

From many-body quantum dynamics with singular potentials to Vlasov equation

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Setting

- N -body Schrödinger, Hartree-Fock equation, Vlasov equation¹

$$i\hbar \partial_t \rho_N = [H_N, \rho_N] \quad \text{with} \quad H_N = \sum_{j=1}^N -\frac{\hbar^2 \Delta_{x_j}}{2} + \sum_{1 \leq j < k \leq N} K(x_j - x_k)$$

$$i\hbar \partial_t \rho = [H, \rho] \quad \text{with} \quad H = \frac{-\hbar^2 \Delta}{2} + V_\rho - X_\rho$$

$$\partial_t f = \{H_f, f\} \quad \text{with} \quad H_f = \frac{|v|^2}{2} + V_f$$

Potential

$$K(x) = \frac{\pm 1}{|x|^a} \mathbb{1}_{x \neq 0} \quad a \in (0, 1]$$

Mean-field potential $V = K * \varrho$ where

$$\varrho_\rho(x) = h^3 \rho(x, x) \quad \varrho_f(x) = \int_{\mathbb{R}^3} f(x, v) dv$$

1. Can also be written $\partial_t f + v \cdot \nabla_x f - \nabla V_f \cdot \nabla_v f = 0$

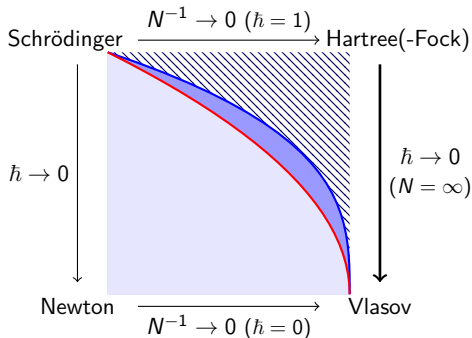
Combined mean-field and semiclassical limit

Smooth interactions

- bosons (Narnhofer-Sewell '81, Spohn '91, Graffi et al. '03, Golse-Paul '17-19)
- fermions, $\hbar = N^{-1/3}$ (Elgart et al. '04, Benedikter et al. '14-16, Petrat-Pickl '16)

Singular interactions (fermions, $\hbar = N^{-1/3}$)

- conditional (Porta et al '17, Saffirio '18), cutoff (Chen-Lee-Liew '21)



Classical case : weak-strong uniqueness

Theorem

$$\|f_1 - f_2\|_{L^1(\mathbb{R}^6)} \leq \|f_1^{\text{in}} - f_2^{\text{in}}\|_{L^1(\mathbb{R}^6)} \exp\left(C \int_0^T \|\nabla_v f_2\|_{L_x^{3,1} L_v^1} dt\right),$$

Proof :

$$(\partial_t + v \cdot \nabla_x - \nabla V_1 \cdot \nabla_v)(f_1 - f_2) = (\nabla V_1 - \nabla V_2) \cdot \nabla_v f_2,$$

so that, since $V = K * \varrho$, we obtain

$$\begin{aligned} \partial_t \int_{\mathbb{R}^6} |f_1 - f_2| dx dv &= - \int_{\mathbb{R}^6} (\varrho_1 - \varrho_2) \nabla K * \int_{\mathbb{R}^3} \text{sgn}(f) \nabla_v f_2 dv dx \\ &\leq \|f\|_{L^1} \left\| \nabla K * \int_{\mathbb{R}^3} |\nabla_v f_2| dv \right\|_{L^\infty}, \end{aligned}$$

From classical to quantum

Classical

$$\varrho_f = \int_{\mathbb{R}^3} f \, dv, \quad \int_{\mathbb{R}^3} \varrho_f = 1$$

$$M_n = \int_{\mathbb{R}^6} f |v|^n \, dx \, dv$$

$$\|f\|_{L^p(\mathbb{R}^6)}$$

$$\nabla_x f = \{-v, f\}, \quad \nabla_v f = \{x, f\}$$

Quantum

$$\varrho_\rho = h^3 \rho(x, x), \quad h^3 \text{Tr}(\rho) = 1$$

$$h^3 \text{Tr}(|\rho|^n \rho)$$

$$\|\rho\|_{\mathcal{L}^p} = h^{\frac{3}{p}} \text{Tr}(|\rho|^p)^{\frac{1}{p}}$$

$$\nabla_x \rho = [\nabla, \rho], \quad \nabla_\xi \rho = \left[\frac{x}{i\hbar}, \rho \right]$$

\implies Analogous semiclassical inequalities. Example :

$$\int_{\mathbb{R}^6} \frac{|\nabla_v f|}{|x - x_0|^2} \, dx \, dv \lesssim \|\nabla_v f\|_{L_x^{3,1} L_v^1} \quad h^2 \text{Tr} \left| \left[\frac{1}{|x - x_0|}, \rho \right] \right| \lesssim \|\varrho_{|\nabla_\xi \rho|}\|_{L^{3\pm\epsilon}}$$

Results

Theorem (LL, Saffirio '20)

Let f solution of Vlasov equation initially sufficiently smooth, ρ a solution of Hartree-(Fock) equation and ρ_f be the Weyl quantization of f . Then

$$\|\rho - \rho_f\|_{\mathcal{L}^1} \leq \left(\|\rho^{\text{in}} - \rho_f^{\text{in}}\|_{\mathcal{L}^1} + C_f(t) \hbar \right) e^{\lambda_f(t)}$$

Theorem (Chong, LL, Saffirio '21)

Let ρ be a solution of the Hartree-Fock equation initially smooth in a semiclassical sense. Then there exists $k, T > 0$, $\rho_{N,\rho}^{\text{in}} \in \mathcal{L}^1(\mathcal{F})$ such that for any ρ_N solution of Schrödinger equation with initial condition $\rho_N^{\text{in}} \in \mathcal{L}^1(\mathcal{F})$ commuting with \mathcal{N} , for any $t \in [0, T]$

$$\|\rho_{N:1} - \rho\|_{\mathcal{L}^1} \lesssim \frac{C e^{\lambda t}}{N^{1/2}} \left(1 + \|(\mathcal{N} + N)^k (\rho_N^{\text{in}} - \rho_{N,\rho}^{\text{in}})\|_{\mathcal{L}^1(\mathcal{F})} \right)$$

where λ is independent of \hbar if $a < 1/2$.

Ideas of the proofs :

- First theorem
 - ▶ Weak-strong uniqueness + regularity of f
- Second theorem
 - ▶ Propagate regularity for ρ

$$\rho, \nabla_x \rho, \nabla_\xi \rho \in \mathcal{L}^p(1 + |\rho|^n)$$

- ▶ Purification of mixed states

$$\rho_N \in \mathcal{L}^1(\mathcal{F}(L^2)) \rightarrow \Psi_{\rho_N} \in \mathcal{F}(L^2 \oplus L^2)$$

- ▶ Bogoliubov transformation $R_t = R_{\rho_t}$

$$\Psi_t = R_t^* e^{itL_N} R_t \Psi^{\text{in}}$$

- ▶ Number of particles outside the Bogoliubov state $\rho_{N,\rho}$ such that

$$\Psi_{\rho_{N,\rho}} = R_t^* e^{itL_N} R_t \Omega$$

Questions ?