The hyperbolic dead water phenomenon

Virtual Summer school on Kinetic and fluid equations for collective dynamics

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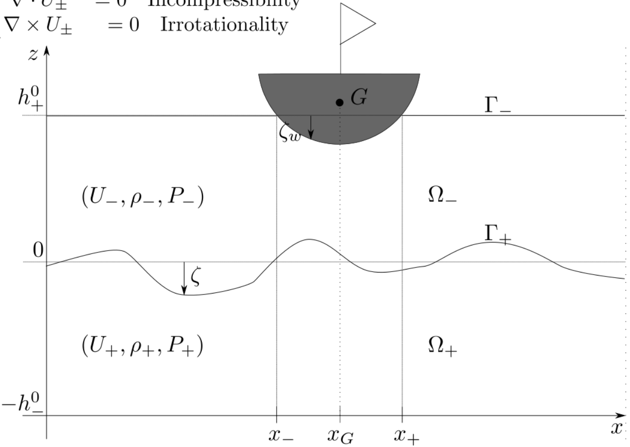
Notations

$$\begin{cases} \nabla \cdot U_{\pm} = 0 & \text{Incompressibility} \\ \nabla \times U_{\pm} = 0 & \text{Irrotationality} \end{cases}$$



Outline of the talk

Definition of the unknowns $(h_\pm, Q_\pm)$ and equations

$$\begin{cases} h_+ = h_+^0 + \zeta, \quad h_- = h_-^0 + \zeta_w - \zeta, \quad \zeta_w(t,x) = \zeta_e(x - x_G(t)), \\ Q_+ = \int_{-h_+^0}^{\zeta} U_+ \cdot e_x \mathrm{d}x, \quad Q_- = \int_{\zeta}^{h_-^0 + \zeta_w} U_- \cdot e_x \mathrm{d}x. \end{cases} \tag{1}$$

Theorem (Nonconservative hyperbolic system in $(h_\pm, Q_\pm)$)

$$\begin{cases} \partial_t h_\pm + \partial_x Q_\pm = 0, \\ \partial_t Q_+ + g h_+ \partial_x h_+ + \partial_x \left(\frac{Q_+^2}{h_+}\right) = -\frac{h_+ \partial_x(\underline{P})}{\rho_+} \\ \partial_t Q_- - g h_- \partial_x h_- + \partial_x \left(\frac{Q_-^2}{h_-}\right) = -\frac{h_- \partial_x(\underline{P})}{\rho_-} - g h_- \partial_x \zeta_w \end{cases} \tag{2}$$

where $\underline{P} = P_{|z=\zeta}$ *is the pressure at the interface.*

Role of the pressure

Theorem (Expression of the pressure at the interface)

$$\begin{cases} -\nabla \cdot \left(\left[\frac{h_+}{\rho_+} + \frac{h_-}{\rho_-} \right] \nabla(\underline{P}) \right) = -\partial_t^2 \zeta_w + a_{\mathrm{FS}}, \quad \forall X \in \mathbb{R}^d, \\ \lim_{|X| \to \infty} \underline{P} = P_{atm} + \rho_- g h_-^0. \end{cases} \tag{3}$$

with $a_{\mathrm{FS}} = a_{\mathrm{FS},+} + a_{\mathrm{FS},-}$, *where* $a_{\mathrm{FS},i} = -\partial_t \partial_x Q_i$ *when there is no boat.*

The hyperbolic system

New unknown : $V(t,x) = \frac{Q_+(t,x+x_G(t))}{h_+(t,x+x_G(t))} - \gamma \frac{Q_-(t,x+x_G(t))}{h_-(t,x+x_G(t))}$ with $\gamma = \frac{\rho_-}{\rho_+}$.

Theorem (Hyperbolic system in (ζ, V))

With the notation $\overline{V}_\pm = \frac{Q_\pm}{h_\pm}$

$$\begin{cases} \partial_t \zeta + \partial_x (h_+(\overline{V}_+ - \dot{x}_G)) = 0 \\ \partial_t V - \dot{x}_G \partial_x V + (1-\gamma) g \partial_x \zeta + \overline{V}_+ \partial_x \overline{V}_+ - \gamma \overline{V}_- \partial_x \overline{V}_- = 0 \end{cases} \quad (4)$$

Reconstruction of $\overline{V}_+$ and $\overline{V}_-$

Using the constraint $h_+ + h_- = \zeta_e(x - x_G)$, we can obtain

Theorem (Reconstruction formula)

$$\begin{cases} \overline{V}_+ & = \frac{h_- V + \gamma \dot{x}_G \zeta_w}{h_- + \gamma h_+}, \\ \overline{V}_- & = \frac{-h_+ V + \dot{x}_G \zeta_w}{h_- + \gamma h_+}. \end{cases} \tag{5}$$

The 1D case allows for explicit computation. Proving that there is a reconstruction formula in 2D is non trivial.

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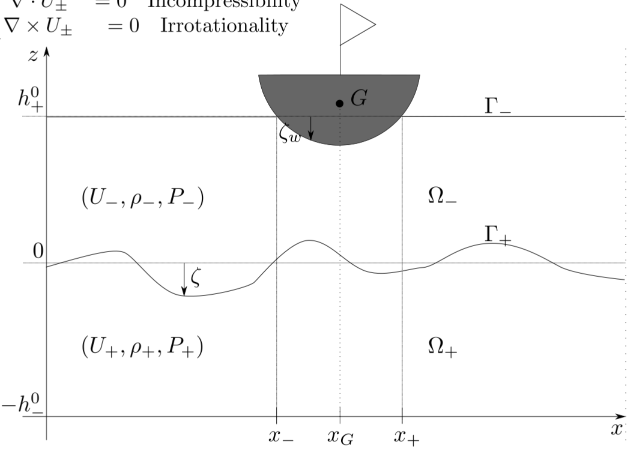

The Newton's equation

Original Newton's equation : $m\ddot{x}_G = F_0 + F_P$ with
$F_P = -\int_{\mathbb{R}} P_{z=h_-^0+\zeta_w}\partial_x\zeta_w \mathrm{d}x$
Shallow water regime, we have $P_{|z=\zeta_w} = \underline{P} + \rho_- g h_-$, i.e. the non-hydrostatic component is neglected.

$$\begin{cases} M(\zeta,\zeta_e) = & \int_{\mathbb{R}} \frac{\zeta_w^2}{h_+/\rho_+ + h_-/\rho_-}\mathrm{d}x, \\ F_w = & \rho_- g\int_{\mathbb{R}}\zeta_e\partial_x\zeta \mathrm{d}x - g\int_{\mathbb{R}}\frac{\zeta_e(h_+ + h_-)}{h_+/\rho_+ + h_-/\rho_-}\partial_x\zeta_e \mathrm{d}x \\ & -\int_{\mathbb{R}}\frac{\zeta_e}{h_+/\rho_+ + h_-/\rho_-}\partial_x\left(h_+^2\overline{V}_+ + h_-^2\overline{V}_-\right)\mathrm{d}x. \end{cases} \tag{6}$$

Theorem (The Newton's equation with the added mass)

$$(m + M(\zeta,\zeta_e))\ddot{x}_G = F_0 + F_w \tag{7}$$

With periodic boundary conditions and a large grid, using a finite volume method and a Lax-Friedrichs scheme we obtain the following numerical results.

Simulation

Perspectives

- Add transparent boundary conditions and study the IBVP associated, coupled with the ODE -> long time existence result
- Take into account dispersive effects -> dispersive IBVP : appearance of boundary layers
- Extend the study to the case of a non flat surface -> four equations and the contact points between the boat and the wave are no longer fixed
- Extend our study to the 2D case -> non explicit computations

Thank you for your attention !

Bibliography

Tatsuo Iguchi and David Lannes.
Hyperbolic free boundary problems and applications to wave-structure interactions.
2018.

David Lannes.
On the dynamics of floating structures.
Annals of PDE, 3(1):11, 2017.

Matthieu J Mercier, Romain Vasseur, and Thierry Dauxois.
Resurrecting dead-water phenomenon.
Nonlinear Processes in Geophysics, 18(2):193–208, 2011.

Tatsuo Iguchi and David Lannes.
Hyperbolic free boundary problems and applications to wave-structure interactions.
2018.

David Lannes.
On the dynamics of floating structures.
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