Relaxation operators in kinetic theory

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Plan

Return to the 90^s

- Find deterministic methods for the Boltzmann equation
- Discrete velocity models have good properties but computation of $Q(f, f)$ for "N" velocities in $O(N^2 \log N)$.
- Look for simpler model rather than for numerical method.
- Find a compromise between the numerical cost and the accuracy of the model.

The Boltzmann equation

The Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) = Q^+(f,f) - \nu(f)f, \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$$

Notations

$f(t,x,v)$	:	distribution function
$n = \int f \, dv$	:	number of molecules per unit volume
$nu = \int vf \, dv$	:	momentum (m=1)
$E = \frac{nu^2}{2} + \frac{3}{2}nT = \frac{1}{2}\int v^2 \, dv$	:	total energy

Main properties I

1) Preservation of positivity : under suitable assumptions the solution exists and $f(t,x,v)$ remains a density function i.e. $f(t,x,v) \geq 0$
2) Collision invariants (mass, momentum and energy are conserved during a collision)

$$\int Q(f,f)(1,v,v^2)\,dv = 0$$

3) $\exists \eta$ entropy density and

$$H(f) = \int \eta(f)dv \quad \text{s.t.} \quad \int \eta'(f)Q(f,f)dv \leq 0.$$

3′) For the Boltzmann equation $\eta(x) = x\ln(x) - x$,

$$\partial_t H(f) + div \int v\eta(f)\,dv \leq 0$$

Main properties II

4) Extended H theorem

$$\int_{\mathbb{R}^3} \eta'(f)Q(f,f)dv = 0 \Leftrightarrow Q(f,f) = 0 \Leftrightarrow \eta'(f) \in \ Span\{1, v, v^2\}$$

$$\Rightarrow f = \mathcal{M} = \frac{n}{(2\pi T)^{\frac{3}{2}}} \exp(-\frac{(v-u)^2}{2T})$$

5) Galilean invariance
6) Correctness of the hydrodynamic limit : Right properties on the linearized operator (Fredholm, kernel $= \mathbb{K}$
Chapmann-Engskog expansion : $f = \mathcal{M}(1 + \varepsilon g) + \mathcal{O}(\varepsilon^2)$ into

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f,f).$$

$\Rightarrow$ Euler and Navier-Stokes

Entropic approximation and first models

A hierarchy of models

Main idea : E. M. Shakhov, MEKHIANIKA ZHIDKOSTI I GAZA, 1968

Replace $Q(f,f) = Q^+(f,f) - \nu f$ with $R(f) = \nu(G - f)$ where $G \approx Q^+(f,f)/\nu$.

Constraints for Maxwell molecules :

$$\int G\,\mathbf{m}(v) = \nu^{-1} \int Q^+(f,f)\,\mathbf{m}(v)$$

where $\mathbf{m}(v) = (m_0, ..., m_N)$ is a generating vector of a (suitable) polynomial space $\mathbb{P}$

"Closure" : $G = \mathcal{M} * p(v)$ where $p(v) \in \mathbb{P}$

Problem : G is not nonnegative everywhere and the solution might become negative !

Entropic Approximation 1

[Levermore, JSP, 1996]

• Replace $f \longrightarrow G = exp(\boldsymbol{\alpha}.\mathbf{m}(v))$ solution of

$$H(G) = \min_{g \in C_f} H(g)$$

with

$$C_f = \{g \geq 0, \int g\mathbf{m}(v) = \int f\,\mathbf{m}(v)dv\}.$$

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Question : $f \longrightarrow G$ well-defined ?

Entropic Approximation 2

Junk says not always ! M3AS, 2000.

But it is possible if the constraint of highest degree is relaxed
J.S., M2AN, 2004.

1. Example : $\mathbb{P} = \mathbb{P}_{N-1} \oplus \mathbb{R}.|v|^N$, just ask $\int g|v|^N \leq \int f|v|^N$
2. The **entropic "approximation"** $f \longrightarrow G(f)$ is well defined.
3. **New closure in Shakhov model :** G = entropic approximation $Q^+(f,f)/\nu$. The model satisfies all properties !

   $\Longrightarrow$ model restricted to Maxwellian molecules.

Method of moment relaxation

Relaxation operator

Aim : Construct relaxation operator

$$R(f) = \nu(G - f)$$

that « behaves as » linear operator while pertaining positivity

Definition

If $R(f)$ satisfies properties $1 - 6$, $R(f)$ is a well defined operator.

Remark

We do not ask $H(f) = \int \eta(f) dv$ to be a Lyapounov Functional.

Relaxation constraints

Let $\boldsymbol{m}(v) = (m_1(v), \ldots, m_N(v))$ be a set of tensors.

$$\int R(f) m_i(v) dv = -\nu_i \int_{\mathbb{R}^3} f\, m_i(v) dv$$

$(\nu_i)_{i=1,\ldots,N}$ are nonnegative relaxation coefficients (frequency)

First example

Set of tensors : $\mathbb{P} = \mathbb{K} \oplus^{\perp} \mathbb{A}$, where

$$\mathbb{A}(v) = (v-u) \otimes (v-u) - \frac{1}{3} \|v-u\|^2 I_d$$

$\Rightarrow$ Rigorous derivation of the ESBGK model [S. Brull, J.S., 2008]

Main example

Set of tensors : $\mathbb{P} = \mathbb{K} \oplus^{\perp} \mathbb{A} \oplus^{\perp} b$, where

$$\boldsymbol{b}(v) = (v-u)\left(\frac{1}{2}(v-u)^2 - \frac{5}{2}T\right)$$

Constraints

$$\begin{aligned}
\int_{\mathbb{R}^3} G\,(1, v, v^2)\,dv &= \int_{\mathbb{R}^3} f(1, v, v^2)\,dv \\
\int_{\mathbb{R}^3} G\,\mathbb{A}(v-u)\,dv &= (1 - \frac{\nu_{\mathbb{A}}}{\nu}) \int_{\mathbb{R}^3} f\,\mathbb{A}(v-u)\,dv \\
\int_{\mathbb{R}^3} G\,\boldsymbol{b}(v-u)\,dv &= (1 - \frac{\nu_b}{\nu}) \int_{\mathbb{R}^3} f\,\boldsymbol{b}(v-u)\,dv
\end{aligned}$$

Conservation laws $\Rightarrow \nu_i = 0$ on $\mathbb{K}$

How to define ν, G, $\nu_{\mathbb{A}}$ and ν_b ?

Variational principle

Choose a strictly convex function η with domain in $\mathbb{R}_+$ and set

$$H(g) = \int \eta(g)\, dv$$

For $\boldsymbol{\rho}_f = \int f\, \boldsymbol{m}(v) dv$, define $L(\boldsymbol{\rho}_f)$, with the relaxation constraints

$$(L(\boldsymbol{\rho}_f))_i = (1 - \frac{\nu_i}{\nu}) \int_{\mathbb{R}^3} f\, m_i(v)\, dv$$

Problem

For $\boldsymbol{\rho}_f \in \mathbb{R}^q$ (q : dimension of span$\{m_i\}$), find if possible a function G such that

1. $\int G\mathbf{m}(v)\, dv = L(\boldsymbol{\rho}_f)$
2. $H(G) = \min\limits_{\int g\mathbf{m}(v)\, dv = L(\boldsymbol{\rho}_f)} H(g).$

Definition of $\nu_{\mathbb{A}}$ and $\nu_{\boldsymbol{b}}$

If $f \mapsto G$ is sufficiently smooth and $R(f)$ is well-posed (H theorem)

$$\partial_t f^\varepsilon + v \cdot \nabla f^\varepsilon = \frac{1}{\varepsilon} R(f^\varepsilon).$$

Order -1 : H theorem $\Rightarrow f_0 = \mathcal{M} \Rightarrow$ Euler equation in $O(\varepsilon)$
Setting $f_1 = \mathcal{M} g_1$
Equation at order 0 :

$$\mathcal{L}_R(g_1) = \mathbb{A}(v-u) : \mathbb{D}(u) + \boldsymbol{b}(v-u) \cdot \nabla_x(-\frac{1}{T}),$$

where $\mathbb{D}(u)$ is the Reynolds tensor and

$$\mathcal{L}_R(g) = \frac{1}{\mathcal{M}} \lim_{\varepsilon \to 0} \frac{R(\mathcal{M}(1+\varepsilon g))}{\varepsilon}$$

Definition of $\nu_{\mathbb{A}}$ and $\nu_{\boldsymbol{b}}$

Compute

$$\mathcal{L}_R(g) = \nu\Big(\sum(1 - \frac{\nu_i}{\nu})\mathbb{P}_{m_i} + \mathbb{P}_{\mathbb{K}} - Id\Big)(g)$$

$$\mu_R = \frac{T}{10}\langle \mathcal{L}_R^{-1}(\mathbb{A}), \mathbb{A}\rangle = \frac{nT}{\nu_{\mathbb{A}}}, \quad \kappa_R = \frac{5nT}{2\nu_{\boldsymbol{b}}}$$

where $\langle \cdot, \cdot \rangle$ is the $L^2(\mathcal{M})$ dot product with the full contraction for tensor.

Definition of $\nu_{\mathbb{A}}$ and $\nu_{\boldsymbol{b}}$

$$\nu_{\mathbb{A}} = \frac{nT}{\mu}, \quad \nu_{\boldsymbol{b}} = \frac{5}{2}\frac{nT}{\kappa} \Longrightarrow Pr = \frac{5}{2}\frac{\mu}{\kappa} = \frac{\nu_{\boldsymbol{b}}}{\nu_{\mathbb{A}}}$$

***R* is designed such that $\mathcal{L}_R^{-1} \sim \mathcal{L}_B^{-1}$ and not $\mathcal{L}_R \sim \mathcal{L}_B$**

Problems to be solved

We assume that $f \geq 0$ and

$$\int_{\mathbb{R}^3} f\,|m_i(v)|\,dv < +\infty, \quad \forall i = 1, \ldots, N.$$

G must satisfy $G \geq 0$ and $\int G\boldsymbol{m}(v) = L(\boldsymbol{\rho}_f)$

Condition

$$C_f = \left\{ g \geq 0 \;/\; \int_{\mathbb{R}^3} g\, m_i(v)\, dv = \left(1 - \frac{\nu_i}{\nu}\right) \int_{\mathbb{R}^3} f\, m_i(v)\, dv \right\} \neq \emptyset.$$

Questions

1. What is the shape of the set of realizable moments

$$\mathcal{R}_{\mathbf{m}}^{+} = \left\{ \int_{\mathbb{R}^3} f \, \mathbf{m}(v) \, dv, \ f \geq 0, \ \int_{\mathbb{R}^3} f \, |m_i(v)| \, dv < +\infty \right\}$$

2. Optimization problem :
   Solve

$$\min_{g \in C_f} \int_{\mathbb{R}^3} \eta(g) \, dv$$

   for some entropy density η.

   The choice of η is crucial for
   - The existence of a (unique) minimizer
   - The H theorem

Remark

No solution (in general) when $\eta(x) = x\ln(x)$ under the constraints

$$\begin{aligned}
\int_{\mathbb{R}^3} g\,(1,\mathbf{v},\mathbf{v}^2)\,dv &= \int_{\mathbb{R}^3} f\,(1,\mathbf{v},\mathbf{v}^2)\,dv \\
\int_{\mathbb{R}^3} g\,\mathbb{A}(v-u)\,dv &= (1-\frac{\lambda_{\mathbb{A}}}{\nu})\int_{\mathbb{R}^3} f\,\mathbb{A}(v-u)\,dv \\
\int_{\mathbb{R}^3} g\,\boldsymbol{b}(v-u)\,dv &= (1-\frac{\lambda_{\boldsymbol{b}}}{\nu})\int_{\mathbb{R}^3} f\,\boldsymbol{b}(v-u)\,dv
\end{aligned}$$

Artificial condition on $\int g|v|^4 dv$?

The problem might not be well posed ?

See [Junk, 1998, 2000], [J.S., 2004], [Hauck et all, 2008], [Pavan, 2011]

Shape of the set $\mathcal{R}_{\mathbf{m}}^{+}$

In 1d : Hamburger moment problem

Given $\rho_0, \ldots, \rho_n \in \mathbb{R}$ and $m_i(x) = x^i$ is there a measure μ such that

$$\int_{\mathbb{R}} x^i d\mu = \rho_i$$

Theorem (Akhiezer, Krein, 1962)

If $n = 2p$ there exists a measure μ iff the Hankel matrix

$$H := (\rho_{i+j})_{0 \le i,j \le p}$$

is positive definite.

The measure $d\mu$ can be changed into $f(x)\, dx$ with $f \in L^1$

Generalisation

Definition

Let $\mathbf{m}(\mathbf{v}) := (\mathbf{m}_0(\mathbf{v}), \cdots, \mathbf{m}_k(\mathbf{v}), \cdots, \mathbf{m}_n(\mathbf{v}))$ be a list of tensors where $v \in \mathbb{R}^d$. $(\mathbf{m}_k)_k$ is pseudo-Haar when :

$$\forall \boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n), \quad [\boldsymbol{\alpha} \neq \mathbf{0} \Rightarrow \boldsymbol{\alpha} \cdot \mathbf{m}(\mathbf{v}) \neq 0], \quad \lambda.a.e\ \mathbf{v} \in \mathbb{R}^d \tag{1}$$

$$\boldsymbol{\alpha} \cdot \mathbf{m}(\mathbf{v}) := \sum_k \alpha_k : \mathbf{m}_k(\mathbf{v}) \tag{2}$$

Problem

Let $\boldsymbol{\rho} = (\rho_0, \ldots, \rho_n)$ a list of tensor. Is there a nonnegative function f in $L^1(\mathbb{R}^d)$ s.t.

$$\int f\, m_i(v) dv = \rho_i$$

Examples in kinetic theory

Example

One may consider the following Pseudo-Haar basis :

1. "Euler" $\mathbf{m}(\mathbf{v}) = \left(1, \mathbf{v}, \mathbf{v}^2\right)$
2. "Gauss"' : $\mathbf{m}(\mathbf{v}) = (1, \mathbf{v}, \mathbf{v} \otimes \mathbf{v})$
3. Grad : $\mathbf{m}(\mathbf{v}) = \left(1, \mathbf{v}, \mathbf{v} \otimes \mathbf{v}, \mathbf{v}^2\mathbf{v}\right)$
4. Levermore : $\mathbf{m}(\mathbf{v}) = \left(1, \mathbf{v}, \mathbf{v} \otimes \mathbf{v}, \mathbf{v}^2\mathbf{v}, \mathbf{v}^4\right)$

Theorem of Junk

Theorem

[Junk, 2000]

1. $\boldsymbol{\rho} \in \mathcal{R}_{\mathbf{m}}^{+} \backslash \{0\} \Leftrightarrow \forall\, \boldsymbol{\alpha} \neq 0$ *such that* $\boldsymbol{\alpha} \cdot \mathbf{m}(v) \leq 0$ *a.e. there is* $\boldsymbol{\rho} \cdot \boldsymbol{\alpha} < 0$
2. $\mathcal{R}_{\mathbf{m}}^{+,*}$ *is an open convex set*
3. $\forall\, \boldsymbol{\rho} \in \mathcal{R}_{\mathbf{m}}^{+}, \exists\, \psi \geq 0, \in C_c^{\infty}(\mathbb{R}^3))$ *such that* $\boldsymbol{\rho} = \int_{\mathbb{R}^3} \mathbf{m}(v)\, \psi(v)\, dv$

Remark

The set of realizable moment $\mathcal{R}_{\mathbf{m}}^{+} \backslash \{0\}$ *is characterized by the set of (non positive) nonnegative polynomials : all* $\boldsymbol{\alpha} \neq 0$ *such that* $\boldsymbol{\alpha} \cdot \mathbf{m}(v) \geq 0$.

Proof I

Definition (Cone spanned by the pseudo-Haar basis)

Let $\mathbf{m}(\mathbf{v})$ pseudo-Haar on $\mathbb{R}^d$ and q dimension of $span\,(m_0(\mathbf{v}), \cdots, m_N(\mathbf{v}))$. Define $\mathcal{C}$ the positive cone spanned by $\mathbf{m}(\mathbf{v})$ in $\mathbb{R}^q$:

$$\mathcal{C} = \left\{ \sum_i \lambda_i \mathbf{m}(\mathbf{v}_i), \lambda_i \geq 0, \quad \mathbf{v}_i \in \mathbb{R}^d \right\} \tag{3}$$

Proof II

First remark

If $m(\mathbf{v})$ is continuous w.r.t $\mathbf{v}$ and $\psi_\epsilon \in C_0^\infty$ such that $\psi_\epsilon \to \delta$ then

$$\boldsymbol{\rho}_\epsilon := \int m(\mathbf{w})\, \psi_\epsilon (\mathbf{w} - \mathbf{v})\ d\mathbf{v} \in \mathcal{R}^+_{\mathbf{m}} \quad \text{et} \quad \boldsymbol{\rho}_\epsilon \longrightarrow m(\mathbf{v})\,. \tag{4}$$

Same thing for each element of $\mathcal{C} \Longrightarrow \mathcal{C} \subset \overline{\mathcal{R}}^+_{\mathbf{m}}$.
Converse statement ?

Second remark

Let $\boldsymbol{\alpha} \neq 0$ such that $\boldsymbol{\alpha} \cdot \mathbf{m}(v) \leq 0,\ \forall v$ and $f \geq 0\ (\neq 0)\ f \in \mathbb{L}^1_{\mathbf{m}}$ then

$$\boldsymbol{\alpha} \cdot \boldsymbol{\rho}_f = \int \boldsymbol{\alpha} \cdot \mathbf{m}(\mathbf{v})\ f(\mathbf{v})\ d\mathbf{v} < 0 \tag{5}$$

Proof III

Definition (Polar cone of $\mathcal{C}$)

$$\begin{aligned} \mathcal{C}^{\circ} &:= \{\boldsymbol{\alpha} \, : \, \boldsymbol{\alpha} \cdot \boldsymbol{\eta} \leq 0 \quad \forall \boldsymbol{\eta} \in \mathcal{C}\}, \\ &= \{\boldsymbol{\alpha} \, : \, \boldsymbol{\alpha} \cdot \mathbf{m}(\mathbf{v}) \leq 0 \quad \forall \mathbf{v} \in \mathbb{R}^{d}\} \end{aligned}$$

Theorem (Interior of a solid convex cone)

If $\mathcal{X}$ is a convex cone with non-empty interior in $\mathbb{R}^q$, then

$$x \in \mathtt{int}(\mathcal{X}) \Leftrightarrow \forall \boldsymbol{y} \in \mathcal{X}^{\circ}, \; [\boldsymbol{y} \neq \mathbf{0} \Rightarrow \boldsymbol{y} \cdot \mathbf{x} < 0] \tag{6}$$

$\Rightarrow$ if $\mathtt{int}(\mathcal{C}) \neq \emptyset$ then $\mathcal{R}_{\mathbf{m}}^{*+} \subset \mathtt{int}(\mathcal{C})$

Proof IV

Proposition

Let $\mathbf{m}(\mathbf{v})$ *pseudo-Haar basis and* $\mathcal{C}$ *the positive cone associated to it, then*

1. $\mathtt{int}(\mathcal{C}) \neq \emptyset \Longrightarrow \mathcal{R}_{\mathbf{m}}^{*+} \subset \mathtt{int}(\mathcal{C})$
2. $\forall \boldsymbol{\rho} \in \mathtt{int}(\mathcal{C}), \exists \psi \geq 0, \in C_c^{\infty}(\mathbb{R}^3))$ *such that*

$$\boldsymbol{\rho} = \int \mathbf{m}(\mathbf{v})\,\psi(\mathbf{v})\,d\mathbf{v} \tag{7}$$

$\Longrightarrow \mathtt{int}(\mathcal{C}) \subset \mathcal{R}_{\mathbf{m}}^{*+}$

Characterizing nonnegative polynomials

XVIIth Hilbert's problem

Show that every nonnegative polynomial with coefficient in $\mathbb{R}$ is a sum of square rational functions.

One of the important question about this problem :

- If $p(\mathbf{v}) = \boldsymbol{\alpha} \cdot \boldsymbol{m}(v)$ is nonnegative, is it a sum of square (S.O.S) polynomials ?

Hilbert problem for quadratic structured space

Example (Levermore space)

The Levermore space can be identified as a product of
$\left(1, \mathbf{v}, \mathbf{v}^2\right) \vee \left(1, \mathbf{v}, \mathbf{v}^2\right) = \left(1, \mathbf{v}, \mathbf{v} \otimes \mathbf{v}, \mathbf{v}^2\mathbf{v}, \ \mathbf{v}^4\right)$

A square polynomial $P(v) = \left(a + \boldsymbol{b} \cdot \boldsymbol{v} + c\boldsymbol{v}^2\right)^2$ can be written as

$$\boldsymbol{\beta}^T M\boldsymbol{\beta}, \text{with} \quad \boldsymbol{\beta} = (a, \boldsymbol{b}, c)^T \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$$

$$M = \begin{bmatrix} 1 & \mathbf{v}^T & \mathbf{v}^2 \\ \mathbf{v} & \mathbf{v} \otimes \mathbf{v} & \mathbf{v}^2\mathbf{v} \\ \mathbf{v}^2 & \mathbf{v}^2\mathbf{v}^T & \mathbf{v}^4 \end{bmatrix}$$

What about Grad space ?

$\mathbf{G}r\,(\mathbf{v}) = \left(1, \mathbf{v}, \mathbf{v} \otimes \mathbf{v}, \mathbf{v}^2\mathbf{v}\right)$ has no quadratic structure

Hankel matrix

Definition

Likewise, for $f \geq 0$ in L^1_{Lev} define a Hankel matrix H as

$$H = \int_{\mathbb{R}^3} M\, f(v)\, dv.$$

$$\Rightarrow \int \left(a + \boldsymbol{b} \cdot \boldsymbol{v} + c\boldsymbol{v}^2\right)^2 f dv = \boldsymbol{\beta}^T H \boldsymbol{\beta} > 0$$

Necessary condition :

H must be definite positive.

Converse statement ?

True if every positive polynomial is a SOS ($\Leftarrow \boldsymbol{\beta}^T H \boldsymbol{\beta} = \boldsymbol{\rho}_f \cdot \boldsymbol{\alpha}_{\boldsymbol{\beta}}$)

Known results between positive polynomials and S.O.S in $\mathbb{R}^d$

1. $d = 1$: every positive polynomial is a SOS
2. $d = 2$: true for polynomial of degree $n \leq 4$ but not always if $n \geq 6$ (Hilbert 1893)
3. $d \geq 3$: true for polynomial of degree $n = 2$ but not always if $n \geq 4$

The first explicit counterexample for non S.O.S polynomial in dimension 2 was only found in 1966 !

Theorem

Artin (1927) Every nonnegative polynomial is a sum of square rational functions.

The Grad space

Every positive polynomial $\boldsymbol{\alpha} \cdot \boldsymbol{m}(v)$ writes as

$$(\boldsymbol{\beta}, 0) \cdot (1, \boldsymbol{v}, \boldsymbol{v} \otimes \boldsymbol{v}, \boldsymbol{v}^2 \boldsymbol{v})$$

with $\boldsymbol{\beta} = (\alpha_0, \alpha_1, \boldsymbol{\alpha}_2)$ and $\alpha_0 \in \mathbb{R}$, $\alpha_1 \in \mathbb{R}^3$, $\boldsymbol{\alpha}_2 \in \mathbb{R}^3 \times \mathbb{R}^3$
$\Rightarrow$ Characterization by S.O.S. in the Gauss space and of realizable moment by the Hankel matrix

Proposition

$\boldsymbol{\rho} = (n, nu, \Pi, \mathcal{Q}) \in \mathcal{R}^+_{Grad}$ *iff* $n > 0$, $\Pi - u \otimes u > 0$.

$$\Pi = \int \boldsymbol{v} \otimes \boldsymbol{v} f \, dv, \quad \mathcal{Q} = \int \boldsymbol{v}^2 \boldsymbol{v} f \, dv.$$

Grad basis

For $f \geq 0$ s.t. $\int_{\mathbb{R}^3} f(1 + |v|^3) < +\infty$, *span* $\left(1, \mathbf{v}, \mathbf{v} \otimes \mathbf{v}, \mathbf{v}^2\mathbf{v}\right)$ is the space generated by

$$\mathbf{a}\left(\mathbf{v}-\mathbf{u}\right) := \left(1, \left(\mathbf{v}-\mathbf{u}\right), \left(\mathbf{v}-\mathbf{u}\right)^2 - 3T, \mathbb{A}\left(\mathbf{v}-\mathbf{u}\right), \mathbf{b}\left(\mathbf{v}-\mathbf{u}\right)\right)$$

$$\boldsymbol{\rho}_f = (n,\, 0,\, 0,\, \overline{\mathbb{P}},\, \boldsymbol{q})$$

where $\overline{\mathbb{P}}$ is the traceless pressure tensor and $\boldsymbol{q}$ is the heat flux

Proposition

$(n,\, 0,\, 0,\, \lambda_{\mathbb{A}}\, \overline{\mathbb{P}},\, \lambda_{\boldsymbol{b}}\, \boldsymbol{q}) \in \mathcal{R}^{+,*}_{Grad}$ $\forall 0 \leq \lambda_{\mathbb{A}} \leq 1$ *and* $\forall \lambda_{\boldsymbol{b}} \in \mathbb{R}$
$\forall -\frac{1}{2} \leq \lambda_{\mathbb{A}} \leq 1,\, \lambda_{\boldsymbol{b}} = 0$ *(Ellipsoidal)*

Remark

The heat flux can take any value

Variational problem

History

Remark

The study of « complex » variational or dual principle in kinetic theorey is quite recent (1990') compared to the study of variational problem in general

- have a look at earlier results
- opens a wide and rich field of research

Rational Extended Thermodynamics and its connection with kinetic and its connections with kinetic theorey (Muller-Ruggeri)

Formulation of duality on a moment closure : [Levermore, JSP, 1996]

Study of the dual expression $\boldsymbol{\rho} \leftrightarrow \boldsymbol{\alpha}$?
[Junk, 1998, 2000], [Schneider, 2004], [Hauck et all, 2008], [Pavan, 2011]

The classical problem

$\boldsymbol{m}(v) = (1, v, v^2, m_3(v) \dots, m_N(v))$ pseudo-Haar basis

$$f \in L^1_{\boldsymbol{m}} \Leftrightarrow \int |f|(1 + |v|^{2p})\, dv < +\infty$$

$$\eta : \ x \longmapsto \begin{cases} x \ln(x) & \text{if } x \geq 0 \\ +\infty & \text{else} \end{cases}$$

Entropy

$$H(g) = \int \eta(g) dv$$

Primal problem

Definition (Entropy density)

(q : dimension of $span\{m_i\}$), for $\boldsymbol{\rho} \in \mathbb{R}^q$, $h : \mathbb{R}^q \to \overline{\mathbb{R}}$

$$h(\boldsymbol{\rho}) = \min_{\int g\mathbf{m}(v)\,dv=\boldsymbol{\rho}} H(g).$$

Definition (Primal problem)

For $\boldsymbol{\rho} \in dom(h)$ find if possible a function G such that

- $\int G\,\mathbf{m}(v)\,dv = \boldsymbol{\rho}$
- $H(G) = h(\boldsymbol{\rho})$

Domain of definition

• If $\boldsymbol{\rho} \in \mathcal{R}_{\mathbf{m}}^{+}$, from Junk's theorem $\exists \Psi \in C_c^\infty(\mathbb{R}^d)$ s.t. $\int \boldsymbol{m}(v)\, \Psi dv = \boldsymbol{\rho}$
$\Rightarrow H(\Psi) < +\infty$

• H is bounded from below (and above) in the set

$$D(\boldsymbol{\rho}) = \{g \geq 0, \int g\mathbf{m}(v)\, dv = \boldsymbol{\rho}, \int g \ln(g)\, dv \leq H(\Psi)\}$$

Proposition

$dom(h) = \mathcal{R}_{\mathbf{m}}^{+}$ i.e. the infimum exists $\forall \boldsymbol{\rho} \in \mathcal{R}_{\mathbf{m}}^{+}$ ($= +\infty$ outside) and h is convex l.s.c. in $\mathbb{R}^q$

Solution of the primal problem

Let g_n be a minimizing sequence in $D(\boldsymbol{\rho})$ i.e. $\lim_{n\to+\infty} H(g_n) = h(\boldsymbol{\rho})$

Dunford-Pettis in $D(\boldsymbol{\rho}) \Rightarrow g_n \rightharpoonup G$ in L^1 but not in $L^1_{\boldsymbol{m}}$

$$\not\Rightarrow \int g_n \boldsymbol{m}(v)\, dv \to \int G\boldsymbol{m}(v)\, dv$$

The infimum may not satisfy $\int G\mathbf{m}(v)\, dv = \boldsymbol{\rho}$

Duality

« Formally »,

$$\frac{\partial}{\partial g}\left(H(g) - \boldsymbol{\alpha}\cdot\Big(\int \boldsymbol{m} g\, dv - \boldsymbol{\rho}\Big)\right) = 0$$

at the infimum, where $\boldsymbol{\alpha}$ = Lagrange multipliers.

$$\Rightarrow \eta'(G) = \boldsymbol{\alpha}\cdot\mathbf{m}(v)$$

$$G = (\eta')^{-1}(\boldsymbol{\alpha}\cdot\mathbf{m}(v)) = (\eta^*)'(\boldsymbol{\alpha}\cdot\boldsymbol{m}(v))$$

Legendre-Fenchel transform : $\eta^*(x^*) = \max_x(x\cdot x^* - h(x)) = \exp(x^*)$

Duality for h

Fenchel-Legendre transform of h

$$h^*(\boldsymbol{\alpha}) = \max_{\boldsymbol{\rho}}(\boldsymbol{\alpha} \cdot \boldsymbol{\rho} - h(\boldsymbol{\rho}))$$

Computation : $h^*(\boldsymbol{\alpha}) = \int \exp(\boldsymbol{\alpha} \cdot \boldsymbol{m}(v))dv$ defined only for $\exp(\boldsymbol{\alpha} \cdot \boldsymbol{m}(v)) \in L^1$

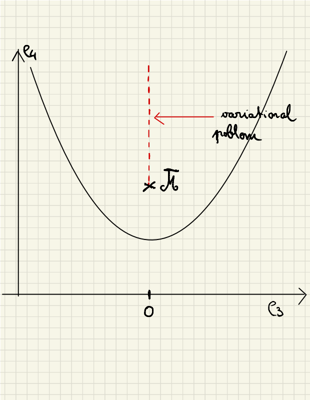
Duality : $\forall \boldsymbol{\alpha}, \boldsymbol{\rho}$ s.t. $h^*(\boldsymbol{\alpha}) + h(\boldsymbol{\rho}) = \boldsymbol{\alpha} \cdot \boldsymbol{\rho}$
$\boldsymbol{\rho} \in \partial h^*(\boldsymbol{\alpha})$ = "slopes of the lines below h*" at $\boldsymbol{\alpha}$ in 1D
and $\boldsymbol{\alpha} \in \partial h(\boldsymbol{\rho})$.
If $\nabla h^*(\boldsymbol{\alpha})$ exists, $\boldsymbol{\rho} = \nabla h^*(\boldsymbol{\alpha})$

Problem : converse statement i.e if $\boldsymbol{\rho} \in \mathcal{R}^+_{\mathbf{m}}$, $\exists \boldsymbol{\alpha}$ such that $\boldsymbol{\rho} = \nabla h^*(\boldsymbol{\alpha})$?

Problem on the boundary

If $dom(h^*) = \Lambda$ and $\Lambda \cap \partial\Lambda \neq \emptyset$, then ∇h^* is not defined on $\Lambda \cap \partial\Lambda \neq \emptyset$.

Figure – Domain of definition 5 moments [Junk, 1998]



Normalized Gaussian $\overline{\mathcal{M}} = exp(\alpha_{\overline{\mathcal{M}}} \cdot \boldsymbol{m}(v))$,

$\partial h^*(\boldsymbol{\alpha}_{\overline{\mathcal{M}}})$ = half-line above $\overline{\mathcal{M}}$

Comparison with classical projection

$$\forall \boldsymbol{\rho} \in \mathcal{R}_{\boldsymbol{m}}^{+,*}, \quad D(\boldsymbol{\rho}) = \left\{ f \geq 0, \int f\, \boldsymbol{m}(v)\, dv = \boldsymbol{\rho} \right\}$$

- $\mathcal{M}$ = Maxwellian such that $\int \mathcal{M}(1, \boldsymbol{v}, \boldsymbol{v}^2) dv = (\rho_0, \rho_1, \rho_2)$
- $\forall f \in D(\boldsymbol{\rho})$ define the distance $d(f, \mathcal{M}) = \int f \ln(\frac{f}{\mathcal{M}})\, dv$ (Csiszar, 1972).

$\Longrightarrow$ I-projection of $\mathcal{M}$ onto $D(\boldsymbol{\rho})$ (Csiszar, 1975)

$$d(G, \mathcal{M}) = \inf_{f \in D(\boldsymbol{\rho})} d(f, \mathcal{M})$$

is not always in $D(\boldsymbol{\rho})$.

Classical entropy is not compatible with moments approach

ϕ divergence

ϕ divergence

« Renormalisation » map of Abdel-Malik and Van Brummelen [2015, *JSP*]

One starts from $(1+\frac{x}{N})^N \to \exp(x)$ and looks for solutions of the form

$$G = \overline{\mathcal{M}}(1+\frac{g}{N})_+^N,$$

$g \in$ $span\{m_0, \dots, m_N\}$, $(x)_+ = \frac{1}{2}(x+|x|)$

$\overline{\mathcal{M}}$: prescribed Gaussian

Remark

1. *polynomial growth* $(1+\frac{g}{N})_+^N$ *instead of exponential.*
2. *measure* $\longrightarrow \overline{\mathcal{M}} dv$

ϕ divergence

Inverse function of $(1+\frac{x}{N})_+^N$: $\widetilde{\ln}(y) = Ny^{1/N} - N$.
H is replaced by

$$H_N = \int \overline{\mathcal{M}} \phi_N(f/\overline{\mathcal{M}}) dv, \quad \text{with} \;\; \phi_N(x) = x\widetilde{\ln}(x)$$

Theorem (Csiszar 1995)

Let ϕ strictly convex, differentiable on $\mathbb{R}^{+,*}$,
$\phi(1) = \phi'(1) = 0$, $\phi'(x) \to +\infty$,
$\phi^*(\boldsymbol{\alpha} \cdot \mathbf{m}(v)) \in L^1(\overline{\mathcal{M}}\, dv)$.
For $\boldsymbol{\rho} \in \mathcal{R}^+_{\mathbf{m}}$, there exists a unique solution to the primal problem

$$\inf_{\int g\mathbf{m}(v)\, dv = \boldsymbol{\rho}} \int \overline{\mathcal{M}} \phi(f/\overline{\mathcal{M}}) dv$$

Moreover, this solution satisfies all constraints.

Comments

1. $\eta(x) = x\ln(x)$ does not satisfy $\phi^*(\boldsymbol{\alpha}\cdot\mathbf{m}(v)) \in L^1(\overline{\mathcal{M}}\,dv)$ for any $\boldsymbol{\alpha}$.
2. Very general but difficult proof
3. No strict proof as concerns the shape of the solution
4. For the above approximation

$$\int \overline{\mathcal{M}}\phi_N(f/\overline{\mathcal{M}})dv \sim \int \overline{\mathcal{M}}(f/\overline{\mathcal{M}})^{1+\frac{1}{N}}dv$$

i.e. weighted $L^{1+\frac{1}{N}}$ space.

Hölder inequality for $g_n \in D(\boldsymbol{\rho}) \Rightarrow$ minimizing sequence $g_n \to G$ in $L^1_{\mathbf{m}}$

General result by convex analysis

Theorem

1. $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *stricly convex and differentiable of* $\mathbb{R}^+$
2. $\phi(0) = 0$, $\lim_{p \to 0} \dfrac{\phi(p)}{p} \in \mathbb{R}$,
3. $\forall \boldsymbol{\alpha}$, $\phi^*(\boldsymbol{\alpha} \cdot \mathbf{m}(v)) \in L^1(\overline{\mathcal{M}}\, dv)$

Then

1. $\forall \boldsymbol{\rho} \in \mathcal{R}_{\boldsymbol{m}}^{+,*}$, $\exists\, !\, \boldsymbol{\alpha}$ *such that* $\nabla h^*(\boldsymbol{\alpha}) = \boldsymbol{\rho}$, $\nabla h(\boldsymbol{\rho}) = \boldsymbol{\alpha}$
2. $G = \overline{\mathcal{M}}(\phi^*)'(\boldsymbol{\alpha} \cdot \mathbf{m}(v))$ *is the unique solution to the primal problem. Moreover* $h(\boldsymbol{\rho})$ *is strictly convex on* $\mathcal{R}_{\boldsymbol{m}}^+$.

Sketch of the proof

1. $dom(h) = \mathcal{R}_{\boldsymbol{m}}^{+}$, with $\mathcal{R}_{\boldsymbol{m}}^{+,*}$ convex and open $\Rightarrow$ h is continuous on $\mathcal{R}_{\boldsymbol{m}}^{+,*}$
2. $h^*(\boldsymbol{\alpha}) = \int \overline{\mathcal{M}} \phi^*(\boldsymbol{\alpha} \cdot \boldsymbol{m}(v))\, dv$ is differentiable in $\mathbb{R}^q$
3. h is continous at $\boldsymbol{\rho} \Rightarrow \exists \boldsymbol{\alpha} \in \partial h(\boldsymbol{\rho}) \neq \emptyset$ (subdifferential at $\boldsymbol{\rho}$) s.t.

$$\Leftrightarrow h(\boldsymbol{\rho}) + h^*(\boldsymbol{\alpha}) = \boldsymbol{\rho} \cdot \boldsymbol{\alpha}$$

$$\Leftrightarrow \boldsymbol{\rho} \in \partial h^*(\boldsymbol{\alpha}) \text{ is reduced to one point } \boldsymbol{\rho} = \nabla h^*(\boldsymbol{\alpha})$$

Remark

The highest degree in $\boldsymbol{m}(v)$ is not necessary even

Method of construction

Step 1 : For $f \geq 0$, $f \in L^1_{Grad}$, consider $\boldsymbol{\rho}_f = \int f\, \boldsymbol{a}(v-u)dv$ and the Maxwellian

$$\mathcal{M}_f = \frac{n}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(v-u)^2}{2T}\right)$$

Step 2 : Consider a given relaxation on $\boldsymbol{\rho}_f$: $L(\boldsymbol{\rho}_f) = (n, 0, 0, \lambda_{\mathbb{A}}\overline{\mathbb{P}}, \lambda_{\boldsymbol{b}}\boldsymbol{q})$

Step 3 : Change $\overline{\mathcal{M}}$ with $\mathcal{M}_f$ for some ϕ divergence Theorem $\Rightarrow \exists\, !\, \boldsymbol{\alpha}$ s.t.

$$G = \mathcal{M}_f(\phi^*)'(\boldsymbol{\alpha} \cdot \boldsymbol{a}(v-u)),$$

with

$$\int G\, \boldsymbol{a}(v-u)\, dv = L(\boldsymbol{\rho}_f), \quad \text{and} \quad H(G) = h(L(\boldsymbol{\rho}_f))$$

Properties

1. $L(\boldsymbol{\rho}_f)$ chosen such as mass, momentum and energy are conserved
2. $G \geq 0$, as soon as ϕ satisfies the assumptions of theorem $\Rightarrow$ $(\phi^*)'(x) \geq 0$ on $\mathbb{R}$.
3. (Extended) H theorem : h is strictly convex in $\mathcal{R}^+_{Grad}$

$$\Longrightarrow (\lambda_{\mathbb{A}}, \lambda_b) \in [0,1]^2 \longrightarrow h(n,\, 0,\, 0,\, \lambda_{\mathbb{A}}\overline{\mathbb{P}},\, \lambda_q q)$$

is strictly convex with a unique minimum in

$$h(n,\, 0,\, 0,\, 0,\, 0) = H(\mathcal{M}_f)$$

• $\int \mathcal{M}_f \phi(f/\mathcal{M}_f)\, dv$ is not a Lyapunov functional (in general) in inhomogeneous case

Galilean invariance

In few words, $\mathbb{P} = \{1,\ \boldsymbol{v},\ \boldsymbol{v}\otimes\boldsymbol{v},\ |\boldsymbol{v}|^2\boldsymbol{v}\}$ is invariant under the action of

$$T = \begin{cases} \tau_u v = v - u \quad \forall u\,\mathbb{R}^3 \\ \Theta v \quad \forall\Theta \text{ rotation} \end{cases}$$

and

$$H(g(Tv)) = \int \mathcal{M}_f(T(v))\phi(\frac{g(Tv)}{\mathcal{M}_f(T(v))})\,dv = \int \mathcal{M}_f(v)\phi(\frac{g(v)}{\mathcal{M}_f(v)})\,dv$$

$\Rightarrow TG(v) = G(T(v))$

Invariance of the measure in the entropy under Galilean transform

Linearized operator

Theorem

$$\mathcal{L}_{\mathcal{M}}(g) = \nu(\sum_i (1 - \frac{\nu_i}{\nu})\mathbb{P}_{m_i} + \mathbb{P}_{\mathbb{K}} - I_d)(g)$$

$\Rightarrow \mathcal{L}_{\mathcal{M}}$ *is Fredholm, self-adjoint and negative on* $\mathbb{K}^{\perp}$

Consequences : $\mathcal{L}_{\mathcal{M}}^{-1}$ is defined on $\mathbb{K}^{\perp}$ and $\mu_K = \frac{P}{\nu_{\mathbb{A}}}$, $\kappa_K = \frac{5}{2}\frac{P}{\nu_b}$

Multiple choices for the relaxations (i.e on ν) $(1 - \frac{\nu_{\mathbb{A}}}{\nu})$, $(1 - \frac{\nu_b}{\nu})$ as soon as : $\mu_K = \mu$, $\kappa_K = \kappa$, $\frac{\nu_b}{\nu_{\mathbb{A}}} = Pr$

Also one can prove from the properties of $\mathcal{L}_{\mathcal{M}}$ that

$$\int_{\mathbb{R}^3} K(g)\phi(v)dv = 0 \ \forall g \quad \Rightarrow \phi \in \mathbb{K}$$

Examples

BGK models

$\phi(x) = x\ln(x)$,

$$\int_{\mathbb{R}^3} \mathcal{M}\phi(\frac{f}{\mathcal{M}})dv = \int_{\mathbb{R}^3} f\ln(\frac{f}{\mathcal{M}})dv$$

the variational problem well-posed in $\mathbb{P} = \{1, \boldsymbol{v}, \boldsymbol{v}\otimes\boldsymbol{v}\}$ for

$$L(\boldsymbol{\rho}_f) = (n,\ 0,\ 0,\ \lambda_{\mathbb{A}}\overline{\mathbb{P}}), \quad \text{with} \quad -\frac{1}{2} \leq \lambda_{\mathbb{A}} \leq 1$$

$\lambda_{\mathbb{A}} = 0 \Leftrightarrow \nu_{\mathbb{A}} = \nu$: BGK model (1954)

Else : ESBGK model [Holway, 1966] from $Pr = \frac{2}{3}$ to $+\infty$

$\Rightarrow$ ESBGK mono and poly [Andries et al, 2000], [Brull, J.S. 2008, 2009]

Possibility to extend the variational principle to $\nu(c)$ $(c = v - u)$ dependant collision frequency [Bouchut, Perthame, 1993], [Struchtrup, 1997]

Shakhov model, 1968

ϕ divergence for the Shakhov model

$$\phi_{\chi^2}(x) = \frac{1}{2}(x-1)^2, \ \ \forall x \in \mathbb{R}$$

$$(\phi^*_{\chi^2})'(x^*) = 1 + x^* \ \ \forall x^* \in \mathbb{R}$$

Minimization « problem » : projection of $\frac{f}{\mathcal{M}} - 1$ on Grad space in $\mathcal{L}^2(\mathcal{M})$
In the Grad space, the choice

$$\nu_{\mathbb{A}} = \nu = \frac{P}{\mu} \quad \text{and} \quad \nu_{\boldsymbol{b}} = \nu Pr$$

$$\Rightarrow G_S = \mathcal{M}\left(1 + \frac{1-Pr}{\rho T}(v-u)\cdot\overrightarrow{q}\left(\frac{(v-u)^2}{T} - 5\right)\right)$$

Remarks on Shakhov model

1. Contrarily to ESBGK model

$$\int_{\mathbb{R}^3} G_s\, \boldsymbol{a}(v-u)\, dv = \frac{1}{\nu} \int_{\mathbb{R}^3} Q^+(f,f)\, \boldsymbol{a}(v-u)\, dv$$

   for Maxwellian molecules.

2. H Theorem $\forall Pr > 0$

3. $\boldsymbol{\rho}_{Gs} \in \mathcal{R}^+_{Grad} \Rightarrow$ this may bring some stability (at least close to a given $\overline{\mathcal{M}}$)

Polynomial approximation of exp

$$\phi_N(x) = Nx^{1/N} - N$$

$$\forall \boldsymbol{\rho}_f \in \mathcal{R}^+_{Grad},\ \exists! \boldsymbol{\alpha}\ s.t.\ G = \mathcal{M}(1 + \frac{\alpha \cdot \mathbf{a}(v-u) - 1}{N})^N_+$$
$$H(G) = H(L(\boldsymbol{\rho}_f)), \quad \int G\, \mathbf{a}(v-u)\, dv = L(\boldsymbol{\rho}_f)$$

Well-posed operator not restricted to polynomial space with even higher order rank

Remark

$\int f \ln f$ is almost a Lyapunov functional for the inhomogeneous equation since

$$\int R(f)\, \widetilde{\ln}(f/\mathcal{M}) dv \approx \int R(f)\, \ln(f/\mathcal{M}) dv$$

The Levermore operator

Let $\mathbb{M} = \mathbb{M}_1 \subset \mathbb{M}_2 \subset \ldots \subset \mathbb{M}_N$ and $0 < \nu_1 < \nu_2 < \ldots < \nu_N$ and

$$\mathcal{M}_k = Argmin\left\{\int g\ln(g) \;/\; \int g m_i(v)\,dv = \int f m_i(v)\,dv,\ \forall m_i \in \mathbb{M}_k\right\}$$

$$K(f) = \nu_1(\mathcal{M} - f) + \sum_{k=2}^{N} (\nu_k - \nu_{k-1})(\mathcal{M}_k - f)$$

Then

$$(\forall \phi \in \mathbb{M}_k \backslash \mathbb{M}_{k-1}), \quad \mathcal{L}_{\mathcal{M}}(m) = -\nu_{k-1} m(v)$$

$\Leftrightarrow$ Grad's relaxation

Problems and questions

1. Variational problem may be not well posed
2. $\mathbb{M} = \mathbb{K} +$ Gauss space : $\nu_1 < \nu_2$ does not give the right Prandtl number. (Mieussens Phd Thesis 1999)
3. If $\phi \in \mathbb{M}_k \backslash \mathbb{M}_{k-1}$, $(k \geq 2)$

$$\int K(f)\phi(v)dv \neq -\nu_{k-1} \int f\phi dv$$

   $((\phi_i)$ orthogonal in $L^2(\mathcal{M})$, not in $L^2(\mathcal{M}_{k-1}))$

4. however

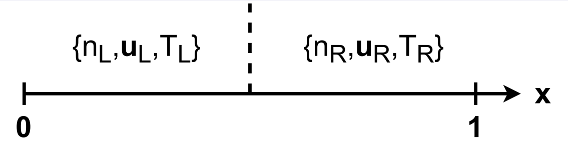
$$\mathcal{L}_{\mathcal{M}}(g) = \nu_N\Big(\sum_{k=1}^{N}(1 - \frac{\nu_{k-1}}{\nu})\mathbb{P}_{\mathbb{M}_k \backslash \mathbb{M}_{k-1}} + \mathbb{P}_{\mathbb{K}} - Id\Big)(g)$$

Stationary shock wave

Stationary normal shock wave - 1

Purpose : the 1D domain is divided in 2 regions, with different gas states, and let the gas relaxed to the stationary state.

Domain initialization



Boundary conditions

- Left at 0 : $\{n_L, u_L, T_L\}$
- Right at 1 : $\{n_R, u_R, T_R\}$

Stationary normal shock wave - 2

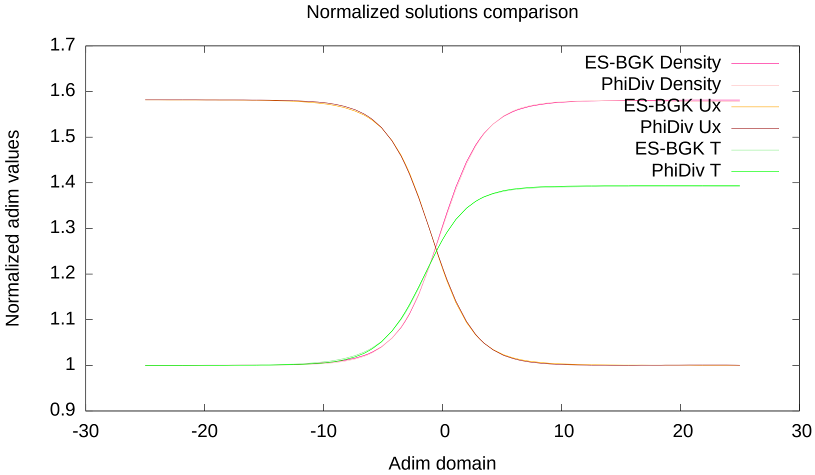
Values :

Mach	Boundary	n	u	T
1.4	left	1	1.278	1
	right	1.581	0.808	1.392

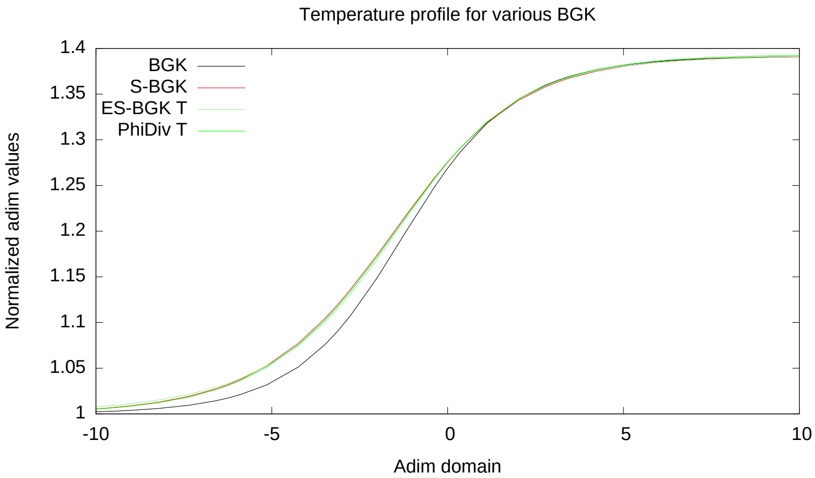
Code characteristics

- DVM on top of a Discontinuous Galerkin Advection solver
- 1D physical, 3D molecular velocities
- BGK model : BGK, S-BGK, ES-BGK, phi-div

Stationary normal shock wave - Results



Stationary normal shock wave - Results



Existing results and opened problems

Existing results

- Fick matrix gas mixtures : [Brull, Pavan, J.S., 2012]
- ESBGK models for mono and polyatomic gas mixtures : viscosity (and shear viscosity), heat conductivity : [Brull,2015, 2021]
- Polyatomic reacting gases, discrete energy, Fick matrix [S. Brull, J.S., 2014], [J.S. 2015]
- Fick matrix poly (and mono) gas mixtures : 2 viscosities and Fick matrix (see talk of K. Guillon)

Existence theorem

- BGK [Perthame 1989], [Perthame-Pulvirenti 1993], . . .
- The many results on ESBGK : Yun et all. Pb $Pr = 2/3$?
- Shakhov near global equilibrium (see talk of Gi-Chan Bae)

Perspectives and open problems

- Numerical comparisons between the different relaxation models Developpement of a code (in progress)
- Relaxation model for multispecies (mono and poly) leading to the full set of transport coefficients (Phenomenological or Onsager matrix)
- Can we construct a single operator based on general Grad's relaxation $L(\boldsymbol{\rho}) \in \mathcal{R}^{+}_{\boldsymbol{m}}$?
- Existence theorem of solutions based on relaxation and variational principles.

THANKS FOR YOUR ATTENTION !