A mean-field limit of the Cucker-Smale model on complete Riemannian manifolds

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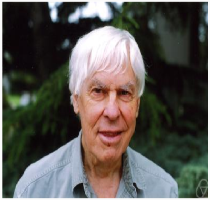
What is Flocking?

A collective behavior that can be observed in birds flying in group.

https://www.naturettl.com/photograph-flocks-birds

Cucker-Smale model



Cucker-Smale model (IEEE Automatic Control 2007): Each bird adjusts its velocity by a weighted sum of relative velocities:

$$\begin{cases} \dot{x}_i = v_i, \quad i = 1, \cdots, N, \\ \dot{v}_i = \sum_{j=1}^{N} \phi(\|x_i - x_j\|)(v_j - v_i), \\ (x_i^{in}, v_i^{in}) \in \mathbb{R}^d \times \mathbb{R}^d, \quad i = 1, \cdots, N. \end{cases} \tag{1}$$

Cucker-Smale model on Riemannian Manifolds

We assume $(\mathcal{G})$: for given (complete) Riemannian manifold $(\mathcal{M}, g)$, there are smooth maps ϕ, G such that

- $(x, v, y, w) \mapsto (x, G(x, v, y, w))$ is a smooth map from $T\mathcal{M} \times T\mathcal{M}$ to $T\mathcal{M}$,
- $\phi(\cdot, \cdot)$ is symmetric, nonnegative, bounded and smooth,
- $G(x, v, y, w) = \phi(x, y)(P_{xy}w - v)$ whenever there is a unique length minimizing geodesic between x and y,
- Otherwise, $G(x, v, y, w) = 0$ and $\phi(x, y) = 0$,

$\Rightarrow$ Cucker-Smale model on $(\mathcal{M}, g)$:

$$\begin{cases} \dot{x}_i = v_i, \quad i = 1, \cdots, N, \\ \nabla_{\dot{x}_i} v_i = \displaystyle\sum_{j=1}^{N} a_{ij} G(x_i, v_i, x_j, v_j), \\ (x_i^{in}, v_i^{in}) \in T\mathcal{M}, \quad i = 1, \cdots, N. \end{cases} \tag{2}$$

Example

For the unit d-sphere $\mathbb{S}^d = \left\{\mathbf{x} \in \mathbb{R}^{d+1} : \|\mathbf{x}\|_2^2 = 1\right\}$ isometrically embedded in $\mathbb{R}^{d+1}$, the parallel transport operator $P_{\mathbf{xy}}$ along the length minimizing geodesic

$$t \mapsto \mathbf{x}\cos t + \frac{\mathbf{y} - (\mathbf{x}\cdot\mathbf{y})\mathbf{x}}{\|\mathbf{y} - (\mathbf{x}\cdot\mathbf{y})\mathbf{x}\|_2}\sin t, \quad \mathbf{x} \neq -\mathbf{y},$$

is given by

$$P_{\mathbf{xy}}\mathbf{w} = \mathbf{w} - \frac{\mathbf{w}\cdot\mathbf{x}}{1 + \mathbf{x}\cdot\mathbf{y}}(\mathbf{x} + \mathbf{y}), \quad \forall\ \mathbf{w} \in T_{\mathbf{y}}\mathcal{M}.$$

In this case, we can consider $\phi(\mathbf{x},\mathbf{y}) = (1 + \mathbf{x}\cdot\mathbf{y})^{p+1}\psi(\mathbf{x},\mathbf{y})$ to get

$$G(\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{w}) := (1+\mathbf{x}\cdot\mathbf{y})^p\psi(\mathbf{x},\mathbf{y})\Big[(1+\mathbf{x}\cdot\mathbf{y})(\mathbf{w}-\mathbf{v}) - (\mathbf{w}\cdot\mathbf{x})(\mathbf{x}+\mathbf{y})\Big],$$

where p is a nonnegative integer and ψ is a smooth function.

Cucker-Smale model on Riemannian manifolds

Basic properties

- Dissipation of kinetic energy: if $a_{ij} = a_{ji}$ and $a_{ij} \geq 0$,

$$\frac{d}{dt}\left(\sum_{i=1}^{N} \|v_i\|_{x_i}^2\right) = -\sum_{i,j} a_{ij} \frac{\|G(x_i, v_i, x_j, v_j)\|_{x_i}^2}{\phi(x_i, x_j)} \leq 0.$$

- Monotone decrease of maximal speed: if $a_{ij} \geq 0$,

$$\max_{1\leq i\leq N} \|v_i(t)\|_{x_i(t)} \leq \max_{1\leq i\leq N} \|v_i^{in}\|, \quad t > 0.$$

Formal derivation of kinetic Cucker-Smale model

Liouville's Equation on manifolds

For given smooth manifold $\mathcal{M}$, consider

- ξ: smooth vector field on $\mathcal{M}$.
- ρ_t: time-dependent d-form on $\mathcal{M}$ such that the fraction of representative particles $\mathcal{F}_t(R)$ contained in any phase space region $R \subset \mathcal{M}$ at time t moving by the equation of motion

$$\dot{x} = \xi(x),$$

can be written as an integral of ρ_t in R:

$$\begin{aligned} \mathcal{F}_t(R) &:= \text{ fraction of particles contained in } R \text{ at time } t \\ &= \int_R \rho_t, \quad \forall\ R: \text{ space region}, \quad t \geq 0. \end{aligned} \tag{3}$$

Liouville's Equation on manifolds

Continuity equation for ρ_t:

$$\frac{\partial \rho_t}{\partial t} + \mathcal{L}_\xi \rho_t = 0. \tag{4}$$

If ρ_t can be expressed by the product of distribution function f and a non-vanishing time-independent d-form ω,

$$\rho_t(x) = f(t, x)\omega(x),$$

we can further simplify the continuity equation (4) to

$$\left[\frac{\partial f}{\partial t} + \mathcal{L}_\xi f + f \mathrm{div}_\omega(\xi)\right]\omega = \left[\frac{\partial f}{\partial t} + \mathrm{div}_\omega(f\xi)\right]\omega = 0,$$

which yields the Liouville equation on $\mathcal{M}$:

$$\frac{\partial f}{\partial t} + \mathrm{div}_\omega(f\xi) = 0.$$

Formal Derivation of Kinetic Equation

Then, we rewrite the Cucker-Smale model on $(\mathcal{M}, g)$,

$$\begin{cases} \dot{x}_i = v_i, \quad i = 1, \cdots, N, \\ \nabla_{\dot{x}_i} v_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} G(x_i, v_i, x_j, v_j), \\ (x_i^{in}, v_i^{in}) \in T\mathcal{M}, \quad i = 1, \cdots, N, \end{cases}$$

as a first-order ODE:

$$\dot{z} = \xi(z), \quad z^{in} \in (T\mathcal{M})^N = \underbrace{T\mathcal{M} \times \cdots \times T\mathcal{M}}_{N \text{ times}}.$$

Curves on $T\mathcal{M}$

For $\mathcal{M} = \mathbb{R}^d$, equation

$$\begin{cases} \dot{x} = v, \\ \dot{v} = F(x, v), \end{cases}$$

can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ F(x, v) \end{pmatrix},$$

where we used the canonical identification of tangent spaces and Euclidean spaces:

$$v, F(x, v) \in \mathbb{R}^d \simeq T_x\mathbb{R}^d, \quad \begin{pmatrix} v \\ F(x, v) \end{pmatrix} \in \mathbb{R}^{2d} \simeq T_{(x,v)}T\mathbb{R}^d.$$

Curves on $T\mathcal{M}$

Therefore, we are looking for two linear homomorphisms $L_1, L_2 : T_x\mathcal{M} \to T_{(x,v)}T\mathcal{M}$ which correspond to

$$L_1 : u \mapsto \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad L_2 : w \mapsto \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

Note: push-foward map π_* sends $\frac{d}{dt}(x, v)$ to $\dot{x}$, and

$$L_2(T_x\mathcal{M}) = \left\{ \frac{d}{dt}(x, v) \middle| \dot{x} = 0 \right\} = \mathrm{Ker}(\pi_*).$$

$\Rightarrow$ What is a linear map $\frac{d}{dt}(x, v) \mapsto \nabla_{\dot{x}} v$?

Connection map

Definition (Sasaki '1958, Dombroski '1962)

For given $(x, v) \in T\mathcal{M}$, let V be a totally normal neighborhood of x, and V' be a neighborhood of 0 such that $\exp_x : V' \to V$ is a diffeomorphism. Moreover, let $\tau : \pi^{-1}(V) \to T_x\mathcal{M}$ be a smooth map satisfying

$$\tau(y, u) = P_{xy} u \quad \forall\, (y, u) \in \pi^{-1}(V).$$

Then, a connection map $K_{(x,v)} : T_{(x,v)} T\mathcal{M} \to T_x\mathcal{M}$ is defined as

$$K_{(x,v)}(A) := (\exp_x \circ R_{-v} \circ \tau)_*(A),$$

where $R_{-v} : T_x\mathcal{M} \to T_x\mathcal{M}$ is a translation map $X \mapsto X - v$.

Proposition (Dombroski '1962)

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold, and $z(t) = (x(t), u(t))$ be a smooth curve in $T\mathcal{M}$ through (x_0, u_0) at $t = 0$. Then, the connection map K satisfies

$$K_{(x_0,u_0)}(\dot{z}(0)) = \nabla_{\dot{x}} u(0).$$

Decomposition of $T_{(x,v)}T\mathcal{M}$

Definition (Horizontal and Vertical subspaces)

Let $z = (x, v)$ be a point on $T\mathcal{M}$.

1. (Horizontal subspace of $T_zT\mathcal{M}$): The horizontal subspace $\mathcal{H}_zT\mathcal{M}$ is the kernel of the connection map K_z:

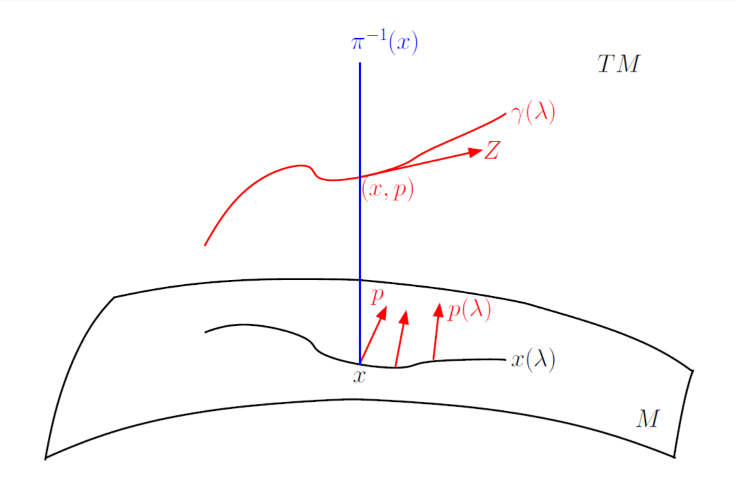
$$\mathcal{H}_zT\mathcal{M} := \mathrm{Ker}(K_z) \ (= L_1(T_x\mathcal{M})).$$

2. (Vertical subspace of $T_zT\mathcal{M}$): The vertical subspace $\mathcal{V}_zT\mathcal{M}$ is the kernel of the linear map $\pi_* : T_zT\mathcal{M} \to T_x\mathcal{M}$:

$$\mathcal{V}_zT\mathcal{M} := \mathrm{Ker}\left(\pi_*\right) \ (= L_2(T_x\mathcal{M})).$$

$$\Longrightarrow T_zT\mathcal{M} = \mathcal{H}_zT\mathcal{M} \oplus \mathcal{V}_zT\mathcal{M}!$$

Horizontal and Vertical lifts



O. Sarbach and T. Zannias.: *The geometry of the tangent bundle and the relativistic kinetic theory of gases*, Classical and Quantum Gravity **31** (2014), 085013.

Definition (Horizontal and Vertical lifts)

For each tangent vector $u \in T_x\mathcal{M}$ and $z = (x, v) \in T\mathcal{M}$, we define the horizontal lift u_z^{hor} and the vertical lift u_z^{ver} as

$$u_z^{hor} := \pi_*|_{\mathcal{H}_z T\mathcal{M}}^{-1}(u), \quad u_z^{ver} := K_z|_{\mathcal{V}_z T\mathcal{M}}^{-1}(u).$$

Proposition ($\mathcal{H}$-$\mathcal{V}$ decomposition of $\dot{z}$)

Let $z = (x, v) : I \to T\mathcal{M}$ be a smooth curve on $T\mathcal{M}$. Then, the tangent vector $\dot{z}$ in $T_z T\mathcal{M}$ can be uniquely written as

$$\dot{z} = \dot{x}_z^{hor} + (\nabla_{\dot{x}} v)_z^{ver}.$$

Cucker-Smale model on manifold: revisited

One can rewrite the Cucker-Smale model (2) as

$$\begin{cases} \dot{z} = \xi(z), \quad z = (z_1, \cdots, z_N), \quad \xi = (\xi_1, \cdots, \xi_N), \\ z_i = (x_i, v_i), \quad \xi_i(z) = (v_i)^{hor}_{z_i} + \left(\dfrac{1}{N} \displaystyle\sum_{j=1}^{N} G(z_i, z_j) \right)^{ver}_{z_i}, \\ z_i^{in} \in T\mathcal{M}, \quad i = 1, \cdots, N, \end{cases} \tag{5}$$

and the corresponding mean-field limit model on $(\mathcal{M}, g)$ is:

$$\begin{aligned} &\partial_t f + \mathrm{div}_{\hat{g}} \left[f \left\{ v_z^{hor} + (F[f](z,t))_z^{ver} \right\} \right] = 0, \\ &F[f](z,t) := \int_{T\mathcal{M}} G(z, z_*) f(z_*, t) \mathrm{Vol}_{\hat{g}}(z_*). \end{aligned} \tag{6}$$

Flocking estimate and Existence of measure-valued solution

Propagation of moments

We set two velocity moments, m_0 and m_2:

$$m_0(t) := \int_{T\mathcal{M}} f(x,v,t)\mathrm{Vol}_{\hat{g}}(x,v),$$

$$m_2(t) := \int_{T\mathcal{M}} \|v\|_x^2 f(x,v,t)\mathrm{Vol}_{\hat{g}}(x,v).$$

Proposition (Propagation of mass and kinetic energy)

Let $f = f(x,v,t)$ be a nonnegative smooth solution to (6), and assume that the support of $f_t(\cdot,\cdot) := f(\cdot,\cdot,t)$ is a compact subset of $T\mathcal{M}$ for all $t \geq 0$. Then, the moments m_0 and m_2 satisfy

$$\begin{aligned} &(i)\ \frac{dm_0(t)}{dt} = 0, \quad \forall\ t > 0. \\ &(ii)\ \frac{dm_2(t)}{dt} = -\iint_{(T\mathcal{M})^2} \frac{\|G(z,z_*)\|_x^2}{\phi(x,x_*)} ff_* \mathrm{Vol}_{\hat{g}}(z_*)\mathrm{Vol}_{\hat{g}}(z). \end{aligned} \tag{7}$$

$$(15): \quad \partial_t f + \mathrm{div}_{\hat{g}} \left[\left\{ v_z^{hor} + (F[f](z,t))_z^{ver} \right\} f \right] = 0.$$

Sketch of Proof:

1. For a compactly supported function φ and vector field U,

$$\int_{\mathcal{M}} \left(g(\mathrm{grad}_g \varphi, U) + \varphi \mathrm{div}_g U \right) \mathrm{Vol}_g = 0.$$

2. From the definition of grad_g, we have

$$g(\mathrm{grad}_g \varphi, U) = d\varphi(U) = U\varphi.$$

3. We multiply $\|v\|_x^2$ to (6), integrate by $\mathrm{Vol}_{\hat{g}}(z) = \mathrm{Vol}_{\hat{g}}(x,v)$ and obtain

$$\begin{aligned} \frac{dm_2}{dt} &= \int_{T\mathcal{M}} \hat{g} \left(\mathrm{grad}_{\hat{g}}(\|v\|_x^2), f \left[v_z^{hor} + (F[f](z,t))_z^{ver} \right] \right) \mathrm{Vol}_{\hat{g}}(z) \\ &= \int_{T\mathcal{M}} f \left[v_z^{hor} + (F[f](z,t))_z^{ver} \right] (\|v\|_x^2) \mathrm{Vol}_{\hat{g}}(z). \end{aligned}$$

$$\frac{dm_2}{dt} = \int_{T\mathcal{M}} f \left[v_z^{hor} + (F[f](z,t))_z^{ver} \right] (\|v\|_x^2) \mathrm{Vol}_{\hat{g}}(z).$$

- v_z^{hor}: tangent vector of the curve $z := (x, v)$ on $T\mathcal{M}$ such that $\dot{x} = v, \ \nabla_{\dot{x}} v = 0$. Therefore, v is a parallel vector field along x and $[v_z^{hor}](\|v\|_x^2) = \dot{z}(\|v\|_x^2) = 0$.
- $(F[f](z,t))_z^{ver}$: tangent vector of the curve $z := (x, v)$ on $T\mathcal{M}$ such that $\dot{x} = 0, \ \nabla_{\dot{x}} v = F[f](z,t)$. Therefore, $[(F[f](z,t))_z^{ver}](\|v\|_x^2) = \dot{z}(\|v\|_x^2) = 2g_x(v, F[f](z,t))$.

In fact, in $\mathbb{R}^d$, the above argument becomes

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{2d}} \|v\|^2 \left(\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (F[f]f) \right) dxdv \\ &= \frac{dm_2}{dt} - 2 \int_{\mathbb{R}^{2d}} v \cdot (F[f]f) dxdv. \end{aligned}$$

Propagation of moments

Therefore, by using the relation

$$\begin{aligned} & g(v, G(z, z_*)) + g(v_*, G(z_*, z)) \\ & = \phi(x, x_*) \left(g(P_{xx_*} v_*, v) - \|v\|_x^2 + g(P_{x_* x} v, v_*) - \|v_*\|_{x_*}^2 \right) \\ & = -\phi(x, x_*) \|P_{xx_*} v_* - v\|_x^2 = -\frac{\|G(z, z_*)\|_x^2}{\phi(x, x_*)}, \end{aligned}$$

we have the desired estimate:

$$\begin{aligned} \frac{dm_2}{dt} & = 2 \int_{T\mathcal{M}} g(v, F[f](z, t)) f(z) \mathrm{Vol}_{\hat{g}}(z) \\ & = 2 \int_{T\mathcal{M}} \int_{T\mathcal{M}} g(v, G(z, z_*)) f(z_*, t) f(z, t) \mathrm{Vol}_{\hat{g}}(z_*) \mathrm{Vol}_{\hat{g}}(z) \\ & = -\int_{T\mathcal{M}} \int_{T\mathcal{M}} \frac{\|G(z, z_*)\|_x^2}{\phi(x, x_*)} f(z, t) f(z_*, t) \mathrm{Vol}_{\hat{g}}(z_*) \mathrm{Vol}_{\hat{g}}(z). \end{aligned}$$

Measure-Valued solution

If $f \in C^\infty(T\mathcal{M} \times [0, T))$ satisfies (6):

$$\partial_t f + \mathrm{div}_{\hat{g}} \left[\left\{ v_z^{hor} + (F[f](z,t))_z^{ver} \right\} f \right] = 0,$$

we multiply an arbitrary test function $h \in C_0^1(T\mathcal{M} \times [0, T))$ to (6), integrate by $\mathrm{Vol}_{\hat{g}}$ and obtain

$$\begin{aligned} &\frac{d}{dt} \int_{T\mathcal{M}} h(f\mathrm{Vol}_{\hat{g}}) - \int_{T\mathcal{M}} \partial_t h(f\mathrm{Vol}_{\hat{g}}) \\ &= \int_{T\mathcal{M}} \hat{g} \left(\mathrm{grad}_{\hat{g}} h, v_z^{hor} + (F[f](z,t))_z^{ver} \right) (f\mathrm{Vol}_{\hat{g}}) \\ &= \int_{T\mathcal{M}} \left[v_z^{hor}(h) + (F[f](z,t))_z^{ver}(h) \right] (f\mathrm{Vol}_{\hat{g}}). \end{aligned}$$

In addition, $F[f](z,t) = \int_{T\mathcal{M}} G(z, z_*)(f\mathrm{Vol}_{\hat{g}}(z_*))$.

Measure-Valued solution

For every Hausdorff topological space X, let $\mathfrak{M}(X)$ be the set of nonnegative Radon measures on X. For a Radon measure $\nu \in \mathfrak{M}(T\mathcal{M})$, we use the following standard duality notation:

$$\langle \nu, h \rangle := \int_{T\mathcal{M}} h(z) d\nu(z), \quad h \in C_0(T\mathcal{M}),$$

where $C_0(T\mathcal{M})$ is the set of all continuous function f on $T\mathcal{M}$ such that $\overline{\{f > \varepsilon\}}$ is compact for every positive number ε.

Measure-Valued solution: Definition

Definition (measure-valued solution)

For $T > 0$, let $\mu : L^\infty([0,T); \mathfrak{M}(T\mathcal{M}))$ be a measure-valued solution to (6) with the initial Radon measure $\mu_0 \in \mathfrak{M}(T\mathcal{M})$ if and only if μ satisfies the following conditions:

1. μ is weakly continuous:

$$t \mapsto \langle \mu_t, h \rangle \quad \text{is continuous,} \quad \forall\ h \in C_0(T\mathcal{M}).$$

2. For any $h \in C_0^1(T\mathcal{M} \times [0,T))$,

$$\begin{aligned}&\langle \mu_t, h(\cdot,t) \rangle - \langle \mu_0, h(\cdot,0) \rangle \\ &\quad = \int_0^t \Big\langle \mu_s, \partial_s h + v_z^{hor}(h) + (F[\mu](z,s))_z^{ver}(h) \Big\rangle \, ds,\end{aligned}$$

where $F[\mu](z,t) := \int_{T\mathcal{M}} G(z,z_*) d\mu_t(z_*) \in T_{\pi(z)}\mathcal{M}$.

Measure-Valued solution

1. The function $f \in L^1(T\mathcal{M} \times [0, T))$ is a distributional weak solution to (6) if and only if $\mu_t(z) := f(z,t)|\mathrm{Vol}_{\hat{g}}(z)|$ is a measure-valued solution to (6).

2. For any solution $\{(x_i, v_i)\}_{i=1}^N$ to the following ODE system:

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad 1 \le i \le N, \\ \nabla_{\dot{x}_i} v_i = \sum_{j=1}^N m_j G(x_i, v_i, x_j, v_j), \\ (x_i(0), v_i(0)) = (x_i^{in}, v_i^{in}) \in T\mathcal{M}, \end{cases}$$

   the empirical measure $\mu_t := \sum_{i=1}^N m_i \delta_{(x_i(t), v_i(t))}$ is a measure-valued solution to (6).

Measure-Valued solution: propagation of moments

For a measure-valued solution $(\mu_t)_{0\leq t<T}$ to (6), we set

$$m_0(t) := \int_{T\mathcal{M}} d\mu_t(x,v),$$
$$m_2(t) := \int_{T\mathcal{M}} \|v\|_x^2 d\mu_t(x,v).$$

Then, similar to (7), the moments m_0 and m_2 satisfy

$$(i)\ \frac{dm_0(t)}{dt} = 0, \quad \forall\ t > 0.$$
$$(ii)\ \frac{dm_2(t)}{dt} = -\iint_{(T\mathcal{M})^2} \frac{\|G(z,z_*)\|_x^2}{\phi(x,x_*)} d\mu_t(z_*) d\mu_t(z),$$

provided that μ_t is compactly supported for each t.

Existence of measure-valued solution

Question: When does the measure-valued solution exists?

Ha-Liu (2009): Existence of solution $(\mu_t)_{0 \le t < T}$ for $\mathcal{M} = \mathbb{R}^d$.

Existence of measure-valued solution in $\mathbb{R}^d$

1. For given compactly supported radon probability measure $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$, we approximate μ_0 by $\mu_0^h = \sum_{i=1}^N m_i \delta_{(x_i^{in}, v_i^{in})}$, where m_i is the μ_0-measure of i-th cube $R^h(i)$ with width h in $\mathbb{R}^{2d}$.

2. Set $\mu_t^h := \sum_{i=1}^N m_i \delta_{(x_i(t), v_i(t))}$, where $\{(x_i(t), v_i(t))\}$ is the solution of

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \ \ 1 \le i \le N, \\ \dot{v}_i = \displaystyle\sum_{j=1}^N m_j G(x_i, v_i, x_j, v_j), \\ (x_i(0), v_i(0)) = (x_i^{in}, v_i^{in}) \in \mathbb{R}^{2d}. \end{cases}$$

Existence of measure-valued solution in $\mathbb{R}^d$

3. For every compactly supported measure valued solutions $(\mu_t)_{0 \le t < T}, (\nu_t)_{0 \le t < T}$, we show

$$W_1(\mu_t, \nu_t) \le C(T) W_1(\mu_0, \nu_0), \quad 0 \le t < T.$$

4. As $h \to 0$, the measure μ_0^h approaches to μ_0 in W_1-distance, and therefore $(\mu_t^h)_{h>0}$ is a Cauchy net in a Polish space $W_1(\mathbb{R}^{2d})$. We denote the limit by μ_t.

5. Since $W_1(\mu_t^h, \mu_t) \to 0$, one can easily verify that $(\mu_t)_{0 \le t < T}$ is the desired measure-valued solution.

Sasaki metric

Definition (Sasaki '1958)

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold without boundary. Then, the Sasaki metric tensor field $\hat{g}$ on $T\mathcal{M}$ is the unique nondegenerate symmetric bilinear form such that for every $h_1, h_2, v_1, v_2 \in T_x\mathcal{M}$ and $z = (x, v) \in T\mathcal{M}$,

$$\hat{g}_z\left((h_1)_z^{hor} + (v_1)_z^{ver}, (h_2)_z^{hor} + (v_2)_z^{ver}\right) = g_x(h_1, h_2) + g_x(v_1, v_2).$$

Sasaki metric

Proposition (Sasaki metric distance)

For a given Riemannian manifold $(\mathcal{M}, g)$, the tangent bundle $T\mathcal{M}$ is also a Riemannian manifold for the Sasaki metric tensor $\hat{g}$. Moreover, if d and $\hat{d}$ are the geodesic distances on $(\mathcal{M}, g)$ and $(T\mathcal{M}, \hat{g})$, respectively, then for every $(x_1, v_1), (x_2, v_2) \in T\mathcal{M}$, we have

$$d(x_1, x_2) \leq \hat{d}\left((x_1, v_1), (x_2, v_2)\right) \leq \sqrt{d(x_1, x_2)^2 + \|P_{x_1 x_2} v_2 - v_1\|_{x_1}^2}.$$

Here, $P_{x_1 x_2}$ is not necessarily unique.

Existence of measure-valued solution

$$3 : W_1(\mu_t, \nu_t) \leq C(T) W_1(\mu_0, \nu_0), \quad 0 \leq t < T,$$

Key estimate in 3: For each measure-valued solution μ, consider a particle trajectory $(X_\mu(s), V_\mu(s)) := (X_\mu(s; t, x, v), V_\mu(s; t, x, v))$ satisfying

$$\begin{aligned} &\dot{X}_\mu(s) = V_\mu(s), \\ &\nabla_{\dot{X}_\mu} V_\mu(s) = F[\mu](X_\mu(s), V_\mu(s), s), \\ &X_\mu(t) = x, \quad V_\mu(t) = v. \end{aligned}$$

Existence of measure-valued solution

Consider a test function $h \in C_0(T\mathcal{M})$ with Lipschitz constant ≤ 1:

$$\frac{|h(z_1) - h(z_2)|}{\hat{d}(z_1, z_2)} \leq 1, \quad \forall\, z_1, z_2 \in T\mathcal{M}.$$

Then, we have

$$\begin{aligned}
&\left| \int_{T\mathcal{M}} h(z) d\mu_t(z) - \int_{T\mathcal{M}} h(z) d\nu_t(z) \right| \\
&= \left| \int_{T\mathcal{M}} h(X_\mu(t;0,z), V_\mu(t;0,z)) d\mu_0(z) - \int_{T\mathcal{M}} h(X_\nu(t;0,z), V_\nu(t;0,z)) d\nu_0(z) \right| \\
&\leq \int_{T\mathcal{M}} \Big| h(X_\mu(t;0,z), V_\mu(t;0,z)) - h(X_\nu(t;0,z), V_\nu(t;0,z)) \Big| d\mu_0(z) \\
&\quad + \left| \int_{T\mathcal{M}} h(X_\nu(t;0,z), V_\nu(t;0,z)) d(\mu_0 - \nu_0)(z) \right| \\
&\leq \hat{d}\Big((X_\mu(t;0,z), V_\mu(t;0,z)), (X_\nu(t;0,z), V_\nu(t;0,z)) \Big) + \mathrm{Lip}(X_\nu, V_\nu) W_1(\mu_0, \nu_0),
\end{aligned}$$

where $\mathrm{Lip}(X_\nu, V_\nu)$ denotes the Lipschitz constant of the map:

$$z \mapsto (X_\nu(t;0,z), V_\nu(t;0,z))$$

for all points z in $\in$ supp μ_0 and $0 \leq t \leq T$.

Existence of measure-valued solution

$$d(x_1, x_2) \le \hat{d}\left((x_1, v_1), (x_2, v_2)\right) \le \sqrt{d(x_1, x_2)^2 + \|P_{x_1 x_2} v_2 - v_1\|_{x_1}^2}.$$

For a manifold $T\mathcal{M}$ endowed with Sasaki metric, we estimate

$$\tilde{d}(x, v, y, w) = \sqrt{d(x, y)^2 + \|P_{xy} w - v\|_x^2}, \quad \forall\, (x, v), (y, w) \in T\mathcal{M}.$$

instead of $\hat{d}$, and we need the following Grönwall-type inequality:

$$\begin{aligned} &\frac{d}{ds}\left(\tilde{d}(X_\mu(s), V_\mu(s), X_\nu(s), V_\nu(s))^2\right) \\ &\quad \lesssim W_1(\mu_s, \nu_s) + \left(\tilde{d}(X_\mu(s), V_\mu(s), X_\nu(s), V_\nu(s))^2\right). \end{aligned}$$

Existence of measure-valued solution

For the Lipschitz constant of the map

$$z \mapsto (X_\mu(s;t,z), V_\mu(s;t,z)),$$

we need the following Grönwall-type inequality: for two characteristic curves (X_μ^1, V_μ^1) and (X_μ^2, V_μ^2) given by the relations

$$X_\mu^1(s) := X_\mu(s;0,x,v), \quad V_\mu^1 = \dot{X}_\mu^1$$
$$X_\mu^2(s) := X_\mu(s;0,y,u), \quad V_\mu^2 = \dot{X}_\mu^2$$

we have

$$\frac{d}{ds}\tilde{d}(X_\mu^1, V_\mu^1, X_\mu^2, V_\mu^2) \lesssim \tilde{d}(X_\mu^1, V_\mu^1, X_\mu^2, V_\mu^2).$$

Standing assumptions

◇ **Standing Assumptions** $(\mathcal{A})$:

- $(\mathcal{A}1)$: The mappings ϕ and G satisfy $(\mathcal{G})$.
- $(\mathcal{A}2)$: $\phi(x,x) > 0$ for every $x \in \mathcal{M}$.
- $(\mathcal{A}3)$: For every compact set $\mathcal{K} \subset \mathcal{M}$ and $x, y, z \in \mathcal{K}$, we have

$$\|id - P_{xz}P_{zy}P_{yx}\| \leq C(\mathcal{K})d(y,z),$$

  where $\|\cdot\|$ is the operator norm.
- $(\mathcal{A}4)$: If x_1 and x_2 are smooth curves on $\mathcal{M}$, whose speeds are uniformly bounded by a constant c, then we have

$$\frac{d}{dt}\left(\tilde{d}(x_1,\dot{x}_1,x_2,\dot{x}_2)^2\right)$$
$$\leq C(c)\left(\tilde{d}(x_1,\dot{x}_1,x_2,\dot{x}_2)^2 + g\left(P_{x_1x_2}\dot{x}_2 - \dot{x}_1, P_{x_1x_2}\nabla_{\dot{x}_2}\dot{x}_2 - \nabla_{\dot{x}_1}\dot{x}_1\right)\right).$$

Standing assumptions

If $\mathcal{M} = \mathbb{R}^d$, we have

$$P_{xy} = \mathrm{id}, \quad \nabla_{\dot{x}}\dot{x} = \ddot{x}, \quad \tilde{d}(x_1, \dot{x}_1, x_2, \dot{x}_2)^2 = \|x_1 - x_2\|^2 + \|\dot{x}_1 - \dot{x}_2\|^2,$$

which imply

$$\|id - P_{xz}P_{zy}P_{yx}\| = 0, \quad \forall\, x, y, z \in \mathbb{R}^d,$$

and

$$\begin{aligned}
&\frac{d}{dt}\left(\tilde{d}(x_1, \dot{x}_1, x_2, \dot{x}_2)^2\right) \\
&\qquad = 2(x_1 - x_2)\cdot(\dot{x}_1 - \dot{x}_2) + 2(\dot{x}_1 - \dot{x}_2)\cdot(\ddot{x}_1 - \ddot{x}_2) \\
&\qquad \le \tilde{d}(x_1, \dot{x}_1, x_2, \dot{x}_2)^2 + 2(\dot{x}_1 - \dot{x}_2)\cdot(\ddot{x}_1 - \ddot{x}_2).
\end{aligned}$$

Standing assumptions

If $\mathcal{M} = \mathbb{H}^d$, one can show:

$$\|P_{xz}P_{zy}P_{yx}v - v\|_x \leq 2\|v\|_x \tanh\frac{d(z,x)}{2}\tanh\frac{d(x,y)}{2}, \quad \forall\ v \in T_x\mathbb{H}^d,$$

and

$$\frac{d}{dt}\left(\tilde{d}(x_1,\dot{x}_1,x_2,\dot{x}_2)^2\right)$$
$$\leq (2c^2+1)\left(\tilde{d}(x_1,\dot{x}_1,x_2,\dot{x}_2)^2 + g\left(P_{x_1x_2}\dot{x}_2 - \dot{x}_1, P_{x_1x_2}\nabla_{\dot{x}_2}\dot{x}_2 - \nabla_{\dot{x}_1}\dot{x}_1\right)\right).$$

Moreover, P_{xy} uniquely exists for all $x, y \in \mathbb{H}^d$, and therefore ϕ, G satisfying $(\mathcal{A}1) - (\mathcal{A}2)$ can be found easily, for instance $\phi \equiv 1$.

Existence of measure-valued solution

If (X^1_μ, V^1_μ) and (X^2_μ, V^2_μ) are two characteristic curves with same measure-valued solution μ and different initial data,

$$\begin{aligned} & P_{X^1_\mu X^2_\mu} \nabla_{\dot{X}^2_\mu} V^2_\mu - \nabla_{\dot{X}^1_\mu} V^1_\mu \\ &= P_{X^1_\mu X^2_\mu} F[\mu](X^2_\mu, V^2_\mu, s) - F[\mu](X^1_\mu, V^1_\mu, s) \\ &= \int_{T\mathcal{M}} \left(P_{X^1_\mu X^2_\mu} G(X^2_\mu, V^2_\mu, z_*) - G(X^1_\mu, V^1_\mu, z_*) \right) d\mu_s(z_*). \end{aligned}$$

Then, the condition $(\mathcal{A}3)$,

$$\|P_{xy} P_{yz_*} v_* - P_{xz_*} v_*\| = \|P_{z_* x} P_{xy} P_{yz_*} v_* - v_*\| = O(\|v_*\| d(x, y)),$$

can be used to show

$$\begin{aligned} & \left\| P_{X^1_\mu X^2_\mu} (P_{X^2_\mu x_*} v_* - V^2_\mu) - (P_{X^1_\mu x_*} v_* - V^1_\mu) \right\|_{X^1_\mu} \\ & \leq \|P_{X^1_\mu X^2_\mu} V^2_\mu - V^1_\mu\|_{X^1_\mu} + O(\|v_*\| d(x, y)) \\ & \leq O(\tilde{d}(X^1_\mu, V^1_\mu, X^2_\mu, V^2_\mu)). \end{aligned}$$

Existence of measure-valued solution

If (X_μ, V_μ) and (X_ν, V_ν) are two characteristic curves with same initial data,

$$\begin{aligned}
&P_{X_\mu X_\nu} \nabla_{\dot{X}_\nu} V_\nu - \nabla_{\dot{X}_\mu} V_\mu \\
&= P_{X_\mu X_\nu} F[\nu](X_\nu, V_\nu, s) - F[\mu](X_\mu, V_\mu, s) \\
&= \int_{T\mathcal{M}} P_{X_\mu X_\nu} G(X_\nu, V_\nu, z_*) d\nu_s(z_*) - \int_{T\mathcal{M}} G(X_\mu, V_\mu, z_*) d\mu_s(z_*) \\
&= \int_{T\mathcal{M}} \left(P_{X_\mu X_\nu} G(X_\nu, V_\nu, z_*) - G(X_\mu, V_\mu, z_*) \right) d\nu_s(z_*) \\
&\qquad + \int_{T\mathcal{M}} G(X_\mu, V_\mu, z_*) d(\nu_s - \mu_s)(z_*).
\end{aligned}$$

Then, the condition $(\mathcal{A}3)$,

$$\|P_{xy} P_{yz_*} v_* - P_{xz_*} v_*\| = \|P_{z_* x} P_{xy} P_{yz_*} v_* - v_*\| = O(\|v_*\| d(x, y)),$$

and the (local) Lipschitz continuity of

$$z \mapsto (X_\mu, V_\mu)(s; t, z), \quad z \mapsto G(z, z_*)$$

can be used to obtain

$$\frac{d}{ds} \left(\tilde{d}(Z_\mu(s), Z_\nu(s))^2 \right) \lesssim W_1(\mu_s, \nu_s) + \left(\tilde{d}(Z_\mu(s), Z_\nu(s))^2 \right).$$

Main Theorem: Existence of measure-valued solution

Theorem (Ahn-Ha-Kim-Schlöder-S, Submitted)

Suppose that the conditions $(\mathcal{A})$ hold, and let μ_0 be a Radon probability measure with compact support in $T\mathcal{M}$. Then, there exists a unique probability measure-valued solution $(\mu_t)_{0\leq t<T}$ in $L^\infty([0,T);\mathcal{P}(T\mathcal{M}))$ with the initial data μ_0 such that μ is weakly Lipschitz continuous and has compact support for each time slice.

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Thank you!