Decay estimates of solutions to the fluid equations with rotation or stratification

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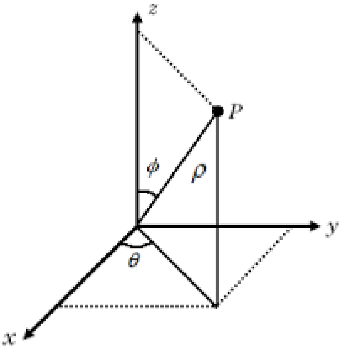
Rotation of the Fluids

Let Ω: speed of rotation. Consider spherical coordinates



$\theta = \Omega t$

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absolute velocity = (velocity relative to the Earth)
+(change of particle position due to Earth's rotation)

$$= v + \vec{\Omega}\times\vec{r}.$$

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If we consider the acceleration, then we deduce the Coriolis force $\vec{\Omega} \times v = \Omega e_3 \times v$.

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Rotating Navier-Stokes

$$\begin{cases} \partial_t \mathrm{u} + (\mathrm{u} \cdot \nabla)\mathrm{u} + \Omega \mathrm{e}_3 \times \mathrm{u} - \nu \Delta \mathrm{u} = -\nabla p + g \\ \nabla \cdot \mathrm{u} = 0. \end{cases} \tag{1}$$

Rotating Navier-Stokes or Euler equations

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- Iwabuchi-Takada('13) : global existence of smooth solution if

$$\|u_0\|_{\dot{H}^s(\mathbb{R}^3)} \leq C|\Omega|^{\frac{s}{2}-\frac{1}{4}}, \qquad \frac{1}{2} < s < \frac{3}{4}.$$

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• Koh-Lee-Takada('14) : extension to the case $\frac{1}{2} < s < \frac{9}{10}$

• Koh-Lee-Takada('14) : Long time existence of solution to 3D rotating Euler equations (maximal time of existence T depends on $|\Omega|$)

3D Generalized NS

$$(RGNS)\begin{cases} \partial_t \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \Omega \mathrm{e}_3 \times \mathrm{u} + (-\Delta)^{\alpha}\mathrm{u} = -\nabla p \\ \nabla\cdot \mathrm{u} = 0. \end{cases} \qquad (2)$$

L^2 energy :

$$\sup_t \|\mathrm{u}(t)\|_{L^2}^2 + 2\int_0^T \|\mathrm{u}\|_{\dot{H}^{\alpha}}^2\, dt \le \|\mathrm{u}_0\|_{L^2}^2.$$

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The condition of θ

$$\begin{aligned} &\frac{3}{4\alpha}\left(1-\frac{2}{p}\right) \le \frac{1}{\theta} \le \frac{1}{2} + \frac{3}{4\alpha} - \frac{1}{p} - \frac{3}{2\alpha p}, \\ &-\frac{1}{2\alpha} - \frac{3}{2\alpha p} + \frac{2}{p} + \frac{s}{2\alpha} < \frac{1}{\theta} < \frac{1}{2} + \frac{1}{8\alpha} - \frac{3}{2\alpha p} + \frac{s}{4\alpha}. \end{aligned} \tag{3}$$

Main Results

Theorem(J. Ahn, J. Kim, L.)

Let $1/2 < \alpha < 5/2$. Suppose that s, p, and θ satisfy

$$\max\left\{0, \frac{5}{2} - 2\alpha\right\} < s < \frac{3(5-2\alpha)}{2(2\alpha+3)}, \qquad \frac{1}{3}+\frac{s}{9} \leq \frac{1}{p} < \frac{5}{12\alpha}+\frac{1}{6}-\frac{s}{6\alpha}, \tag{4}$$

θ satisfies the previous condition. Then $\exists C > 0$ s.t. $u_0 \in \dot{H}^s(\mathbb{R}^3)$

$$\|u_0\|_{\dot{H}^s(\mathbb{R}^3)} \leq C|\Omega|^{\frac{s}{2\alpha}+1-\frac{5}{4\alpha}}, \tag{5}$$

(RGNS) possesses a unique global solution
$u \in \mathcal{C}([0,\infty); \dot{H}^s(\mathbb{R}^3)) \cap L^\theta(0,\infty; \dot{H}^s_p(\mathbb{R}^3))$.

Main Results : Decay estimates

Theorem(J. Ahn, J. Kim , L.)

Let $1/2 < \alpha < 5/2$. Suppose that s and p satisfy (4), $1/p < 1/3 + s/6$, and that $\tilde{s}$ and $\tilde{p}$ satisfy

$$\tilde{s} \geq 0, \qquad \frac{1}{2} - \frac{1}{2p} + \frac{s}{6} \leq \frac{1}{\tilde{p}} \leq \frac{1}{p}.$$

Assume further that $\tilde{q} = (1/3 + 1/p + 1/\tilde{p} - s/3)^{-1}$ satisfies $1 < \tilde{q} \leq \tilde{p}/(\tilde{p} - 1)$. Then, there exists a positive constant C such that for any $u_0 \in (\dot{H}^s \cap \dot{H}^{\tilde{s}}_{\tilde{q}})(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, (RGNS) possesses a unique and global solution satisfying

$$\|u(\cdot,t)\|_{\dot{H}^{\tilde{s}}_{\tilde{p}}(\mathbb{R}^3)} \leq C_* \|u_0\|_{\dot{H}^{\tilde{s}}_{\tilde{q}}(\mathbb{R}^3)} \, t^{-\frac{3}{2\alpha}\left(\frac{1}{\tilde{q}} - \frac{1}{\tilde{p}}\right)} (1+|\Omega| t)^{-\left(1-\frac{2}{\tilde{p}}\right)}, \qquad t > 0, \tag{6}$$

Leray Projection

Let $\mathbb{P}$ be a Leray projection operator in 3D.

$$\mathbb{P} = \{P_{ij}\}, \qquad P_{ij} = \delta_{ij} + R_j R_j, \qquad R_i : \text{Riesz operator}$$

J is a matrix s.t. $J\vec{a} = \Omega e_3 \times \vec{a}$. Then

$$J = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A_α is a fractional Stokes operator s.t. $A_\alpha u = \mathbb{P}(-\Delta)^\alpha u$.

Leray Projection

Then (RGNS) system can be written as

$$u_t + A_\alpha(\Omega)u + \mathbb{P}(u \cdot \nabla)u = 0$$

where $A_\alpha(\Omega)u = A_\alpha u + \mathbb{P}J\mathbb{P}u$. If we consider the symbol of the Fourier transform of the Linear equations, then

$$\partial_t \Phi + |\xi|^{2\alpha}\Phi + S(\xi)\Phi = 0$$

S is a symbol of $\mathbb{P}J\mathbb{P}$.

$$\Phi(t,\xi) = e^{-|\xi|^{2\alpha}t}\left(\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)I - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)R(\xi)\right)\Phi_0$$

RGNS

Equivalent Integral equation

$$u(t) = T_\Omega^\alpha(t)u_0 - \int_0^t T_\Omega^\alpha(t-\tau)\mathbb{P}\mathrm{div}\,(u \otimes u)\,(\tau)d\tau \qquad (7)$$

$$T_\Omega^\alpha(t)f := \mathcal{F}^{-1}\Bigg(\frac{1}{2}e^{i\frac{\xi_3}{|\xi|}\Omega t}e^{-|\xi|^{2\alpha}t}\,(I+\mathcal{R}(\xi))\,\mathcal{F}f + \frac{1}{2}e^{-i\frac{\xi_3}{|\xi|}\Omega t}e^{-|\xi|^{2\alpha}t}\,(I-\mathcal{R}(\xi))\,\mathcal{F}f\Bigg).$$

RGNS

$\mathcal{R}(\xi)$ denotes the skew-symmetric matrix related to the Riesz transforms,

$$\mathcal{R}(\xi) := \begin{pmatrix} 0 & -i\xi_3/|\xi| & i\xi_2/|\xi| \\ i\xi_3/|\xi| & 0 & -i\xi_1/|\xi| \\ -i\xi_2/|\xi| & i\xi_1/|\xi| & 0 \end{pmatrix} \quad \text{for } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Define $\mathcal{G}_\pm(\tau)$ by

$$\mathcal{G}_\pm(\tau)[f](x) := \mathcal{F}^{-1} e^{\pm i\tau \frac{\xi_3}{|\xi|}} \mathcal{F} f = \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\tau \frac{\xi_3}{|\xi|}} \mathcal{F} f(\xi) \mathrm{d}\xi,$$

RGNS

Proposition(Koh-Lee-Takada, '14)

$$\|\mathcal{G}_{\pm}(\tau)f\|_{\dot{B}^{\sigma}_{p,2}(\mathbb{R}^3)} \leq C(1+|\tau|)^{-(1-\frac{2}{p})}\,\|f\|_{\dot{B}^{\sigma+3(1-\frac{2}{p})}_{\frac{p}{p-1},2}(\mathbb{R}^3)}. \tag{8}$$

Brief Sketch of Proof

- The estimates of

$$\|T_\Omega^\alpha(t)f\|_{\dot{H}_p^\sigma(\mathbb{R}^3)}$$

$$\|T_\Omega^\alpha(\cdot)f\|_{L^\theta(0,\infty;L^p(\mathbb{R}^3))}$$

$$\left\|\int_0^t T_\Omega^\alpha(t-\tau)\mathbb{P}\mathrm{div} f(\tau)d\tau\right\|_{L^\theta(0,\infty;\dot{H}_p^s(\mathbb{R}^3))}$$

$$\|T_\Omega^\alpha(t)f\|_{\dot{H}_p^\sigma(\mathbb{R}^3)} \leq C(1+|\Omega|t)^{-(1-2/p)}t^{-\frac{\sigma}{2\alpha}-\frac{3}{2\alpha}(1/r-1/p)}\|f\|_{L^r}$$

Brief Sketch of Proof

- Global Existence

$$\Psi(u)(t) := T_\Omega^\alpha(t)u_0 - \int_0^t T_\Omega^\alpha(t-\tau)\mathbb{P}\mathrm{div}\,(u\otimes u)\,(\tau)d\tau,$$

$$Y := \left\{ u \mid \|u\|_{L^\theta(0,\infty;\dot{H}^s_p)} \leq 2C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4\alpha}(1-\frac{2}{p})}\,\|u_0\|_{\dot{H}^s}\,,\ \mathrm{div}\,u = 0 \right\},$$

Very Brief Sketch of Proof

- For the decay estimates,

$$\|u\|_{X^k_{l,q}}(t) := \sup_{\tau\le t}[\tau^{\frac{3}{2\alpha}(\frac{1}{q}-\frac{1}{l})}(1+|\Omega|\tau)^{1-\frac{2}{l}}\|u(\cdot,\tau)\|_{\dot{H}^k_l(\mathbb{R}^3)}], \qquad t>0. \tag{9}$$

- Estimates

$$\int_{\frac{t}{2}}^{t}\|T^{\alpha}_{\Omega}(t-\tau)\mathbb{P}\mathrm{div}\,(u\otimes u)(\tau)\|_{\dot{H}^k_l(\mathbb{R}^3)}\,d\tau,$$

$$\int_0^t\|T^{\alpha}_{\Omega}(t-\tau)\mathbb{P}\mathrm{div}\,(u\otimes u)(\tau)\|_{\dot{H}^{\tilde{s}}_{\tilde{p}}}\,d\tau$$

Stratification

Dictionary Definition

stratification : the arrangement or classification of something into different groups.

Stratified Fluid

Stratified Fluids, Wikipedia

A stratified fluid may be defined as the fluid with density variations in the vertical direction. For example, air and water; both are fluids and if we consider them together then they can be seen as a stratified fluid system. Density variations in the atmosphere profoundly affect the motion of water and air.

3D Boussinesq equations

$$\begin{cases} \partial_t \mathrm{u} + (\mathrm{u} \cdot \nabla)\mathrm{u} - \nu \Delta \mathrm{u} = -\nabla p + \rho \mathrm{e}_3 \\ \partial_t \rho + (\mathrm{u} \cdot \nabla)\rho = \kappa \Delta \rho \\ \nabla \cdot \mathrm{u} = 0. \end{cases} \tag{10}$$

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Strong stratification

$$(u_s, \rho_s, p_s) = (0,\ N^2 x_3,\ \frac{N^2 x_3^2}{2}).$$

Boussinesq equations with strong stratification

Setting

$$\theta = \rho - \rho_s, \quad P = p - p_s$$

Boussinesq equations with strong stratification

Setting

$$\theta = \rho - \rho_s, \quad P = p - p_s$$

Rewrite equation by initial perturbations

$$\begin{cases} \partial_t \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} - \nu\Delta\mathrm{u} = -\nabla P + \theta \mathrm{e}_3 \\ \partial_t\theta + (\mathrm{u}\cdot\nabla)\theta = \kappa\Delta\theta - N^2 u_3 \\ \nabla\cdot\mathrm{u} = 0. \end{cases} \tag{11}$$

Boussinesq equations with strong stratification

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• Takada('18) : long time existence of solutions for the inviscid 3D stratified Boussinesq equations and convergence to the solution to 2D type Euler equations

Boussinesq equations with strong stratification

- Castro, Córdoba and Lear('19) : global existence of solutions to 2D damped stratified Boussinesq equations in $\mathbb{T} \times [-1, 1]$, with $\mathrm{v} \cdot \mathrm{n} = 0$ on the boundary if the initial data is sufficiently small and also obtained some decay estimates

$$\begin{cases} \mathrm{u}_t + \mathrm{u} + (\mathrm{u} \cdot \nabla)\mathrm{u} = -\nabla P + \theta \mathrm{e}_3, \qquad \mathrm{div}\, \mathrm{v} = 0 \\ \theta_t + (\mathrm{u} \cdot \nabla)\theta = -N^2 u_3 \end{cases} \tag{12}$$

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• Wan('19) obtained the global existence of solutions for the corresponding Cauchy problem.

Equations

$$(SDB)\begin{cases} \mathrm{v}_t + \mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v} = -\nabla P + \theta \mathrm{e}_3, \qquad \operatorname{div}\mathrm{v} = 0 \\ \theta_t + (\mathrm{v}\cdot\nabla)\theta = -N^2 v_3 \end{cases} \tag{13}$$

Energy Inequality :

$$\|\mathrm{v}(t)\|_{L^2}^2 + \frac{1}{N^2}\|\theta(t)\|_{L^2}^2 + 2\int_0^T \|\mathrm{v}\|_{L^2}^2 ds \le \|\mathrm{v}_0\|_{L^2}^2 + \frac{1}{N^2}\|\theta_0\|_{L^2}^2.$$

Main Results

Theorem(J. Kim , L.)

Let $2 < p < 3$ and $m > 7/2$. Then $s > 1 + 3/p$, $\delta > 0$ and $C > 0$ s.t. if an initial data $\|\mathrm{v}_0\|^2_{H^m} + \|\theta_0\|^2_{H^m} < \delta^2$ and $\|\langle \Phi_- \mathrm{u}_0, \mathrm{e}_4 \rangle\|^2_{W^{s+3(1-\frac{2}{p}),p'}} < \delta^2$, then global solution $(\mathrm{v}, \theta) \in C([0,\infty); H^m(\mathbb{R}^3))$ which satisfies (SDB). Moreover if $m \geq 6 + 3/2 - 3/p$, then (v, θ) satisfies

$$\|\mathrm{v}(t)\|_{W^{2,p}} \leq 2\delta \min\{1, t^{-(\frac{3}{2}-\frac{2}{p})}\{\log(1+t)\}^{1-\frac{2}{p}}\} \qquad \text{for all } t \geq 0,$$

$$\|v_3(t)\|_{L^p} \leq 2\delta \min\{1, t^{-(2-\frac{2}{p})}\{\log(1+t)\}^{1-\frac{2}{p}}\} \qquad \text{for all } t \geq 0$$

and

$$\|\theta(t)\|_{W^{2,p}} \leq 2\delta \min\{1, t^{-(1-\frac{2}{p})}\{\log(1+t)\}^{1-\frac{2}{p}}\} \qquad \text{for all } t \geq 0.$$

SDB

Let $\mathrm{u} := (\mathrm{v}, \theta/N)^\top$. Then

$$\mathrm{u}_t + \mathrm{u} + \widetilde{\mathbb{P}} J \widetilde{\mathbb{P}} \mathrm{u} + \widetilde{\mathbb{P}}(\mathrm{v} \cdot \nabla)\mathrm{u} = 0, \qquad \widetilde{\nabla} \cdot \mathrm{u} = 0, \tag{14}$$

where

$$\widetilde{\nabla} := \begin{pmatrix} \nabla \\ 0 \end{pmatrix}, \qquad \widetilde{\mathbb{P}} := \left(\begin{array}{c|c} \mathbb{P} & 0 \\ \hline 0 & 1 \end{array} \right), \qquad \text{and} \qquad J := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & N & -1 \end{pmatrix}.$$

SDB

$$\mathrm{w}_t + \mathrm{w} + \widetilde{\mathbb{P}} J \widetilde{\mathbb{P}} \mathrm{w} = 0, \qquad \widetilde{\nabla} \cdot \mathrm{w} = 0. \tag{15}$$

$$\partial_t \mathcal{F}\mathrm{w} + \mathcal{F}\mathrm{w} + \widetilde{P}(\xi) J \widetilde{P}(\xi) \mathcal{F}\mathrm{w} = 0, \qquad \xi \cdot \mathcal{F}\mathrm{w} = 0, \tag{16}$$

$$\widetilde{P}(\xi) := \left(\begin{array}{c|c} P(\xi) & 0 \\ \hline 0 & 1 \end{array}\right) = \frac{1}{|\xi|^2} \begin{pmatrix} \xi_2^2 + \xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 & 0 \\ -\xi_2\xi_1 & \xi_1^2 + \xi_3^2 & -\xi_2\xi_3 & 0 \\ -\xi_3\xi_1 & -\xi_3\xi_2 & \xi_1^2 + \xi_2^2 & 0 \\ 0 & 0 & 0 & |\xi|^2 \end{pmatrix}.$$

$\xi_h = (\xi_1, \xi_2, 0, 0)^\top$

$$\widetilde{P}(\xi) J \widetilde{P}(\xi) = \frac{1}{|\xi|^2} \begin{pmatrix} 0 & 0 & 0 & N\xi_1\xi_3 \\ 0 & 0 & 0 & N\xi_2\xi_3 \\ 0 & 0 & 0 & -N|\xi_h|^2 \\ -N\xi_1\xi_3 & -N\xi_2\xi_3 & N|\xi_h|^2 & -|\xi|^2 \end{pmatrix}.$$

Eigenvalues and Eigenvectors of Linear Operator

$$\lambda_0(\xi) = \lambda_1(\xi) = 1, \qquad \lambda_\pm(\xi) = \frac{1 \pm \sqrt{1 - 4N^2|\xi_h|^2/|\xi|^2}}{2}$$

and

$$\mathrm{b}_0(\xi) = \frac{1}{|\xi|}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \end{pmatrix}, \qquad \mathrm{b}_1(\xi) = \frac{1}{|\xi_h|}\begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvalues and Eigenvectors of Linear Operator

$$\mathrm{b}_{\pm}(\xi) = \begin{cases} \dfrac{1}{\sqrt{\lambda_{\mp}}|\xi|^2} \begin{pmatrix} N\xi_1\xi_3 \\ N\xi_2\xi_3 \\ -N|\xi_h|^2 \\ \lambda_{\mp}|\xi|^2 \end{pmatrix} & \text{when } 1 - 4N^2|\xi_h|^2/|\xi|^2 \geq 0, \\ \dfrac{1}{\sqrt{2}|\xi_h||\xi|} \begin{pmatrix} N\xi_1\xi_3 \\ N\xi_2\xi_3 \\ -N|\xi_h|^2 \\ \lambda_{\pm}|\xi|^2 \end{pmatrix} & \text{when } 1 - 4N^2|\xi_h|^2/|\xi|^2 < 0, \end{cases}$$

Decay estimates of Linear Operaotr

$$\mathcal{F}\mathrm{u}(t) = \sum_{j=1,\pm} e^{-\lambda_j(\xi)t}\langle \mathcal{F}\mathrm{u}_0, \mathrm{b}_j\rangle \mathrm{b}_j - \sum_{j=1,\pm}\int_0^t e^{-\lambda_j(\xi)(t-\tau)}\langle \mathcal{F}(\mathrm{v}\cdot\nabla)\mathrm{u}(\tau), \mathrm{b}_j\rangle \mathrm{b}$$

$$\mathcal{G}_j(t)[\mathrm{f}] := \int e^{2\pi i x\cdot\xi - \lambda_j(\xi)t}\langle \mathcal{F}\mathrm{f}(\xi), \mathrm{b}_j(\xi)\rangle \mathrm{b}_j(\xi)\mathrm{d}\xi$$

$$\Phi_j \mathrm{f} := \int e^{2\pi i x\cdot\xi}\langle \mathcal{F}\mathrm{f}(\xi), \mathrm{b}_j(\xi)\rangle \mathrm{b}_j(\xi)\mathrm{d}\xi.$$

Decay estimates of Linear Operaotr

$$\|\mathcal{G}_{+}^{0}(t)[\mathrm{f}]\|_{L^p} \leq Ce^{-\frac{1}{2}t}\|\Phi_{+}\mathrm{f}\|_{L^{p'}}, \tag{17}$$

$$\|\mathcal{G}_{-}^{0}(t)[\mathrm{f}]\|_{L^p} \leq C\min\{1, t^{-(1-\frac{2}{p})}\{\log(1+t)\}^{1-\frac{2}{p}}\}\|\Phi_{-}\mathrm{f}\|_{L^{p'}}, \tag{18}$$

$$\|\mathcal{G}_{-}^{0}(t)[\partial_k\mathrm{f}]\|_{L^p} \leq C\min\{1, t^{-(\frac{3}{2}-\frac{2}{p})}\{\log(1+t)\}^{1-\frac{2}{p}}\}\|\Phi_{-}\mathrm{f}\|_{L^{p'}}, \tag{19}$$

$$\|\mathcal{G}_{-}^{0}(t)[R_k^n\mathrm{f}]\|_{L^p} \leq C\min\{1, t^{-(\frac{n}{2}+1-\frac{2}{p})}\{\log(1+t)\}^{1-\frac{2}{p}}\}\|\Phi_{-}\mathrm{f}\|_{L^{p'}}, \tag{20}$$

Brief Idea of the Proof

• Using the first equation, we replace the estimates of θ Replace θ by the terms of ω.

$$E_m(t)^2 := \|\mathrm{v}(t)\|_{H^m}^2 + \|\theta(t)\|_{H^m}^2 \qquad \forall t \geq 0$$

and

$$B_m(T)^2 := \sup_{0\leq t\leq T} E_m(t)^2 + \int_0^T \|\mathrm{v}(t)\|_{H^m}^2 \mathrm{d}t + \sum_{k=1,2} \int_0^T \|\partial_k \theta(t)\|_{H^{m-1}}^2 \mathrm{d}t \tag{21}$$

Estimates

$$\frac{1}{2}B_m(T)^2 - \frac{3}{2}E_m(0)^2 \leq CB_m(T)^3 + CB_m(T)^2 \int_0^T \|v_3(t)\|_{W^{s,p}} \mathrm{d}t. \tag{22}$$

$$\frac{1}{2}\frac{d}{dt}E_m(t)^2 + \|v(t)\|_{H^m}^2 \leq CE_m(t)\|v(t)\|_{H^m}^2 + CE_m(t)\sum_{k=1,2}\|\partial_k\theta(t)\|_{H^{m-1}}^2 \tag{23}$$

$$-\sum_{n=1}^{m}\int_{\mathbb{R}^3}\partial_3 v_3(t)|\partial_3^n\theta(t)|^2 \mathrm{d}x.$$

Estimates

$$\int_0^T \|v_3(t)\|_{W^{s,p}} \mathrm{d}t \leq C\|v_0\|_{H^m} + C\int_0^T \|v(t)\|_{H^m}^2 \mathrm{d}t + \int_0^T \|\langle \mathbb{P}\theta(t)\mathrm{e}_3, \mathrm{e}_3\rangle\|_{W^{s,p}} \mathrm{d}t.$$

Future Studies

Limit Problems and Decay estimates

$$\begin{cases} \partial_t \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \Omega \mathrm{e}_3 \times \mathrm{u} - \nu\Delta \mathrm{u} = -\nabla P + \theta \mathrm{e}_3 \\ \partial_t \theta + (\mathrm{u}\cdot\nabla)\theta = \kappa\Delta\theta - N^2 u_3 \\ \nabla\cdot \mathrm{u} = 0. \end{cases} \tag{24}$$

Thank you for your attention!