Ellipsoidal BGK model with the correct Prandtl number

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Kinetic and fluid equations for collective behavior
France-Korea IRL in Mathematics

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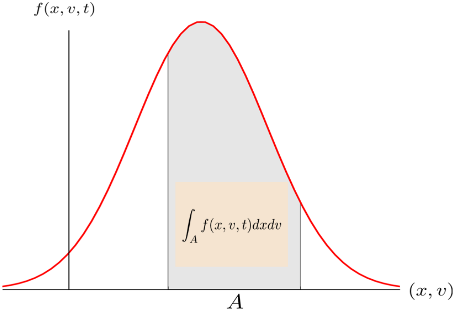
Boltzmann equation

Velocity distribution function

- Given a particle system: gas, plasma,...
- Maxwell(1860), Boltzmann(1872) : How particles are distributed in the phase space?
- $\int_A f(x, v, t)dxdv = \#$ of particles such that $(x, v) \in A$ at time t

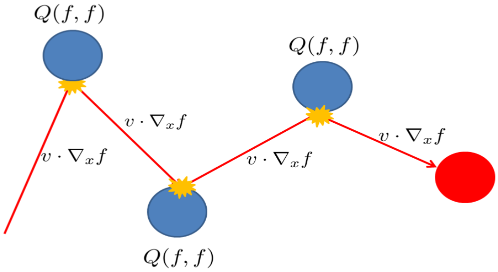


The Boltzmann equation

- For non-ionized monatomic rarefied gas (1872):

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

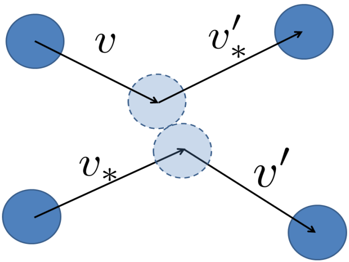
- Transport+collision



Collision Operator

$$Q(f,f)(v) \equiv \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} B(v - v_*, \omega)(f(v')f(v'_*) - f(v)f(v_*))d\omega dv_*.$$

$$v' = v - [(v - v_*) \cdot \omega]\omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega.$$



Q satisfies

- Q satisfies

$$\int_{\mathbb{R}^3} Q(f,f)(1,v,|v|^2)dv = 0$$

and

$$\int_{\mathbb{R}^3} Q(f,f)\ln f dv \le 0$$

which respectively lead to

- Conservation laws

$$\frac{d}{dt}\int_{\mathbb{R}^3} f(x,v,t)(1,v,|v|^2)dxdv = 0.$$

- and H-theorem

$$\frac{d}{dt}\int_{\mathbb{R}^6} f\ln f\, dxdv = \int_{\mathbb{R}^3} Q(f,f)\ln f dv \leq 0$$

Equilibrium: local Maxwellian

- Equilibrium

$$
\begin{aligned}
&Q(f,f)=0\\
&\Leftrightarrow \int_{\mathbb{R}^3} Q(f,f)\ln f dv = 0\\
&\Leftrightarrow \ln f + \ln f_* - \ln f' - \ln f'_* = 0\\
&\Leftrightarrow \ln f = \lambda_1|v|^2 + \lambda_2\cdot v + \lambda_3.
\end{aligned}
$$

- (local) Maxwellian:

$$f = e^{\lambda_1|v|^2+\lambda_2\cdot v+\lambda_3}.$$

Local Maxwellian

- Equilibrium

$$Q(\mathcal{M},\mathcal{M})=0$$

- Due to the conservation laws, we get

$$\mathcal{M}(f)(x,v,t)=\frac{\rho(x,t)}{\sqrt{(2\pi T(x,t))^3}}\exp\Big(-\frac{|v-U(x,t)|^2}{2T(x,t)}\Big).$$

where

$$\begin{aligned}\rho(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)dv\\ \rho(x,t)U(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)vdv\\ \rho(x,t)T(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)|v-U(x,t)|^2dv.\end{aligned}$$

BGK model

BE: fundamental but not practical

- hard to develop fast & efficient numerical methods.

- Most difficulties and costs arise in the computation of Q.

BGK model

The Boltzmann-BGK model

- BGK Model (Bhatnagar-Gross-Krook [1954]):

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\kappa}(\mathcal{M}(f) - f)$$

- $1/\kappa$: collision frequency

- $\mathcal{M}$: Local Maxwellian where

$$\mathcal{M}(f)(x,v,t)=\frac{\rho(x,t)}{\sqrt{(2\pi T(x,t))^3}}\exp\Big(-\frac{|v-U(x,t)|^2}{2T(x,t)}\Big).$$

where

$$\begin{aligned}\rho(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)dv\\ \rho(x,t)U(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)vdv\\ \rho(x,t)T(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)|v-U(x,t)|^2dv.\end{aligned}$$

- Collision process of BE $\Rightarrow$ Relaxation process

- Much lower computational cost compared to BE

- Still shares important features with BE:
  - Conservation laws
  - H-theorem
  - Relaxation to equilibrium.
  - Correct Euler Limit

- Very popular model for numerical experiments in kinetic theory (citation 8800)

- Collision process of BE $\Rightarrow$ Relaxation process

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- Still shares important features with BE:
  - Conservation laws
  - H-theorem
  - Relaxation to equilibrium.
  - Correct Euler Limit
  - Navier-Stokes Limit ?

- Very popular model for numerical experiments in kinetic theory (citation 8800)

Prandtl number

- Compressible Navier-Stokes equation:

$$
\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho U) = 0, \\
&\partial_t(\rho U) + \nabla_x \cdot (\rho U \otimes U + P), = \mu \nabla_x \cdot \sigma \\
&\partial_t E + \nabla_x \cdot (EU + PU + \mu\sigma) = \kappa \triangle T.
\end{aligned}
$$

- Prantl number: The ratio between viscosity and heat conductivity:

$$
\frac{\mu}{\kappa}.
$$

Prandtl number

- Prandtl number: ratio between diffusivity and viscosity.
- Boltzmann equation : 2/3
- BGK model : 1.
- Therefore, compressible NS limit of the BGK model is not correct.

Ellipsoidal BGK model

The Ellipsoidal-BGK model

- ES-BGK Model [Halway, 1964] :

$$\partial_t f + v \cdot \nabla_x f = \frac{\rho}{\tau}(\mathcal{M}_{\nu}(f) - f),$$

- ν: Knudsen parameter: $(-1/2 \leq \nu < 1)$
- τ denotes

$$\tau = \kappa\,(1-\nu)$$

- $\mathcal{M}_{\nu}(f)$: Ellipsoidal Gaussian

Ellipsoidal Gaussian parametrized by ν

- The local Maxwellian is generalized to the ellipsoidal Gaussian:

$$\mathcal{M}_{\nu}(f) = \frac{\rho}{\sqrt{\det(2\pi\mathcal{T}_{\nu})}} \exp\left(-\frac{1}{2}(v-U)^{\top}(\mathcal{T}_{\nu})^{-1}(v-U)\right)$$

- $\mathcal{T}_{\nu}$:Temperature Tensor:

$$\mathcal{T}_{\nu}(x,t) \,=\, (1-\nu)T(x,t)Id + \nu\Theta(x,t)$$

where Θ denotes the stress Tensor:

$$\Theta(x,t) = \frac{1}{\rho}\int_{R^3} f(x,v,t)(v-U)\otimes(v-U)dv.$$

- Prandtl number: $\frac{1}{1-\nu}$.

- 2 important cases:
  - $\nu = 0$: Classical BGK model
  - $\nu = -1/2$: ES-BGK with correct Prandtl number.

- Halway (1964)
- H-theorem: Andries-Le Tallec-Perlat-Perthame (2001)
- Systematic derivation: Brull-Schnieder (2008)

Part I: Stationary solutions in a slab

Stationary BGK model in a slab

- Stationary problem in a slab: $(x, v) \in [0,1] \times \mathbb{R}^3$

$$v_1 \frac{\partial f}{\partial x} = \frac{\rho}{\tau}\big(\mathcal{M}_\nu(f) - f\big),$$

- Mixed boundary conditions ($\delta_1 + \delta_2 = 1$):

$$f(0,v) = \delta_1 f_L(v) + \delta_2 \left(\int_{|v_1|<0} f(0,v)|v_1|dv \right) M_w(0), \quad (v_1 > 0)$$

$$f(1,v) = \delta_1 f_R(v) + \delta_2 \left(\int_{|v_1|>0} f(1,v)|v_1|dv \right) M_w(1). \quad (v_1 < 0)$$

- δ_1: Inflow and δ_2: Diffusive.

Literatures

- BGK
  - Ukai (92): Weak solution with inflow boundary data
  - Nouri (08): QBGK: Weak solution with diffusive boundary data
  - Y. et al (16.18): ES-BGK, QBGK, RBGK.
- Boltzmann
  - Arkeryd-Cercignani-Illner (91): Measure-Valued Solutions.
  - Maslova: Mild Solutions (93)
  - Arkeryd-Nouri (98,99,00...): Weak solutions
  - Brull (08): Gas mixture
  - Guo-Kim-Esposito-Marra (13,18): Near Maxwellian

Norms

- Norm:

$$\sup_x \|f\|_{L^1_{2,}} = \sup_x \left\{ \int_{\mathcal{R}^3} |f(x,v)|(1+|v|^2)dv \right\},$$

- Trace norms ($n(i)$: outward normal):

$$\|f\|_{L^1_{\gamma,|v_1|}} = \sum_{i=0,1} \int_{v\cdot n(i)<0} |f(i,v)||v_1|dv + \int_{v\cdot n(i)>0} |f(i,v)||v_1|dv,$$

$$\|f\|_{L^1_{\gamma,\langle v\rangle}} = \sum_{i=0,1} \int_{v\cdot n(i)<0} |f(i,v)|\langle v\rangle dv + \int_{v\cdot n(i)>0} |f(i,v)|\langle v\rangle dv,$$

where $\langle v\rangle = (1+|v|^2)$.

Conditions on f_{LR}

(P_1) Finite flux $+$ Not too much concentration around $v_1 = 0$:

$$\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \left\|\frac{f_{LR}}{|v_1|}\right\|_{L^1_{\gamma,\langle v\rangle}} < \infty$$

(P_2) No vertical inflow at the boundary:

$$\int_{\mathbb{R}^2} f_L v_i dv = \int_{\mathbb{R}^2} f_R v_i dv = 0 \quad (i = 2, 3)$$

Mild Solution

Definition

$f \in L^1_2([0,1]_x \times \mathbb{R}^3_v)$ is a mild solution if

$$f(x,v) = e^{-\frac{1}{\tau|v_1|}\int_0^x \rho_f(y)dy} f(0,v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \rho_f(z)dz} \rho_f(y)\mathcal{M}(f)dy \quad \text{if } v_1 > 0$$

The mild solution for $v_1 < 0$ is similarly defined.

Mild solution

For $v_1 > 0$

$$|v_1|\partial_x f = \frac{\rho}{\tau}\big(\mathcal{M}_\nu(f) - f\big)$$

$$\partial_x f + \frac{\rho}{\tau|v_1|} f = \frac{\rho}{\tau|v_1|}\mathcal{M}_\nu(f)$$

$$\frac{d}{dx}\left(e^{\frac{\int_0^x \rho(y)dy}{|v_1|\tau}} f(x,v)\right) = \frac{1}{\tau|v_1|} e^{\frac{\int_0^x \rho(y)dy}{|v_1|\tau}} \rho(x)\mathcal{M}_\nu(f).$$

The case for $v_1 < 0$ is the same.

Main result: Inflow dominant case $\delta_2 \ll 1$

- : Non-critical case: $-1/2 < \nu < 1$:

Theorem (Brull-Y. 20)

Let $-1/2 < \nu < 1$. Suppose f_{LR} satisfies (P_1), (P_2). Then, for sufficiantly small δ_2 and τ^{-1}, there exists a unique mild solution $f \geq 0$ for BVP.

- : Critical case: $\nu = -1/2$:

Theorem (Brull-Y. 20)

Let $\nu = -1/2$: Suppose f_{LR} satisfies (P_1), (P_2). Assume further that

$$\left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right| \ll 1,$$

Then, for sufficiantly small δ_2 and τ^{-1}, there exists a unique mild solution $f \geq 0$ for BVP.

Main result: Diffusive dominant case: $\delta_1 \ll 1$

- : Non-critical case: $-1/2 < \nu < 1$:

Theorem (Brull-Y. 20)

Let $-1/2 < \nu < 1$. Suppose f_{LR} satisfies (P_1), (P_2). Assume furthe that f satisfies

$$\int_{v_1<0} f(0,v)|v_1|dv + \int_{v_1>0} f(1,v)|v_1|dv = 1. \tag{3.1}$$

Then, for sufficiantly small δ_2 and τ^{-1}, then there exists a unique mild solution $f \geq 0$ for BVP.

- : Critical case: $\nu = -1/2$:

Theorem (Brull-Y. 20)

Let $\nu = -1/2$: Suppose f_{LR} satisfies (P_1), (P_2). Assume the flux satisfies

$$\int_{v_1<0} f(0,v)|v_1|dv + \int_{v_1>0} f(1,v)|v_1|dv = 1. \tag{3.2}$$

Then, for sufficiantly small δ_2 and τ^{-1}, then there exists a unique mild solution $f \geq 0$ for BVP.

Approximate Scheme

We define our approximate scheme by

$$f^{n+1}(x,v) = e^{-\frac{1}{\tau|v_1|}\int_0^x \rho_n(y)dy} f^{n+1}(0,v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \rho^n(z)dz} \rho_n(y)\mathcal{M}_\nu(f^n)dy \quad \text{if } v_1 > 0$$

and

$$f^{n+1}(0,v) = \delta_1 f_L(v) + \delta_2\left(\int_{|v_1|<0} f^n(0,v)|v_1|dv\right) M_w(0), \quad (v_1 > 0)$$

The scheme for $v_1 < 0$ similiarly defined.

Solution Space

$$\Omega_\nu = \left\{ f \in L^1_2 \;\middle|\; f \text{ satisfies } (\mathcal{A}), (\mathcal{B}), (\mathcal{C}) \right\}$$

where

- $(\mathcal{A})$ f is non-negative:

$$f(x,v) \geq 0 \text{ a.e}$$

- $(\mathcal{B})$ Lower bounds ($|\kappa| = 1$):

$$\rho \geq C_1. \qquad \kappa^\top \{\mathcal{T}_\nu\} \kappa \geq C_2$$

- $(\mathcal{C})$ Norm bounds

$$\|f\|_{L^1_2}, \quad \|f\|_{L^1_{\gamma,|v_1|}}, \; \|f\|_{L^1_{\gamma,\langle v\rangle}} \leq C_3$$

We want $f^n \to f$

- Uniform estimate:

$$f^n \in \Omega_\nu \quad \text{for all } n.$$

- Contractivity:

$$\|f^{n+1} - f^n\| \leq \alpha \|f^{n+1} - f^n\|$$

for appropriate norm and $\alpha < 1$.

Difficulties

- Singularities may arise near $v_1 = 0$:

$$\partial_x f = \frac{\rho}{\tau v_1}(\mathcal{M}_\nu - f).$$

- Singularities may arise near $\mathcal{T}_\nu = 0$:

$$\mathcal{M}_\nu \text{ contains } \mathcal{T}_\nu^{-1} \text{ and } (\det \mathcal{T}_\nu)^{-1}$$

- Dichotomy:

$$(-1/2 < \nu < 1 : \mathcal{T}_\nu \sim T\, Id) \ \text{ VS } \ (\nu = -1/2 : \mathcal{T}_{-1/2} \not\sim T\, Id)$$

1st difficulty: $\frac{1}{|v_1|}$

We can control the singularity: $\frac{1}{|v_1|}$, if we integrate in x and v:

Lemma

Let $f \in \Omega_i$ $(i = 1, 2)$. Then we have

$$\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|} e^{-\frac{\int_y^x \rho_f(z)dz}{\tau|v_1|}} \rho_f(y)\mathcal{M}_\nu(f)dydv \leq C\left(\frac{\ln\tau + 1}{\tau}\right)$$

Proof

For $f \in \Omega_\nu$, we can reduce the integral into

$$\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{a_{\ell,1}(x-y)}{\tau|v_1|}}e^{-Cv_1^2}dydv$$

and divide

$$\left\{\int_0^x\int_{|v_1|<\frac{1}{\tau}}+\int_0^x\int_{\frac{1}{\tau}\le|v_1|<\tau}+\int_0^x\int_{|v_1|\ge\tau}\right\}\frac{1}{\tau|v_1|}e^{-\frac{a_{\ell,1}(x-y)}{\tau|v_1|}}e^{-Cv_1^2}dv_1dy$$
$$\equiv I_1+I_2+I_3.$$

I_1, I_3 are small

- I_1 and I_3 are small:

$$I_1, I_3 = \mathcal{O}(\tau^{-1}).$$

- Estimate of I_2: We first integrate on x:

$$I_2 \leq \frac{1}{a_{\ell,1}} \int_{\frac{1}{\tau} \leq |v_1| \leq \tau} \left(1 - e^{-\frac{a_{\ell,1}x}{\tau|v_1|}}\right) dv_1$$

and apply the Tyalor expasion to $1 - e^{-\frac{a_{\ell,1}}{\tau|v_1|}}$:

$$\begin{aligned} I_2 &= \frac{1}{a_{\ell,1}} \int_{\frac{1}{\tau} < |v_1| < \tau} \left\{ \left(\frac{a_{\ell,1}}{\tau|v_1|}\right) - \frac{1}{2!}\left(\frac{a_{\ell,1}}{\tau|v_1|}\right)^2 + \frac{1}{3!}\left(\frac{a_{\ell,1}}{\tau|v_1|}\right)^3 + \cdots \right\} dv_1 \\ &= \frac{1}{\tau}\ln\tau^2 + \frac{1}{2!}\frac{a_{\ell,1}}{\tau^2}\frac{\tau^2-1}{\tau} + \frac{1}{2\cdot 3!}\frac{a_\ell^2}{\tau^3}\frac{\tau^4-1}{\tau^2} + \frac{1}{3\cdot 4!}\frac{a_\ell^3}{\tau^4}\frac{\tau^6-1}{\tau^3}\cdots \\ &\leq \mathcal{O}\left(\frac{\ln\tau + 1}{\tau}\right). \end{aligned}$$

2nd difficulty: $\mathcal{T}_\nu = 0$, or $(\det \mathcal{T}_\nu) = 0$

We show that this never happens under our assumptions:

Lemma

(1) *Let* $-1/2 \leq \nu < 1$. *Assume* $f^n \in \Omega$. *Then, for sufficiently large* τ, *we have*

$$\kappa^\top \left\{\mathcal{T}_\nu^{n+1}\right\} \kappa \geq C.$$

for some $C > 0$ *indepdent of* n.

- We divide the proof into $-1/2 < \nu < 1$ and $\nu = -1/2$ (3rd difficulty) .

The Proof for $-1/2 < \nu < 1$

- In this case, $\mathcal{T}_\nu$ and T are equivalent:

Lemma

Let $-1/2 \lneq \nu < 1$. Then we have

$$\min\{1-\nu, 1+2\nu\}TId \leq \mathcal{T}_\nu \leq \max\{1-\nu, 1+2\nu\}TId,$$

- Therefore, it is enough to estimate T.

Estimate of T

Therefore, it is enough to estimate T:

$$\begin{aligned}
3\{\rho^{n+1}\}^2 T^{n+1} &= \left(\int_{\mathbb{R}^3} f^{n+1} dv\right)\left(\int_{\mathbb{R}^3} f^{n+1}|v|^2 dv\right) - \left|\int_{\mathbb{R}^3} f^{n+1} v dv\right|^2 \\
&\geq \left(\int_{\mathbb{R}^3} f^{n+1}|v_1| dv\right)^2 - \left(\int_{\mathbb{R}^3} f^{n+1} v_1 dv\right)^2 \quad (\equiv I) \\
&\quad - \sum_{(i,j)\neq(1,1)} \left|\int_{\mathbb{R}^3} f^{n+1} v_i dv\right|\left|\int_{\mathbb{R}^3} f^{n+1} v_j dv\right| \quad (\equiv R) \\
&\equiv I - R.
\end{aligned}$$

I bounded below, and R small

- I is bounded below:

$$I \geq 4\delta_1^2\gamma_{\ell,1}$$

where

$$\gamma_{\ell,1} = \left(\int_{v_1>0} e^{-\frac{a_{u,1}}{\tau|v_1|}} f_L|v_1|dv\right)\left(\int_{v_1<0} e^{-\frac{a_{u,1}}{\tau|v_1|}} f_R|v_1|dv\right).$$

- by the smallness of vertical flow, R is small:

$$R \leq C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right).$$

Estimate of I and R

Therefore, for sufficiently large τ, we can get

$$T^{n+1} \geq \frac{1}{3\{\rho^{n+1}\}^2}\left\{4\delta_1^2\gamma_{\ell,1} - C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right)\right\} \geq C_1 \tag{3.3}$$

where

Critical Case: $\nu = -1/2$ (3rd difficulty)

- In the critical case, we don't have such equivalence type estimate.

$$\mathcal{T}_{-1/2} \not\approx T\, Id.$$

- Therefore, we have to estimate $\mathcal{T}_{-1/2}$ directly.

Computation of $\mathcal{T}_{-1/2}$

- For this, we observe that

$$\begin{aligned}&\rho^{n+1}\left(\kappa^{\top}\left\{\mathcal{T}^{n+1}_{-1/2}\right\}\kappa\right)\\&=\int_{\mathbb{R}^3} f^{n+1}\left\{|v|^2-(v\cdot\kappa)^2\right\}dv-\left\{\rho^{n+1}|U^{n+1}|^2-\rho^{n+1}(U^{n+1}\cdot\kappa)^2\right\}\\&\equiv I-II,\end{aligned}$$

  for $|\kappa|=1$.
- I: Total energy minus directional total energy.
- II: Kinetic energy minus directional kinetic energy.
- We will show that I is bounded below and II is small.

Lower bound of I

- which can be bounded below:

$$I = \int_{\mathbb{R}^3} f^{n+1}\left\{|v|^2 - (v\cdot\kappa)^2 dv\right\} \geq \delta_1 a_{-1/2,1}.$$

where

$$a_{-1/2,1} = \inf_{|\kappa|=1}\int_{\mathbb{R}^3} e^{-\frac{2}{|v_1|}\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}}} f_{LR}\left\{|v|^2 - (v\cdot\kappa)^2\right\} dv.$$

Control of II

- We first need some control on bulk velocity:

Lemma

Let $f^n \in \Omega_\nu$*.*
(1) *For* $i = 1$*, we have*

$$\left|\int_{\mathbb{R}^3} f^{n+1} v_1 dv\right| \leq \left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right| + O(\delta_2, 1/\tau).$$

(2) *For* $i = 2, 3$*, we have*

$$\left|\int_{\mathbb{R}^3} f^{n+1} v_i dv\right| \leq C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right).$$

- Slab flow: $U_1 \sim$: Depends on the discrepance of the boundary flux
- Vertical flow: U_2, U_3: Small

Control of II

The discrepance of the boundary flux, together with the no vertical flows assumptions control II:

$$
\begin{aligned}
II &\approx \left| \int_{\mathbb{R}^3} f^{n+1} v_1 \, dv \right|^2 + \sum_{i=2,3}^{3} \left| \int_{\mathbb{R}^3} f^{n+1} v_i \, dv \right|^2 \\
&\leq C \left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right|^2 + O(\delta_2, \tau^{-1}).
\end{aligned}
$$

Therefore,

$$\kappa^{\top}\left\{\mathcal{T}^{n+1}_{-1/2}\right\}\kappa \geq \delta_1 a_{-1/2} - \left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right|^2 + O(\delta_2, \tau^{-1}).$$

Lip Continuity of $\mathcal{M}_\nu$

Lemma

Let f, g be elements of Ω_i. Then $\mathcal{M}_\nu$ satisfies

$$|\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)| \leq C_{\ell,u} \sup_x \|f - g\|_{L^1_2} e^{-C_{\ell,u}|v|^2}.$$

We expand $\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)$ as

$$\begin{aligned} \mathcal{M}_\nu(f) - \mathcal{M}_\nu(g) &= (\rho_f - \rho_g)\int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \rho} d\theta \\ &+ (U_f - U_g)\int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial U} d\theta \\ &+ (\mathcal{T}_f - \mathcal{T}_g)\int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \mathcal{T}_\nu} d\theta. \end{aligned} \tag{3.4}$$

Roughly,

$$|\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)| \leq C\left(\frac{1}{\rho} + \frac{1}{T^{5/2}}\right)\|f - g\|$$

Contraction

Lemma

Suppose $f^{n+1}, f^n \in \Omega$. Then, under the assumption of Theorem 2.2, we have

$$\sup_x \|f^{n+1} - f^n\|_{L^1_2} + \|f^{n+1} - f^n\|_{L^1_{\gamma,|v_1|}} + \|f^{n+1} - f^n\|_{L^1_{\gamma,\langle v\rangle}}$$
$$\leq K(\delta_1, \tau, f_{LR}) \sup_x \|f_n - f_{n-1}\|_{L^1_2} + \delta_2 C\|f^n - f^{n-1}\|_{L^1_{\gamma,|v_1|}} + \delta_3 C\|f^n - f^{n-1}\|_{L^1_{\gamma,\langle v\rangle}}$$

where $K(\delta_1, \tau, f_{LR})$ denotes

$$K(\delta_1, \tau, f_{LR}) = \frac{\delta_1}{\tau}\left(\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \left\|f_{LR}|v_1|^{-1}\right\|_{L^1_{\gamma,\langle v\rangle}}\right) + \frac{\ln t + 1}{\tau\delta_1^3}.$$

Part II: Entropy Production Estimates

Relative entropy

- We define

$$\begin{aligned} H(f) &= \int f \ln f \, dxdv && : \text{ Entropy} \\ H(f|g) &= H(f) - H(g) && : \text{ Relative entropy} \\ D_{\nu}(f) &= \int \big(\mathcal{M}_{\nu}(f) - f\big) \ln f \, dxdv && : \text{ Entropy Production} \end{aligned}$$

- Multiplying $\ln f$ on ES-BGK and taking integration, we get

$$\frac{d}{dt}\{H(f) - H(\mathcal{M}_0)\} = D_{\nu}(f).$$

Theorem (Kim, Lee, and Y. 2020)

For each $-1/2 \leq \nu < 1$, *define* C_ν *by*

$$C_\nu = \sup_{x>0} \frac{3\ln\left(1+\frac{1}{3}x\right) - \ln\left(1+\frac{1+2\nu}{3}x\right) - 2\ln\left(1+\frac{1-\nu}{3}x\right)}{3\ln\left(1+\frac{1}{3}x\right) - \ln(1+x)}$$

Then, we have

- C_ν *is non-negative and strictly less than 1:*

$$0 \leq C_\nu \leq \frac{1}{3}\nu^2(5-2\nu) < 1, \qquad (-1/2 \leq \nu < 1)$$

- *The following entropy-entropy production estimates holds:*

$$D_\nu(f) \leq -(1-C_\nu)\{H(f) - H(\mathcal{M}_0)\}$$

Previous results

Non-critical case [Y. 2017]

$$D_\nu(f) \leq -\min\{1+2\nu, 1-\nu\}\{H(f) - H(\mathcal{M}_0)\}.$$

Linearized version: [Y. 2018]

$$\langle L_\nu f, f\rangle \leq -(1-|\nu|)\,\|(I-P)f\|^2.$$

Boltzmann equation: [Villani 2004]

$$D_{BE}(f) \leq -C_\epsilon H(f|\mathcal{M}_0)^{1+\epsilon}.$$

Why?

- From ES-BGK model:

$$\frac{d}{dt}\{H(f)-H(\mathcal{M}_0)\} = D_\nu(f) \leq -C\{H(f)-H(\mathcal{M}_0)\}.$$

- Gronwall inequality:

$$\{H(f)-H(\mathcal{M}_0)\} \leq e^{-Ct}\{H(f)-H(\mathcal{M}_0)\}.$$

- Kullback inequality:

$$\frac{1}{2}\|f-\mathcal{M}_0\|^2_{L^1} \leq H(f)-H(\mathcal{M}_0)$$

- Asymptotic Behavior (Homogeneous case)

$$\|f(t)-\mathcal{M}_0\|_{L^1} \leq \sqrt{2}e^{-\frac{1}{2}(1-C_\nu)t}\sqrt{H(f)-H(\mathcal{M}_0)}.$$

Familiar Analny

- Heat equation on Torus with $\int u = 0$.

$$\partial_t u - \triangle_x u = 0.$$

- Energy estimate:

$$\partial_t \|u\|^2_{L^2} = -\|\nabla_x u\|^2_{L^2}.$$

- Poincare inequality

$$\partial_t \|u\|^2_{L^2} \leq -C\|u\|^2_{L^2}.$$

- Asymptotic behavior

$$\|u(t)\|^2_{L^2} \leq e^{-Ct}\|u_0\|^2_{L^2}.$$

Key: Difference between various Maxwellians

Lemma

For $-1/2 \leq \nu < 1$, *we have*

$$H(\mathcal{M}_\nu) - H(\mathcal{M}_0) \leq C_\nu \{H(\mathcal{M}_1) - H(\mathcal{M}_\nu)\}.$$

Proof of the Main Result

- By convexity of $H(f)$,

$$D_\nu(f) = \int H'(f)(\mathcal{M}_\nu - f) \le H(\mathcal{M}_\nu) - H(f).$$

- Apply the lemma to r.h.s:

$$\begin{aligned} H\big(\mathcal{M}_\nu\big) - H(f) &= -\{H(f) - H(\mathcal{M}_0)\} + \underbrace{\{H(\mathcal{M}_\nu) - H(\mathcal{M}_0))\}} \\ &\le -\{H(f) - H(\mathcal{M}_0)\} + \underbrace{C_\nu \{H(\mathcal{M}_1) - H(\mathcal{M}_0)\}} \\ &\le -\{H(f) - H(\mathcal{M}_0)\} + C_\nu \{H(f) - H(\mathcal{M}_0)\} \\ &= -(1 - C_\nu)\{H(f) - H(\mathcal{M}_0)\}, \end{aligned}$$

where we used :

$$H(\mathcal{M}_0) \le H(\mathcal{M}_1) \le H(f).$$

Proof of Key Lemma

- We show that

$$\frac{H(\mathcal{M}_{\nu}) - H(\mathcal{M}_{0})}{H(\mathcal{M}_{1}) - H(\mathcal{M}_{\nu})}.$$

is uniformly bounded in $-1/2 \leq \nu < 1$.

Proof of Key Lemma

- By an explicit computation using conservation laws and diagonalization,

$$H(\mathcal{M}_0) - H(\mathcal{M}_\nu) = \frac{1}{2}\rho \ln \frac{\prod_{i=1}^3 \{(1-\nu)\left(\frac{\theta_1+\theta_2+\theta_3}{3}\right) + \nu\theta_i\}}{\left(\frac{\theta_1+\theta_2+\theta_3}{3}\right)^3}.$$

and

$$H(\mathcal{M}_0) - H(\mathcal{M}_1) = \frac{1}{2}\rho \ln \frac{\theta_1\theta_2\theta_3}{\left(\frac{\theta_1+\theta_2+\theta_3}{3}\right)^3}.$$

where θ_i $(i = 1, 2, 3)$ denotes the eigenfunctions of Θ.

Reduction

Then, the key Lemma turns into

$$\underbrace{\frac{3\ln\left(\frac{\theta_1+\theta_2+\theta_3}{3}\right)-\ln\left[\prod_{i=1}^{3}\left\{(1-\nu)\left(\frac{\theta_1+\theta_2+\theta_3}{3}\right)+\nu\theta_i\right\}\right]}{3\ln\left(\frac{\theta_1+\theta_2+\theta_3}{3}\right)-\ln\theta_1\theta_2\theta_3}}_{\equiv F(\theta_1,\theta_2,\theta_3)}\leq C_\nu$$

Therefore, optimal C_ν is

$$C_\nu=\sup_{\substack{\theta_1,\theta_2,\theta_3>0\\ \exists\ i,j:\ \theta_i\neq\theta_j}} F(\theta_1,\theta_2,\theta_3).$$

Key observation

- Enough to consider only two variables.

$$\sup_{\substack{\theta_1,\theta_2,\theta_3>0 \\ \exists\ i,j:\ \theta_i\neq\theta_j}} F(\theta_1,\theta_2,\theta_3) = \sup_{\theta_1>\theta_2=\theta_3} F(\theta_1,\theta_2,\theta_3)$$

- related to the elementary question:

Fix $X = x+y+z$, $P = xyz$, what is the range of $xy+yz+zx$?

$$\text{Ans:}\quad \frac{1}{4}S^2(4\alpha-3\alpha^2) \leq xy+yz+zx \leq \frac{1}{4}S^2(4\beta-3\beta^2),$$

where α and β are solutions of $x^2-x^3 = S/(27P^3)$.

This, together with the scalability

$$F(\theta_1, \theta_2, \theta_3) = F(k\theta_1, k\theta_2, k\theta_3)$$

enable us to reduce the problem further to

$$\begin{aligned}
&\sup_{\substack{\theta_1,\theta_2,\theta_3>0 \\ \exists\ i,j:\ \theta_i\neq\theta_j}} F(\theta_1, \theta_2, \theta_3) \\
&= \sup_{x>0} \frac{3\ln\left(1+\frac{1}{3}x\right) - \ln\left(1+\frac{1+2\nu}{3}x\right) - 2\ln\left(1+\frac{1-\nu}{3}x\right)}{3\ln\left(1+\frac{1}{3}x\right) - \ln\left(1+x\right)} \\
&\equiv C_{\nu}.
\end{aligned}$$

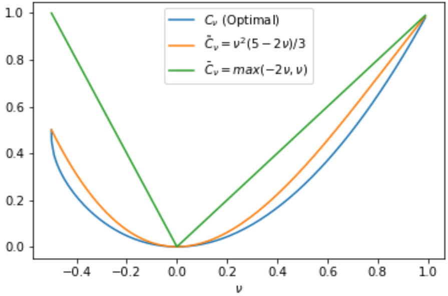
C_ν

$$C_0 = 0, \quad \text{and} \quad C_{-1/2} = 1/2$$

and

$$0 \leq C_\nu \leq \frac{1}{3}\nu^2(5 - 2\nu) < 1$$

on $\nu \in [-1/2, 1)$



Thank You Very Much!

Thank you for your attention!