An optimal transport approach of hypocoercivity for the 1d kinetic Fokker-Planck equation

Samir Salem

CMLS, École Polytechnique

Virtual Summer school on Kinetic and fluid equations for collective dynamics, August 23rd 2021

1. Hypocoercivity

2. Optimal transport and coupling

3. Result

The starting point is the **kinetic Fokker-Planck** equation, set on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla\Psi(x) \cdot \nabla_v f_t = \nabla_v \cdot (\nabla_v f_t + v f_t),$$

for the initial data f_0.
It is the time evolution equation of some **Langevin** process

$$\begin{aligned} dX_t &= V_t dt \\ dV_t &= (-V_t + \nabla\Psi(X_t))dt + \sqrt{2} dB_t. \end{aligned}$$

The theory dealing with the convergence to the equilibrium, given in this case by

$$f_\infty(x,v) = Z^{-1} e^{-\Psi(x) - \frac{|v|^2}{2}}, \quad Z = \sqrt{2\pi}^d \int_{\mathbb{R}^d} e^{-\Psi(x)} dx,$$

of solutions to this *hypoelliptic* equation is called *hypocoercivity*.

H^1 *Sobolev norms* : for $b < \sqrt{ac}$, the new norm, equivalent to the usual $\|\cdot\|_{H^1(f_\infty)}$, is defined as

$$\|h\|_{\tilde{H}}^2 = \|h\|_{L^2(f_\infty)}^2 + a\|\nabla_x h\|_{L^2(f_\infty)}^2 + 2b\,\langle \nabla_x h, \nabla_v h\rangle_{L^2(f_\infty)} + c\|\nabla_v h\|_{L^2(f_\infty)}^2.$$

Under the assumption that $e^{-\Psi}$ satisfies some Poincaré inequality, we obtain exponential convergence in this new norm

$$\left\|\frac{f_t}{f_\infty} - 1\right\|_{\tilde{H}} \leq e^{-\kappa t}\left\|\frac{f_0}{f_\infty} - 1\right\|_{\tilde{H}}.$$

$L \ln L$ *Relative entropy* :
Under the assumption that Ψ satisfies a Logarithmic-Sobolev inequality, one can obtain

$$\int f_t \ln \frac{f_t}{f_\infty} = O(e^{-\kappa t}),$$

and by Talagrand's inequality

$$W_2(f_t, f_\infty) = O(e^{-\kappa t}).$$

Can this estimate be obtained as the consequence of the K.F.P. eq being a contraction in some metric, equivalent to the usual W_2?

1 Hypocoercivity

2 Optimal transport and coupling

3 Result

For a (convex) cost function c, and $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ consider the Kantorovitch problem

$$W_c(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(z_1 - z_2)\pi(dz_1, dz_2),$$

with

$$\begin{aligned}\Pi(\mu,\nu) &= \{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d),\ \forall A, B \subset \mathbb{R}^d,\ \pi(A \times \mathbb{R}^d) = \mu(A),\ \pi(\mathbb{R}^d \times B) = \nu(B)\}\\ &= \{\mathcal{L}((X,Y)), X \sim \mu, Y \sim \nu\}.\end{aligned}$$

Whenever c is strictly convex, and μ, ν are smooth, the Kantorovitch problem coincides with the Monge problem :

$$W_c(\mu,\nu) = \inf_{\mathcal{T}\#\nu=\mu} \int_{\mathbb{R}^d} c(\mathcal{T}(z) - z)\nu(dz),$$

and the infimum is obtained for a unique optimal transport map $\mathcal{T}$ (Brenier's Theorem).
For instance $W_2^2 = W_c$ for $c = |\cdot|^2$.

1 Hypocoercivity

2 Optimal transport and coupling

3 Result

W_2 Bolley, Guillin, Malrieu : define the new distance, equivalent to the usual W_2

$$W_A^2(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} (z_1 - z_2)\cdot A(z_1 - z_2)\pi(dz_1, dz_2), \quad A = \begin{pmatrix} aI_d & bI_d \\ bI_d & cI_d \end{pmatrix}$$

$$= \inf_{(X,V)\sim\mu,(\tilde{X},\tilde{V})\sim\nu} \mathbb{E}\left[a|X-\tilde{X}|^2 + 2b(X-\tilde{X})\cdot(V-\tilde{V}) + c|V-\tilde{V}|^2\right].$$

Under the assumption that Ψ is **strictly convex**, there is contraction in this distance

$$W_A(f_t, f_\infty) \le e^{-\kappa t} W_A(f_0, f_\infty).$$

Idea of the proof :

$$dX_t = V_t dt \qquad d\tilde{X}_t = \tilde{V}_t dt$$

$$dV_t = (-V_t + \nabla\Psi(X_t))dt + \sqrt{2}dB_t \qquad d\tilde{V}_t = (-\tilde{V}_t + \nabla\Psi(\tilde{X}_t))dt + \sqrt{2}d\tilde{B}_t.$$

For instance $(B_t)_{t\ge0} = (\tilde{B}_t)_{t\ge0}$ (synchronous coupling), and by definition

$$W_A^2(f_t, g_t) \le \mathbb{E}\left[a|X_t-\tilde{X}_t|^2 + 2b(X_t-\tilde{X}_t)\cdot(V_t-\tilde{V}_t) + c|V_t-\tilde{V}_t|^2\right].$$

By **optimal transport**, $\forall t > 0$, $\exists T_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ s.t. $T_t \# g_t = f_t$ and

$$W_A^2(f_t, g_t) = \int_{\mathbb{R}^{2d}} (T_t(z) - z) \cdot A(T_t(z) - z) g_t(z) dz.$$

The properties of these optimal transport maps, **and the diffusion in** v, enable to obtain the

Theorem (S. (2021))

For $d = 1$, let $\Psi \in \mathcal{C}^2(\mathbb{R})$ s.t. $\Psi''(x) > 0$ for $|x| \geq R$ and $| [\Psi''(x)]_- | << 1$ otherwise. There is $A \in \mathcal{M}_2(\mathbb{R})$ and $\kappa > 0$ s.t. $\forall f_0 \in \mathcal{P}_2(\mathbb{R}^2)$, $\forall t > 0$, there holds

$$W_A(f_t, f_\infty) \leq e^{-\kappa t} W_A(f_0, f_\infty).$$

Many thanks for your attention!