Hydrodynamic limits of the nonlinear Schrödinger equation with the Chern-Simons gauge fields

Virtual Summer school on Kinetic and fluid equations for collective dynamics

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Chern-Simons-Schrödinger equations

- The Chern-Simons-Schrödinger (CSS) equations is given by

$$
\begin{aligned}
&\mathrm{i}\hbar D_0\psi + \frac{\hbar^2}{2m}(D_1D_1 + D_2D_2)\psi - V'(|\psi|^2)\psi = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2,\\
&\partial_0 A_1 - \partial_1 A_0 = -\hbar \mathrm{Im}(\bar{\psi} D_2 \psi), \quad \partial_0 A_2 - \partial_2 A_0 = \hbar \mathrm{Im}(\bar{\psi} D_1 \psi),\\
&\partial_1 A_2 - \partial_2 A_1 = -m|\psi|^2.
\end{aligned}
$$

- $\partial_0 = \partial_t, \quad \partial_i = \partial_{x_i},$
- $\psi : \mathbb{R}_+ \times \Omega \to \mathbb{C}$:Complex scalar field,
- $A_\mu : \mathbb{R}_+ \times \Omega \to \mathbb{R}$: Gague field,
- $D_\mu = \partial_\mu + \frac{\mathrm{i}}{\hbar} A_\mu$: covariant derivative,
- V: Self-interacting potenital energy density.

Chern-Simons-Schrödinger equations

- The CSS equation is invariant under the gauge transform:

$$\psi \to \psi e^{\mathrm{i}\chi}, \quad A_\mu \to A_\mu - \hbar \partial_\mu \chi.$$

- Therefore, one need to give a gauge condition. Usually, one consider the Coulomb gauge condition $\nabla \cdot A = \partial_1 A_1 + \partial_2 A_2 = 0$.
- One can also consider the other gauge condition:
  -Temporal gauge condition: $A_0 = 0$,
  -Lorentz gauge condition: $\partial_\mu A_\mu = 0$.

Chern-Simons-Schrödinger equations

- Under the Coulomb gauge condition, the CSS equation becomes

$$\begin{aligned}
&\mathrm{i}\hbar\partial_t\psi - A_0\psi + \frac{\hbar^2}{2m}\left(\Delta\psi + \frac{2\mathrm{i}}{\hbar}A\cdot\nabla\psi - \frac{1}{\hbar^2}|A|^2\psi\right) - V'(|\psi|^2)\psi = 0,\\
&\Delta A_0 = \hbar\mathrm{Im}(Q_{12}(\bar{\psi},\psi)) + \partial_1(A_2|\psi|^2) - \partial_2(A_1|\psi|^2),\\
&\Delta A_1 = m\partial_2|\psi|^2, \quad \Delta A_2 = -m\partial_1|\psi|^2,
\end{aligned}$$

  where $Q_{12}(\bar{\psi},\psi) := \partial_1\bar{\psi}\partial_2\psi - \partial_2\bar{\psi}\partial_1\psi$.

- Choosing $m = 1$ and $\hbar = \varepsilon$, we have the family of the scaled CSS equations:

$$\begin{aligned}
&\mathrm{i}\varepsilon\partial_t\psi - A_0\psi + \frac{\hbar^2}{2}\left(\Delta\psi + \frac{2\mathrm{i}}{\varepsilon}A\cdot\nabla\psi - \frac{1}{\varepsilon^2}|A|^2\psi\right) - V'(|\psi|^2)\psi = 0,\\
&\Delta A_0 = \varepsilon\mathrm{Im}(Q_{12}(\bar{\psi},\psi)) + \partial_1(A_2|\psi|^2) - \partial_2(A_1|\psi|^2),\\
&\Delta A_1 = \partial_2|\psi|^2, \quad \Delta A_2 = -\partial_1|\psi|^2,
\end{aligned}$$

Conservation laws

- The CSS system conserves the total charge and the total energy. Define

$$\mathcal{Q}(t) := \int_\Omega |\psi^\varepsilon|^2 \,\mathrm{d}x,$$

$$\mathcal{E}^\varepsilon(t) := \int_\Omega \frac{\varepsilon^2}{2} \sum_{j=1}^{2} |D_j^\varepsilon \psi^\varepsilon(t,x)|^2 + V(|\psi^\varepsilon(t,x)|^2) \,\mathrm{d}x,$$

  where $D_j^\varepsilon := \partial_j + \frac{\mathrm{i}}{\varepsilon} A_j^\varepsilon$.
- Then,

$$\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{E}^\varepsilon}{\mathrm{d}t} = 0.$$

Hydrodynamic formulation : Madelung transformation

- Considering the Madelung transformation

$$\psi^\varepsilon = \sqrt{\rho^\varepsilon}\,\mathsf{exp}\left(\frac{\mathrm{i}}{\varepsilon}S^\varepsilon\right),$$

we introduce the hydrodynamic variables

$$\rho^\varepsilon = |\psi^\varepsilon|^2, \quad \rho^\varepsilon u^\varepsilon := \rho^\varepsilon(\nabla S^\varepsilon + A^\varepsilon) = \frac{\mathrm{i}\varepsilon}{2}(\psi^\varepsilon\nabla\bar{\psi}^\varepsilon - \bar{\psi}^\varepsilon\nabla\psi^\varepsilon) + |\psi^\varepsilon|^2 A^\varepsilon.$$

- Then, the imaginary part of the Schrödinger equation becomes

$$\partial_t\rho^\varepsilon + \nabla\cdot(\rho^\varepsilon u^\varepsilon) = 0,$$

which corresponds to the continuity equation in the classical mechanics.

Hydrodynamic formulation

- On the other hand, the real part of the Schrödinger equation becomes

$$\partial_t S^\varepsilon + A_0^\varepsilon + \frac{1}{2}|\nabla S^\varepsilon + A^\varepsilon|^2 + V'(\rho^\varepsilon) = \frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}.$$

- Taking gradient,

$$\partial_t(\nabla S^\varepsilon) + (u^\varepsilon \cdot \nabla)u^\varepsilon + \rho^\varepsilon (u^\varepsilon)^\perp + \nabla A_0^\varepsilon + \frac{\nabla p(\rho^\varepsilon)}{\rho^\varepsilon} = \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right),$$

where $p(\rho) = \rho V'(\rho) - V(\rho)$. Choosing $V = \frac{1}{\gamma}\rho^\gamma$, $p(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$.

- Using the gauge equation, one can derive

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \frac{\nabla p(\rho^\varepsilon)}{\rho^\varepsilon} = \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right).$$

Hydrodynamic formulation

- To sum up, we have the following hydrodynamic system:

$$\partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0,$$
$$\partial_t(\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla p(\rho^\varepsilon) = \frac{\rho^\varepsilon \varepsilon^2}{2} \nabla \left(\frac{\Delta \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right),$$
$$\Delta A_0^\varepsilon = \nabla \times (\rho^\varepsilon u^\varepsilon), \quad \Delta A^\varepsilon = -(\nabla \rho^\varepsilon)^\perp.$$

- As $\varepsilon \to 0$, the hydrodynamic equations formally converges to the Euler-Chern-Simons equations:

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = 0,$$
$$\Delta A_0 = \nabla \times (\rho u), \quad \Delta A = -(\nabla \rho)^\perp.$$

- The main concern is to provide a rigorous analysis for this convergence.

Main theorem

- Consider the well-prepared initial data condition:

$$\int_\Omega \frac{\rho_0^\varepsilon |u_0^\varepsilon - u_0|^2}{2}\,\mathrm{d}x + \int_\Omega \frac{p(\rho_0^\varepsilon|\rho_0)}{\gamma-1}\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\sqrt{\rho_0^\varepsilon}|^2\,\mathrm{d}x = \mathcal{O}(\varepsilon^\lambda),$$

where $p(n|\rho) := \frac{\gamma-1}{\gamma}(n^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(n-\rho))$.

Theorem

Suppose $\gamma \geq 2$. Let $(\psi^\varepsilon, A_0^\varepsilon, A^\varepsilon)$ be the global solution to the CSS equations. Moreover, let (ρ, u, A_0, A) be the unique local-in-time smooth solution to the Euler-Chern-Simons equations for $0 \leq t \leq T_*$.

Theorem (continued)

Then, for any $0 \le t \le T_*$, we have

$$\begin{aligned}
&\rho^\varepsilon(t,\cdot) \to \rho(t,\cdot), \quad \text{in} \quad L^\gamma(\Omega),\\
&(\rho^\varepsilon u^\varepsilon)(t,\cdot) \to (\rho u)(t,\cdot), \quad \text{in} \quad L^{\frac{2\gamma}{\gamma+1}}(\Omega),\\
&(\sqrt{\rho^\varepsilon} u^\varepsilon)(t,\cdot) \to (\sqrt{\rho} u)(t,\cdot), \quad \text{in} \quad L^2(\Omega),\\
&A_0^\varepsilon \to A_0, \quad \text{in} \quad L^{2\gamma}(\Omega), \quad \nabla A_0^\varepsilon \to \nabla A_0 \quad \text{in} \quad L^{\frac{2\gamma}{\gamma+1}}(\Omega),\\
&A^\varepsilon \to A, \quad \text{in} \quad L^{2\gamma}(\Omega), \quad \nabla A^\varepsilon \to \nabla A, \quad \text{in} \quad L^\gamma(\Omega),
\end{aligned}$$

as $\varepsilon \to 0$.

Relative entropy

- To obtain a hydrodynamic limit (of the classical systems), the relative entropy method is successful.
- Consider the following general system of conservation laws:

$$\partial_t U_i + \sum_{k=1}^{d} \partial_k A_{ik}(U) = 0, \quad U \in \mathbb{R}^m, \quad A \in \mathbb{R}^{m\times d}.$$

- The compressible Euler equation can be written in this form with $U = (\rho, \rho u)$ and

$$A(U) = \frac{1}{\rho}\begin{pmatrix} \rho P_1 & \rho P_2 \\ P_1^2 + \frac{\gamma-1}{\gamma}\rho^{\gamma+1} & P_1 P_2 \\ P_2 P_1 & P_2^2 + \frac{\gamma-1}{\gamma}\rho^{\gamma+1} \end{pmatrix} = \begin{pmatrix} \rho u^\top \\ \rho u \otimes u + \frac{\gamma-1}{\gamma}\rho^\gamma I_2 \end{pmatrix}.$$

Relative entropy

- A usual entropy defined for the compressible Euler equation is

$$\eta(U) := \frac{|P|^2}{2\rho} + \frac{\rho^\gamma}{\gamma} = \frac{\rho|u|^2}{2} + \frac{\rho^\gamma}{\gamma}.$$

- Corresponding relative entropy and relative flux is given as

$$\eta(V|U) := \eta(V) - \eta(U) - D\eta(U)\cdot(V-U),$$
$$A(V|U) := A(V) - A(U) - DA(U)\cdot(V-U).$$

Here,

$$[DA(U)\cdot(V-U)]_{ij} := \sum_{k=1}^{3} \partial_{U_k} A_{ij}(U)(V_k - U_k).$$

Relative entropy method

- The relative entropy method is based on the following key estimate on the relative entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\eta(V|U)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\eta(V)\,\mathrm{d}x - \int_{\mathbb{R}^d}\nabla_x(D\eta(U)) : A(V|U)\,\mathrm{d}x \\ - \int_{\mathbb{R}^d} D\eta(U)\cdot(\partial_t V + \nabla_x\cdot A(V))\,\mathrm{d}x.$$

- We note that the energy functional $\mathcal{E}$ can be written in terms of the hydrodynamic quantities:

$$\mathcal{E}^\varepsilon = \frac{1}{2}\rho^\varepsilon|u^\varepsilon|^2 + \frac{(\rho^\varepsilon)^\gamma}{\gamma} + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho^\varepsilon}|^2 = \eta(U^\varepsilon) + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho^\varepsilon}|^2.$$

Modulated energy

- On the other hand, the hydrodynamic limit of the quantum system is based on the modulated energy estimate.
- The natural modulated energy is

$$\begin{aligned}\mathcal{H}^\varepsilon(t) &:= \int_\Omega \frac{1}{2}|(\varepsilon D^\varepsilon - \mathrm{i}u)\psi^\varepsilon|^2 + \frac{p(\rho^\varepsilon|\rho)}{\gamma-1}\,\mathrm{d}x \\ &= \int_\Omega \frac{1}{2}|(\varepsilon D^\varepsilon - \mathrm{i}u)\psi^\varepsilon|^2 + \frac{(\rho^\varepsilon)^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(\rho^\varepsilon-\rho)}{\gamma}\,\mathrm{d}x.\end{aligned}$$

  After tedious computation, we find

$$\mathcal{H}^\varepsilon = \int_\Omega \eta(U^\varepsilon|U)\,\mathrm{d}x + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho^\varepsilon}|^2\,\mathrm{d}x.$$

- Therefore, the modulated energy and the relative entropy are almost the same quantity, except for the “quantum term”.

Modulated energy estimate

- Using the equivalent relation between modulated energy and the relative entropy, one can use the celebrated theory of relative entropy to modulated energy of the CSS equations.

Proposition

Let $(\psi^\varepsilon, A_0^\varepsilon, A^\varepsilon)$ be the solution to the CSS equations and (ρ, u) be the unique local-in-time smooth solution to the compressible Euler equation. Then,

$$\mathcal{H}^\varepsilon(t) \leq C\varepsilon^{\min\{\lambda,2\}}.$$

- The proof is based on the previous proposition on the relative entropy, and an appropriate estimate for the quantum correction term.
- Therefore, one can conclude that the modulated energy vanishes as $\varepsilon \to 0$.

Proof of Proposition

- We estimate $\frac{\mathrm{d}\mathcal{H}^\varepsilon}{\mathrm{d}t}$ as

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}^\varepsilon}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\varepsilon - \int_\Omega \nabla_x(D\eta(U)) : A(U^\varepsilon|U)\,\mathrm{d}x \\ &\quad - \int_\Omega D\eta(U)\cdot(\partial_t U^\varepsilon + \nabla_x \cdot A(U^\varepsilon))\,\mathrm{d}x \\ &= 0 + I_1 + I_2.\end{aligned}$$

- Using the definition of $D\eta$ and $A(U^\varepsilon|U)$, we have

$$I_1 \leq C\left(\int_\Omega \rho^\varepsilon|u^\varepsilon - u|^2\,\mathrm{d}x + \int_\Omega p(\rho^\varepsilon|\rho)\,\mathrm{d}x\right) \leq \int_\Omega \eta(U^\varepsilon|U)\,\mathrm{d}x.$$

- On the other hand, using the governing equation of U^ε,

$$I_2 = -\frac{\varepsilon^2}{2}\int_\Omega \rho^\varepsilon u\cdot\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right)\mathrm{d}x \leq C\varepsilon^2\int_\Omega |\nabla\sqrt{\rho^\varepsilon}|^2\,\mathrm{d}x + C\varepsilon^2.$$

Proof of Proposition

- Combining the estimates, we have

$$\frac{\mathrm{d}\mathcal{H}^\varepsilon}{\mathrm{d}t} \le C\mathcal{H}^\varepsilon + C\varepsilon^2.$$

- Gronwall's inequality and the assumption of well-prepared initial data imply the desired estimate.
- With the modulated energy estimate in hand, one can obtain the desired convergence.

Lemma

Let $\gamma \ge 2$ be a constant. Then,

$$|\rho^\varepsilon - \rho|^\gamma \le (\rho^\varepsilon)^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(\rho^\varepsilon - \rho) = \frac{\gamma}{\gamma - 1} p(\rho^\varepsilon|\rho).$$

Proof of Theorem

- Convergence of the density:

$$\|\rho^\varepsilon - \rho\|_{L^\gamma}^\gamma \le C \int_{\mathbb{R}^2} p(\rho^\varepsilon|\rho)\,\mathrm{d}x \le \mathcal{H}^\varepsilon \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

- Convergence of the momentum:

$$\begin{aligned}
\|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^{\frac{2\gamma}{\gamma+1}}} &\le \|\rho^\varepsilon(u^\varepsilon - u)\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|(\rho^\varepsilon - \rho)u\|_{L^{\frac{2\gamma}{\gamma+1}}} \\
&\le \|\sqrt{\rho^\varepsilon}\|_{L^{2\gamma}} \|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + \|\rho^\varepsilon - \rho\|_{L^\gamma} \|u\|_{L^{\frac{2\gamma}{\gamma-1}}} \\
&\le C\|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + C\|\rho^\varepsilon - \rho\|_{L^\gamma} \le C\mathcal{H}^\varepsilon \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\|\sqrt{\rho^\varepsilon}u^\varepsilon - \sqrt{\rho}u\|_{L^2} &\le \|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + \|(\sqrt{\rho^\varepsilon} - \sqrt{\rho})|u|\|_{L^2} \\
&\le \|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + \|u\|_{L^{\frac{2\gamma}{\gamma-1}}} \|\sqrt{\rho^\varepsilon} - \sqrt{\rho}\|_{L^{2\gamma}} \\
&\le \mathcal{H}^\varepsilon + C\|\rho^\varepsilon - \rho\|_{L^\gamma}^{1/2} \to 0.
\end{aligned}$$

Proof of Theorem

- To prove the convergence of the gauge fields, we recall that

$$\Delta(A_0^\varepsilon - A_0) = \partial_1(\rho^\varepsilon u_2^\varepsilon - \rho u_2) - \partial_2(\rho^\varepsilon u_1^\varepsilon - \rho u_1).$$

- Using HLS inequality and CZ inequality,

$$\|A_0^\varepsilon - A_0\|_{L^{2\gamma}} \leq \|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^{\frac{2\gamma}{\gamma+1}}} \to 0,$$

and

$$\|\nabla(A_0^\varepsilon - A_0)\|_{L^{\frac{2\gamma}{\gamma+1}}} \leq \|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^{\frac{2\gamma}{\gamma+1}}} \to 0.$$

Proof of Theorem

- Similarly, the gauge difference $A^\varepsilon - A$ satisfies

$$\Delta(A^\varepsilon - A) = (\nabla(\rho - \rho^\varepsilon))^\perp,$$

which implies

$$\begin{aligned}\|A^\varepsilon - A\|_{L^{2\gamma}} &\le \|\rho^\varepsilon - \rho\|_{L^{\frac{2\gamma}{\gamma+1}}} \le (\|\sqrt{\rho^\varepsilon}\|_{L^2} + \|\sqrt{\rho}\|_{L^2})\|\sqrt{\rho^\varepsilon} - \sqrt{\rho}\|_{L^{2\gamma}} \\ &\le C\|\rho^\varepsilon - \rho\|_{L^\gamma} \to 0,\end{aligned}$$

and

$$\|\nabla(A^\varepsilon - A)\|_{L^\gamma} \le \|\rho^\varepsilon - \rho\|_{L^\gamma} \to 0.$$

Thank you very much for attention.