# Large amplitude solution of the Boltzmann equation

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Kinetic and fluid equations for collective behavior

May 14, 2021

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# Introduction to the Boltzmann equation

# The Boltzmann equation

Here, $F = F(t, x, v)$ stands for **the density distribution function of particles** with position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$ at time $t > 0$.

- **Boltzmann equation**

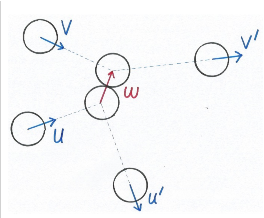
$$\partial_t F + v \cdot \nabla_x F = Q(F, F),$$

describes collisions among particle interactions.

- **Collision operator**

$$\begin{aligned} Q(F_1, F_2) &:= \int_{u \in \mathbb{R}^3} \int_{\omega \in \mathbb{S}^2} B(v - u, \omega) \Big[ F_1(u') F_2(v') - F_1(u) F_2(v) \Big] \, d\omega du \\ &:= Q_+(F_1, F_2) - Q_-(F_1, F_2), \quad \text{if } B \text{ is integrable} \end{aligned}$$

where $B(v - u, \omega)$ is a collision kernel.




We assume these collisions to be perfect elastic : Momentum and energy conservation;

$$\begin{cases} v + u = v' + u' \\ |v|^2 + |u|^2 = |v'|^2 + |u'|^2, \end{cases}$$

where $u' = u + [(v-u)\cdot\omega]\omega, \quad v' = v - [(v-u)\cdot\omega]\omega.$

- **Collision kernel** $B(v-u,\omega)$

$$B(v-u,\omega) = |v-u|^\gamma b(\theta), \quad 0 \le \gamma \le 1 \text{ (hard potential)}, \\ 0 \le b(\theta) \le C|\cos\theta| \text{ (angular cutoff)},$$

where $\cos\theta = \langle \frac{v-u}{|v-u|}, \omega \rangle$.

# Boundary conditions

We denote the phase boundary in the space $\Omega \times \mathbb{R}^3$ as $\gamma = \partial\Omega \times \mathbb{R}^3$, and split it into an outgoing boundary $\gamma_+$, an incoming boundary $\gamma_-$:

$$\gamma_+ := \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x)\cdot v > 0\},$$
$$\gamma_- := \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x)\cdot v < 0\}$$

where $n(x)$ is the outward normal vector at $x \in \partial\Omega$.

1. The in-flow boundary condition: for $(x,v) \in \gamma_-$,

$$F(t,x,v)|_{\gamma_-} = g(t,x,v)$$

2. The bounce-back boundary condition: for $x \in \partial\Omega$,

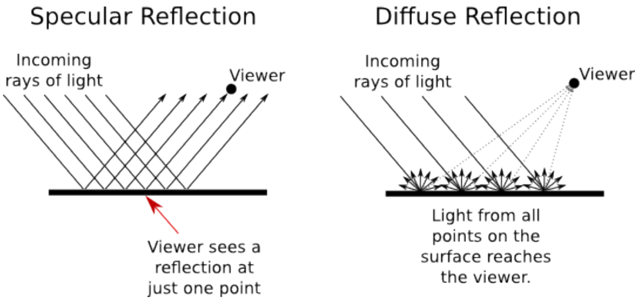
$$F(t,x,v)|_{\gamma_-} = F(t,x,-v)$$

3. Specular reflection: for $x \in \partial\Omega$,

$$F(t,x,v)|_{\gamma_-} = F(t,x,v-2(n(x)\cdot v)n(x)) = F(t,x,R(x)v)$$

4. Diffuse reflection: for $(x,v) \in \gamma_-$,

$$F(t,x,v)|_{\gamma_-} = c_\mu \mu(v) \int_{u\cdot n(x)>0} F(t,x,u)\{n(x)\cdot u\}du$$




[http://math.hws.edu/graphicsbook/c4/s1.html]

# Main Goal

- Does a solution have the **global well-posedness**?

  There exists a unique solution $F(t,x,v)$ which satisfies the system for any time $t > 0$ when an initial distribution function $F_0$ is given.

- Does a solution reach the **physical equilibrium**?

  In Stat Physics, the global Maxwellian $\mu(v) = e^{-\frac{|v|^2}{2}}$ is regarded as an equilibrium state (with proper initial data for conservations). Then, we expected our solution reaches that.

$$\boxed{F(t,x,v) \to \mu(v) \text{ as } t \to \infty}$$

  in some sense.

# Problem and main results

# The Boltzmann equation near Maxwellian

Let $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v) \geq 0$. Then, the Boltzmann equation can be rewritten as

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f,f),$$

where $L$ is a linear operator

$$Lf = \nu(v) f - Kf = -\frac{1}{\sqrt{\mu}} \left[ Q(\sqrt{\mu} f, \mu) + Q(\mu, \sqrt{\mu} f) \right],$$

$$\nu(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\omega) \mu(u) \, d\omega du \sim (1+|v|)^\gamma,$$

and $\Gamma$ is a nonlinear operator

$$\begin{aligned}\Gamma(f,f) := \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} f) &= \frac{1}{\sqrt{\mu}} Q_+(\sqrt{\mu} f, \sqrt{\mu} f) - \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \sqrt{\mu} f) \\ &:= \Gamma_+(f,f) - \Gamma_-(f,f).\end{aligned}$$

# History

**Classical works**

- R. Diperna and P.L.Lion (1989) : Renormalized solution of the Boltzmann equation
- Y. Guo (2002, 2003) : Wellposedness and convergence to equilibrium of (cutoff)VPB, VMB with small perturbation
- L.Desvillettes and C.Villani (2005) : Convergence to equilibrium under high order regularity assumption
- P. Gressman and R.M.Strain (2011) : Wellposedness and convergence to equilibrium of noncutoff Boltzmann

**Low regularity well-posedness** Wellposedness of $L^\infty$-mild solution

*Small amplitude results*

- Y.Guo (2010) : Exponential decay to Maxwellian for boundary problems using double Duhamel iterations. Analytic boundary for specular boundary condition.
- M. Briant and Y. Guo (2016) Maxwellian BC in $C^1$ domain with polynomial weight

# History

- S. Liu and X. Yang (2017) Similar result as Guo(2010) with soft potential. Weight loss for decay (Caflisch's idea).
- C.Kim and L (2018) : Specular reflection with $C^3$ convex domain usnig triple iterations
- C.Kim and L (2018) : Specular reflection in some non-convex domains
- Y. Cao, C.Kim and L (2019) : VPB with diffuse in convex domains

*Large amplitude results*

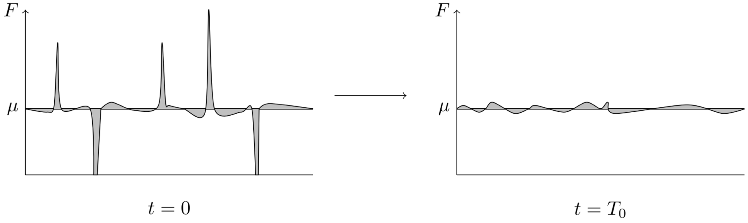
- R.Duan,F.Huang,Y.Wang and T.Yang (2017) : Global well-posedness in a whole space $\mathbb{R}^3$ or periodic domain $\mathbb{T}^3$ with small $L^1_v L^\infty_x$ of initial data
- R.Duan and Y.Wang (2019) : Construct the global solutions in general bounded domains with diffuse reflection BC. Smallness of $L^1_v L^\infty_x$ has removed and assumed small initial $L^2$ data.
- R. Duan, F. Huang,Y. Wang and Z. Zhang (2019) : Soft potential with non-isotheral boundary (diffuse BC), No weight loss, Large amplitude problem

# Large-amplitude initial data problem

**Main question** : Is it possible to construct a global solution allowed to initially have large amplitude?

**Large-amplitude initial data problem**

This allows that initial data could be far from equilibrium state in $L^\infty$ sense even we have global in time solution in $L^\infty$. Instead, we impose smallness to relative entropy or integrable $L^p$ data.



# Relative entropy

Define a relative entropy

$$\mathcal{E}(F) := \int_{\Omega}\int_{\mathbb{R}^3} \left( \frac{F}{\mu} \ln \frac{F}{\mu} - \frac{F}{\mu} + 1 \right) \mu \, dvdx \geq 0.$$

Note that the relative entropy can be reduced to $\mathcal{E}(F) = \int_{\Omega\times\mathbb{R}^3} F \ln \frac{F}{\mu} \, dvdx$ under the mass conservation.

**Lemma (Decay property of relative entropy)**

$$\mathcal{E}(F) \leq \mathcal{E}(F_0),$$

for any $t \geq 0$, where $F$ satisfies the Boltzmann equation and specular BC.

**Lemma ( $L^1$ and $L^2$ control via relative entropy)**

$$\int_{\Omega\times\mathbb{R}^3} \frac{1}{4\mu} |F-\mu|^2 \cdot \mathbf{1}_{|F-\mu|\leq\mu} \, dvdx + \int_{\Omega\times\mathbb{R}^3} \frac{1}{4} |F-\mu| \cdot \mathbf{1}_{|F-\mu|>\mu} \, dvdx \leq \mathcal{E}(F_0).$$

# Main result (R.Duan, G. Ko, and L, 2020 preprint)

## Theorem

*Assume that* $\Omega$ *is a general* $C^3$ *bounded convex domain. Define a weighted function*

$$w = w_\rho(v) = (1 + \rho^2|v|^2)^\beta e^{\varpi|v|^2}$$

*with fixed constants* $0 < \varpi < 1/64$ *and* $\beta \geq 5/2$*, where* $\rho > 0$ *is a constant to be determined later. Then, for any* $M_0 > 0$*, there are* $\rho = \rho(M_0) > 0$ *and* $\epsilon = \epsilon(M_0) > 0$ *such that if initial data satisfy* $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$ *and*

$$\|wf_0\|_{L^\infty_{x,v}} \leq M_0, \quad \mathcal{E}(F_0) \leq \epsilon_0,$$

*then the Boltzmann equation with specular BC admits a unique global-in-time solution* $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$ *satisfying*

$$\|wf(t)\|_{L^\infty_{x,v}} \leq C\left(M_0 + M_0^2\right) \exp\left\{\frac{4}{\nu_0}(M_0 + M_0^2)\right\} e^{-\vartheta t},$$

*for all* $t \geq 0$*, where* $C \geq 1$ *and* $\vartheta > 0$ *are generic constants.*

# Remark : Example of large-amplitude initial data

We choose

$$F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v), \quad \text{with} \quad f_0(x,v) := \frac{\phi(x) - 1}{w}\sqrt{\mu},$$

where $\phi(x)$ is to be chosen such that all conditions on $F_0(x,v)$ hold true.

- $F_0(x,v) \geq 0$

$$F_0(x,v) = \mu \left\{ \left(1 - \frac{1}{w}\right) + \frac{\phi}{w} \right\} \geq 0 \quad \text{if} \quad \phi(x) \geq 0$$

- Mass and energy conservation

$$\int_{\Omega\times\mathbb{R}^3} \sqrt{\mu} f_0 \, dxdv = \int_{\Omega\times\mathbb{R}^3} |v|^2 \sqrt{\mu} f_0 \, dvdx = 0 \quad \text{if} \quad \int_\Omega (\phi(x) - 1)\, dx = 0$$

- Large amplitude initial data

$$M_0 = \|wf_0\|_{L^\infty_{x,v}} = \sup_x |\phi(x) - 1| \cdot \sup_v \sqrt{\mu} \sim \sup_x |\phi(x) - 1| \ (\text{arbitrary})$$

- Relative entropy

$$\mathcal{E}(F_0) = \int_{\Omega\times\mathbb{R}^3} \mu \left(1 + \frac{\phi(x)-1}{w}\right) \ln\left(1 + \frac{\phi(x)-1}{w}\right)\ dvdx \ll 1(WANT)$$

To verify this, we use the convexity of $\Phi(s) := s \ln s$ over $s > 0$ and $0 < \frac{1}{w} < 1$!

$$\Phi\left(1 + \frac{\phi(x)-1}{w}\right) = \Phi\left(\left(1 - \frac{1}{w}\right)\cdot 1 + \frac{1}{w}\phi(x)\right) \leq \left(1 - \frac{1}{w}\right)\Phi(1) + \frac{1}{w}\Phi(\phi(x))$$

$$= \frac{1}{w}\Phi(\phi(x)) = \frac{\phi(x)\ln\phi(x)}{w}.$$

Thus,

$$\mathcal{E}(F_0) \leq \int_\Omega (\phi(x)\ln\phi(x) - \phi(x) + 1)\ dx \int_{\mathbb{R}^3} \frac{\mu}{w}\ dv$$

$$\sim \underbrace{\frac{1}{\rho^3}}_{small} \int_\Omega (\phi(x)\ln\phi(x) - \phi(x) + 1)\ dx \ll 1,$$

if $\int_\Omega (\phi(x)\ln\phi(x) - \phi(x) + 1)\ dx < \infty.$

**Remark)** The relative entropy $\mathcal{E}(F_0)$ does not need to be small!!

# Sketch of proof

# Approach : Characteristics and Duhamel Principle

$X(s;t,x,v)$:=Position of the particle at time $s$, which was at $(t,x,v)$.
$V(s;t,x,v)$:=Velocity of the particle at time $s$, which was at $(t,x,v)$.

Note that we have the characteristics: $\frac{dX(s)}{ds} = V(s)$, $\frac{dV(s)}{ds} = 0$.
Remind the Boltzmann equation

$$f_t + v \cdot \nabla_x f + \nu(v) f = Kf + \Gamma(f,f).$$

Along the characteristics

$$\frac{d}{ds}\left(e^{\nu(v)s} f(s, X(s;t,x,v), V(s;t,x,v))\right) = e^{\nu(v)s}\left[Kf + \Gamma(f,f)\right](s).$$

Taking the time-integration from 0 to t yields

$$\begin{aligned} f(t,x,v) &= e^{-\nu(v)t} f_0(X(0;t,x,v), V(0;t,x,v)) \\ &\quad + \int_0^t e^{-\nu(v)(t-s)}[Kf + \Gamma(f,f)](s, X(s), V(s)) ds. \end{aligned}$$

# Iteration and Change of variable

Since $Kf$ is integrable operaor with "good" kernel $k(v,u)$, our model problem is to consider

$$\int_{|u|\leq N}\int_{|u'|\leq 2N} f(s',X(s';s,X(s),u),u')du'du \lesssim \int_{|u'|}\int_{\Omega} f(s',y,u')dydu'$$

which holds if the mapping $u \mapsto X(s';s,X(s),u)$ is uniformly nondegenerate.

# Treating nonlinear term in large amplitude solution

- How to deal with **the nonlinear term** $\Gamma(f,f)$?

Previously(in small data problem), we treated the nonlinear term as

$$|w(v)\Gamma(f,f)(t)| \le C\nu(v) \underbrace{\|wf(t)\|^2_{L^\infty_{x,v}}}_{\text{not small anymore}} .$$

Alternatively, we use different estimates for $\Gamma(f,f)$, "roughly"

$$|w(v)\Gamma_+(f,f)(t)| \sim \|wf\|_\infty \|f\|_{L^2}$$

since $\|f\|_{L^2}$ is something to do with (small) relative entropy.
However, for about $w(v)\Gamma_-(f,f)$, because of **local term** $f(v)$,

$$|w(v)\Gamma_-(f,f)(t)| \sim \underbrace{\nu(v)} w(v)f(v)\|f\|_{L^1} \to \text{unbounded for hard potential}$$

We need to change the formulation of the Boltzmann equation:

$$\partial_t f + v\cdot\nabla_x f + R(f)(t,x,v)f = Kf + \Gamma_+(f,f),$$

where $R(f)(t,x,v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-u,\omega)\left[\mu(u) + \sqrt{\mu(u)}f(t,x,u)\right] d\omega du.$

# $R(f)$ estimate ($L^\infty_x L^1_v$ estimate)

- Find **the positive lower bound for** $R(f)(t,x,v)$

$$wf(t,x,v) \sim e^{-\int_0^t R(f)(s)ds} wf_0 + \int_0^t e^{-\int_s^t R(f)(\tau)d\tau} \left[wKf + w\Gamma_+(f,f)\right](s)\, ds.$$

But, is $R(f)$ uniformly positive?

$$\begin{aligned} R(f)(t,x,v) &= \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-u,\omega)\left[\mu(u) + \sqrt{\mu(u)}f(t,x,u)\right]\, d\omega du \\ &\geq \nu(v)\left[1 - C_* \int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}} |f(t,x,u)|\, du\right] \geq \frac{\nu(v)}{2} \quad (WANT) \end{aligned}$$

for some constant $C_* > 0$. Thus, it suffices to prove

$$\int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}} |f(t,x,u)|\, du \leq \frac{1}{2C_*}, \tag{1}$$

for all $(t,x) \in [0,T_0) \times \Omega$.
Now let's assume (1) so that $R(f)$ is uniformly positive.

# Why triple iteration cause trouble?

## Lemma (R. Duan and Y. Wang, 2019)

There is a constant $C > 0$ such that

$$|w(v)\Gamma_+(f,f)(t,x,v)| \le C\frac{\|wf(t)\|_{L^\infty_{x,v}}}{1+|v|}\left(\int_{\mathbb{R}^3}(1+|\eta|)^{-4\beta+4}|wf(t,x,\eta)|^2\,d\eta\right)^{\frac{1}{2}}.$$

In $C^3$ convex bounded domain, we need **triple velocity integration** to perform a change of variable! Rewrite as for $h = wf$ (Skip $Kf$)

$$\begin{aligned}
h(t,x,v) &\sim e^{-\nu_0 t}\|h_0\|_{L^\infty_{x,v}} + \int_0^t e^{-\nu_0(t-s)}|w\Gamma_+(f,f)(s)|\,ds \\
&\sim e^{-\nu_0 t}\left(1+\int_0^t \|h(s)\|_{L^\infty_{x,v}}\,ds\right)\|h_0\|_{L^\infty_{x,v}} \\
&\quad + \int_0^t \|h(s)\|_{L^\infty_{x,v}}\int_0^s \|h(s')\|_{L^\infty_{x,v}}\iint_{u,u'}(1+|u|)^{-4\beta+4}(1+|u'|)^{-4\beta+4}h^2 \\
&\sim e^{-\nu_0 t}\left(1+\int_0^t \|h(s)\|_{L^\infty_{x,v}} + \|h(s)\|_{L^\infty_{x,v}}\int_0^s \|h(s')\|_{L^\infty_{x,v}}\right)\|h_0\|^2_{L^\infty_{x,v}} + \dots
\end{aligned}$$

Worst term comes from combination of $\|wf\|_\infty$ in $\Gamma$ estimate and initial term of the next iteration!

# $L^\infty_{x,v}$ estimate-Pointwise estimates for $\Gamma_+(f,f)$

To resolve the problem, we shall mix two different ways of treating $\Gamma_+$.

**Lemma (R.Duan, G. Ko, and L, 2020, preprint)**

*Let $0 < \varpi \leq 1/64$. There is a constant $C > 0$ such that*

$$|w(v)\Gamma_+(f,f)(t,x,v)| \leq C \int_{\mathbb{R}^3} |\tilde{k}(v,\eta)h^2(\eta)|\, d\eta,$$

*for all $v \in \mathbb{R}^3$, where the kernel $\tilde{k}(v,\eta)$ is integrable and has singularity $|v-\eta|^{-1}$ in the hard potential case.*

Above estimate can be obtained by full complicated structure of linearized Boltzmann kernel.

# Exact order of two different Γ estimates

We should mix above two different Γ estimate in exact order as following,

$$h(t,x,v) \overset{L^\infty L^2 (2019)}{\sim} e^{-\nu_0 t}\|h_0\|_{L^\infty_{x,v}} + \int_0^t \|h(s)\|_{L^\infty_{x,v}} \left(\int_u (1+|u|)^{-4\beta+4} h^2\right)^{\frac{1}{2}}$$

$$\overset{(L^2)^2 (2020)}{\sim} e^{-\nu_0 t}\left(1+\int_0^t \|h(s)\|_{L^\infty_{x,v}}\right)\|h_0\|_{L^\infty_{x,v}}$$

$$+ \int_0^t \|h(s)\|_{L^\infty_{x,v}} \left(\int_{u,u'} (1+|u|)^{-4\beta+4} \tilde{k}^2(u,u') h^4(u')\right)^{\frac{1}{2}}$$

$$\overset{L^\infty L^2 (2019)}{\sim} e^{-\nu_0 t}\left(1+\int_0^t \|h(s)\|_{L^\infty_{x,v}}\right)\|h_0\|^2_{L^\infty_{x,v}} + \text{small terms}$$

# $Rf$ issue revisited

To make positive lower bound for $R(f)(t,x,v)$,

$$\int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}} |f(t,x,u)|\, du \lesssim \|wf(t)\|_{L^\infty_{x,v}} \int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}} \frac{1}{w(u)}\, du \lesssim \frac{\|wf(t)\|_{L^\infty_{x,v}}}{\rho^3}$$

- Treat $\int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}}$ as integral operator from iteration step. For diffuse BC, only one furtuer iteration is need, where we still need two more iteration So it is impossible to make LHS generic small without large $\rho$.
- We choose (sufficiently large ) $\rho$ depending on apriori bound $\|wf\|_\infty \leq \bar{M}$. ($\bar{M}$ will be bounded by initial amplitude $\|wf_0\|_\infty$ in the end.) We note that choosing sufficiently large $\rho$ is not allowed in the case of diffuse reflection BC, because even local well-posedness theory requires generic size of $\rho$.

# Grönwall type inequality and bootstrap argument

Triple iteration gives

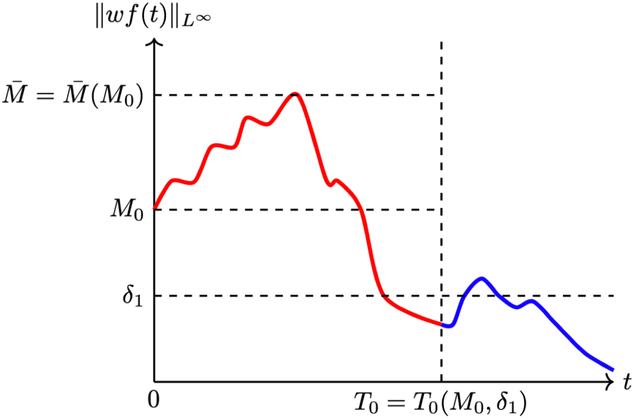
$$\|h(t)\|_{L^\infty_{x,v}} \lesssim e^{-\nu_0 t}\left(1+\int_0^t \|h(s)\|_{L^\infty_{x,v}}\right)\|h_0\|^2_{L^\infty_{x,v}} + \underbrace{\sup_{0\le s\le t}\|h(s)\|^4_{L^\infty_{x,v}}\mathcal{E}(F_0)}_{\text{small part}(=:D)}.$$

If $\sup_{0\le s\le T_0}\|h(s)\|_{L^\infty_{x,v}} \le \bar{M}$, then it follows from Grönwall type inequality

$$\|h(t)\|_{L^\infty_{x,v}} \lesssim_{\|h_0\|_{L^\infty_{x,v}}} \bar{M}\left(1+\frac{2}{\nu_0}D\right)e^{-\nu_0 t} + D,$$

for all $0 \le t \le T_0$! We choose $\bar{M}$ so that depends only on $\|wf_0\|_\infty := M_0$ and also choose $\mathcal{E}(F_0) \ll 1$ depending on $\bar{M}$ (so on $M_0$). After that we use bootstarp argument to extend time interval of well-posedness. After finite step, exponential decaying factor $e^{-\nu_0 t}$ makes $\|h(t)\|_\infty$ sufficiently small so that it satisfies the condition for small data problem.

# $L^\infty_{x,v}$ estimate-Bootstrap argument



# Remark : Issue of initial vacuum

In the case of diffuse BC problem $F_0(x,v)$ may have initial vacuum with generic weight $w_\rho(v) = (1+\rho^2|v|^2)^\beta e^{\varpi|v|^2}$ where $\rho$ is generic large. However, in our result,

$$\rho = \rho(M_0) \quad \text{(Note that this is not allowed for diffuse BC)}$$

so that

$$\sup_{x\in\Omega}\left|\int_{\mathbb{R}^3}|v|^m\sqrt{\mu}f_0(x,v)\,dv\right| \leq M_0\int_{\mathbb{R}^3}\frac{|v|^m\sqrt{\mu}}{w}\,dv \leq C_{\beta,m}\frac{M_0}{\rho^3} \gg 1,$$

where $\|wf_0\|_\infty = M_0$. Hence physical density perturbation should be very small which implies that our initial data $F_0(x,v)$ does not allow local vacuum.

Instead, however, as we mentioned, large $\rho$ can replace small relative entropy condition for some (but general, in fact) kinds of initial data.

Thank you!