 0, \quad \forall\ i,j = 1,\cdots,N.$$

More precisely, there exists a finite time $t_* \geq 0$ and a positive constant Λ_1 such that, for all $t \geq t_*$,

$$\left|\dot{\theta}_i(t) - \dot{\theta}_j(t)\right| \leq \left(\max_{1\leq i,j\leq N} \left|\dot{\theta}_i(t_*) - \dot{\theta}_j(t_*)\right|\right) e^{-\Lambda_1(t-t_*)}, \quad \forall\ i,j = 1,\cdots,N,$$

and therefore, complete phase-locking emerges asymptotically. That is, for all $i,j = 1,\cdots,N$, there exist $\lim_{t\to\infty}(\theta_i(t) - \theta_j(t))$.

Sketch of proof

Let θ_i be a solution of

$$\begin{cases} F(\dot{\theta}_i) = \nu_i + \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad \forall\ t > 0 \\ \theta_i(0) = \theta_i^{in}, \quad \forall\ i = 1, \dots, N. \end{cases}$$

Then, $(\theta_i, \omega_i := \dot{\theta}_i)$ satisfies

$$\begin{cases} \dot{\theta}_i = \omega_i, \quad \forall\ t > 0, \quad \forall\ i = 1, \dots, N, \\ \dot{\omega}_i = \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N} \dfrac{\cos(\theta_j - \theta_i)}{F'(\omega_i)}(\omega_j - \omega_i), \\ \theta_i(0) = \theta_i^{in}, \quad \omega_i(0) = F^{-1}\left(\nu_i + \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N} \sin(\theta_j^{in} - \theta_i^{in})\right). \end{cases}$$

Uniform stability

Notation.

$$G := F^{-1}, \quad a_G := G\Big(-\kappa + \min_{1\le i\le N} \nu_i \Big), \quad b_G := G\Big(\kappa + \max_{1\le i\le N} \nu_i \Big)$$

Theorem

Let (θ_i, ω_i) *and* $(\tilde{\theta}_i, \tilde{\omega}_i)$ *be two global solutions with different initial data* $(\theta_i^{in}, \omega_i^{in})$ *and* $(\tilde{\theta}_i^{in}, \tilde{\omega}_i^{in})$ *respectively. Suppose that*

$$\sup_{(x,y\in[a_G,b_G]^2)} \frac{\partial H}{\partial x} < \infty, \quad H(x,y) := \begin{cases} \dfrac{x-y}{F(x)-F(y)} & x \neq y, \\ \dfrac{1}{F'(x)} & x = y, \end{cases} \quad \mathcal{D}(\Theta^{in}) \le \frac{\pi}{2}.$$

Then, there exists a constant $C = C(\kappa, F)$ *such that*

$$\sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i| + |\omega_i - \tilde{\omega}_i| \le C\Bigg(\sum_{i=1}^{N} |\theta_i^{in} - \tilde{\theta}_i^{in}| + |\omega_i^{in} - \tilde{\omega}_i^{in}| \Bigg), \quad \forall\ t \ge 0.$$

Uniform stability

Theorem

Let θ_i and $\tilde{\theta}_i$ be two global solutions with different initial data θ_i^{in} and $\tilde{\theta}_i^{in}$ respectively. Suppose that

$$\sup_{(x,y\in[a_G,b_G]^2)} \frac{\partial H}{\partial x} < \infty, \quad H(x,y) := \begin{cases} \dfrac{x-y}{F(x)-F(y)} & x \neq y, \\ \dfrac{1}{F'(x)} & x = y, \end{cases} \quad \mathcal{D}(\Theta^{in}) \leq \frac{\pi}{2}.$$

Then, there exists a constant $C = C(\kappa, F)$ such that

$$\sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i| \leq C \sum_{i=1}^{N} |\theta_i^{in} - \tilde{\theta}_i^{in}|, \quad \forall\ t \geq 0.$$

Remark. For general ℓ^p-norm with $p \in [1, \infty]$, both theorems still hold.

Sketch of proof of the first theorem

$$
\begin{aligned}
&\frac{d}{dt}|\omega_i - \tilde{\omega}_i| = \operatorname{sgn}(\omega_i - \tilde{\omega}_i)(\dot{\omega}_i - \dot{\tilde{\omega}}_i) \\
&= \operatorname{sgn}(\omega_i - \tilde{\omega}_i)\frac{\kappa}{N}\sum_{j=1}^{N}\left[\frac{\cos(\theta_j - \theta_i)}{F'(\omega_i)}(\omega_j - \omega_i) - \frac{\cos(\tilde{\theta}_j - \tilde{\theta}_i)}{F'(\tilde{\omega}_i)}(\tilde{\omega}_j - \tilde{\omega}_i)\right] \\
&= \operatorname{sgn}(\omega_i - \tilde{\omega}_i)\frac{\kappa}{N}\sum_{j=1}^{N} G'(\bar{\nu})^2 \cos(\theta_j - \theta_i)\big[(F(\omega_j) - F(\omega_i)) - (F(\tilde{\omega}_j) - F(\tilde{\omega}_i))\big] \\
&+ \operatorname{sgn}(\omega_i - \tilde{\omega}_i)\frac{\kappa}{N}\sum_{j=1}^{N}\left[\frac{H(\omega_j, \omega_i)}{F'(\omega_i)} - G'(\bar{\nu})^2\right]\cos(\theta_j - \theta_i)\big[(F(\omega_j) - F(\omega_i)) - (F(\tilde{\omega}_j) - F(\tilde{\omega}_i))\big] \\
&+ \operatorname{sgn}(\omega_i - \tilde{\omega}_i)\frac{\kappa}{N}\sum_{j=1}^{N}\left[\frac{H(\omega_j, \omega_i)\cos(\theta_j - \theta_i)}{F'(\omega_i)} - \frac{H(\tilde{\omega}_j, \tilde{\omega}_i)\cos(\tilde{\theta}_j - \tilde{\theta}_i)}{F'(\tilde{\omega}_i)}\right](F(\tilde{\omega}_j) - F(\tilde{\omega}_i)) \\
&= \operatorname{sgn}(\omega_i - \tilde{\omega}_i)\frac{\kappa}{N}\sum_{j=1}^{N}\left(\mathcal{I}_{11}^{ij} + \mathcal{I}_{12}^{ij} + \mathcal{I}_{13}^{ij}\right)
\end{aligned}
$$

Sketch of proof of the first theorem

Further estimate yields

$$\sum_{i=1}^{N} \mathrm{sgn}(\omega_i - \tilde{\omega}_i) \frac{\kappa}{N} \sum_{j=1}^{N} \mathcal{I}_{11}^{ij} \leq - \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i|,$$

$$\sum_{i=1}^{N} \mathrm{sgn}(\omega_i - \tilde{\omega}_i) \frac{\kappa}{N} \sum_{j=1}^{N} \mathcal{I}_{12}^{ij} \lesssim e^{-\Lambda_1 t} \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i|,$$

$$\sum_{i=1}^{N} \mathrm{sgn}(\omega_i - \tilde{\omega}_i) \frac{\kappa}{N} \sum_{j=1}^{N} \mathcal{I}_{13}^{ij} \lesssim e^{-\Lambda_1 t} \left(\sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i| + |\omega_i - \tilde{\omega}_i| \right).$$

Combined with $\frac{d}{dt}|\theta_i - \tilde{\theta}_i| = \mathrm{sgn}(\theta_i - \tilde{\theta}_i)(\omega_i - \tilde{\omega}_i) \leq |\omega_i - \tilde{\omega}_i|$, one has

$$\frac{d}{dt} \left(\sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i| + |\omega_i - \tilde{\omega}_i| \right) \lesssim e^{-\Lambda_1 t} \left(\sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i| + |\omega_i - \tilde{\omega}_i| \right).$$

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