

Cyclotomic Hecke L -values of a totally real field

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2021 June 07

Hi !

French-Korean IRL Webinar in Number Theory

Main Theme

Question

Can L -values detect or recover its coefficients:

- in an analytic way,
- in an algebraic way in $\overline{\mathbb{Q}}$, or
- in an algebraic way in $\overline{\mathbb{F}_\ell}$?

For a Dirichlet series $L(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$, one gets

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T L(\sigma + it) n^{\sigma + it} dt = a_n \quad n \in \mathbb{Z}_+$$

for all $\sigma > \sigma_a$. : *abscissa of convergence.*

Question

When a_n are algebraic and $L(s)$ possesses algebraicity, can we recover the algebraic properties of a_n from the L -values?

Previous Results

- Let f be a newform of weight 2 and level N with

f: Hecke eigen cusp form, "primitive" i.e. N : minimal.

$$f(z) = \sum_{n \geq 1} a_f(n) \exp(2\pi i n z).$$

- Hecke field or Coefficient Field.*
- Set $\mathbb{Q}_f := \mathbb{Q}(a_f(n) \mid n \geq 1) \subseteq \overline{\mathbb{Q}}$.
 - Let $L(s, f, \psi)$ the modular L -function twisted by a Dirichlet character ψ .
 - There exist two complex numbers Ω_f^+ and Ω_f^- such that

$$L_f(\psi) := \frac{\tau(\overline{\psi}) L(1, f, \psi)}{\Omega_f^\pm} \in \mathbb{Q}_f(\psi)$$

Adjoining all the values of ψ to \mathbb{Q}_f .

for all ψ with $\psi(-1) = \pm 1$ and the Gauss sum $\tau(\psi)$ of ψ .

- Set $\Xi_p :=$ the set of Dirichlet characters of p -power conductors. *p : a prime.*

Theorem (Luo-Ramakrishnan)

- $\mathbb{Q}_f(\mu_{p^\infty}) = \mathbb{Q}(\mu_{p^\infty}, L_f(\psi), \psi \in \Xi_p)$.
- $\mathbb{Q}_f = \mathbb{Q}(L_f(\chi_D) \mid \chi_D : \text{quadratic})$.

M_{p^∞} = the set of all p -power roots of 1.

Previous Results, II

Theorem (S.)

$\mathbb{Q}_f(\psi) = \mathbb{Q}(L_f(\psi))$ for *almost all* ψ of p -power conductors.

"all but finitely many"

- Let $\ell \neq p$ be primes. Set $\mathbb{F}_f := \mathbb{F}_\ell(a_n(f) : n \geq 1)$.
- If $\bar{\rho}_{f,\ell}$ is irreducible, then there exist $\Omega_f^\pm \in \mathbb{C}$ such that $L_f(\xi)$ are integral for all Dirichlet ξ and $\exists \xi$ s.t. $L_f(\xi) \not\equiv 0 \pmod{\mathcal{L}}$.

residual Galois rep'n assoc. to f .

\mathcal{L} prime in $\bar{\mathbb{Q}}$.

Conjecture

If $\bar{\rho}_{f,\ell}$ is irreducible, then $\mathbb{F}_f(\psi) = \mathbb{F}_\ell(L_f(\psi))$ for almost all $\psi \in \Xi_p$.

Conjecture (Variant of Greenberg's conjecture)

If $\bar{\rho}_{f,\ell}$ is irreducible, then $L_f(\chi) \not\equiv 0 \pmod{\mathcal{L}}$ for almost all $\chi \in \Xi_p$.

Theorem (S.)

$\mathbb{F}_\ell(\psi) = \mathbb{F}_\ell(L(0, \psi))$ for almost all odd $\psi \in \Xi_p$.

Setup

- K : a totally real field with $d = [K : \mathbb{Q}]$.
- p : a rational prime unramified in K .
- $\text{Cl}_K(p^n)$: ray class group of K modulo p^n .
- χ : a ray class character of the form $\chi = \psi \circ N_{K/\mathbb{Q}}$ for a Dirichlet character ψ of p -power conductor. It is called a *cyclotomic* character of K of p -power modulus. Can show " $\text{cond}(\chi) = \text{cond}(\psi)$ ".
- $\chi_f := \psi \circ N_{K/\mathbb{Q}} : (\mathcal{O}/p^n)^\times \rightarrow \overline{\mathbb{Q}}^\times$.
- $\tau(\chi) := \chi(\mathfrak{0}_K) \sum_{\alpha \pmod{p^n}} \chi_f(\alpha) \exp\left(\frac{2\pi i \text{Tr}(\alpha)}{p^n}\right)$.
different of K .

Hecke L -function of χ is

$$L_K(s, \chi) = \sum_{\mathfrak{a} : \substack{\text{integral} \\ \text{ideal} + (0)}} \frac{\chi(\mathfrak{a})}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}$$

It is well-known that

$$L_K(0, \chi) \in \mathbb{Q}(\chi) = \mathbb{Q}(\psi).$$

Main Result

ψ : odd.



Theorem (Jun-Lee-S.)

Let $F = \mathbb{Q}(\exp(\frac{2\pi i}{p(p-1)}))$. Then $F(L_K(0, \chi)) = F(\chi)$ for almost all totally χ .

Remark

- The Leopoldt conjecture is equivalent to saying that all the characters on $\text{Cl}_K(p^\infty)$ are the cyclotomic characters.
- One can obtain a generalization such that for a tame ray class character ξ ,

$$F(L_K(0, \xi\chi)) = F(\xi, \chi)$$

for almost all totally odd χ .

under some mild cond. on cond(ξ).

Proof: reduction to non-vanishing problem

For almost all cyclotomic characters χ of p -power moduli, verify inclusion

$$F(\chi) \subseteq F(L_K(0, \chi)).$$

- Evaluate the following quantity in two different ways:

$$\frac{1}{[F(\chi) : F]} \text{Tr}_{F(\chi)/F} (\chi_f(\alpha) L_K(0, \bar{\chi})),$$

where α is chosen so that $\mathbb{Q}(\chi_f(\alpha)) = \mathbb{Q}(\chi)$.

- Assume that $F(\chi) \subsetneq F(L_K(0, \chi))$ for infinitely many χ .

$$L_\chi := F(L_K(0, \chi)).$$
$$\text{Tr}_{F(\chi)/F} = \text{Tr}_{L_\chi/F} \left(L_K(0, \chi) \text{Tr}_{F(\chi)/L_\chi} (\chi_f(\alpha)) \right) = 0.$$

$\in M_{p^\infty}$

$$\text{Tr}_{F(\chi)/L_\chi} (\chi_f(\alpha)) \neq 0 \iff F(\chi) = L_\chi$$

Next task is to show: the average is nonvanishing for χ with sufficiently large conductor.

More discussion

Need to study the Galois average of

$$\text{Tr} \left(L_K(0, \bar{\chi}) \right) = \left(\frac{i^{3[K:\mathbb{Q}]} \sqrt{d_K}}{\sqrt{\pi}^{[K:\mathbb{Q}]}} \tau(\bar{\chi}) L_K(1, \chi) \right)$$

$$\sum_{\sigma \in \text{Gal}(F(\chi)/F)} \tau(\bar{\chi}^\sigma) L_K(1, \chi^\sigma)$$

- $\text{Tr}_{F(\chi)/F}(\tau(\bar{\chi})\chi(\mathfrak{a}))$ behaves essentially like an additive character.
- The Galois average the average “ $\text{Tr}_{F(\chi)/F}(\tau(\bar{\chi})L_K(1, \chi))$ ” is essentially a special value of a Dirichlet series with additive twists

$$\sum_{\mathfrak{a}} \frac{\text{Tr}_{F(\chi)/F}(\tau(\bar{\chi})\chi(\mathfrak{a}))}{N(\mathfrak{a})^s}.$$

Problem reduces to verify non-vanishing of L -values with additive twists.

- As verification for additive twists is quite similar to multiplicative one, let us focus on non-vanishing of $L_K(\underline{0}, \chi)$.

1.

Approximate functional equation

$$\text{Let } W(\chi) := i^{-d} \frac{\tau(\chi)}{\sqrt{N(p^n)}}.$$

Fast convergent series expression for $L_K(1, \chi)$.

$$L_K(1, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N_{K/\mathbb{Q}}(\mathfrak{a})} F_1 \left(\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{p^{\frac{3n}{4}[K:\mathbb{Q}]}} \right) + W(\chi) \sum_{\mathfrak{a}} \bar{\chi}(\mathfrak{a}) F_2 \left(\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{d_K p^{\frac{n}{4}[K:\mathbb{Q}]}} \right),$$

F_1, F_2 : fast decaying function.

The average $\frac{1}{[F(\chi):F]} \text{Tr}_{F(\chi)/F}(\chi_f(\alpha) L_K(1, \chi))$ is decomposed into two parts:

- First part: For an integer $m \geq 1$, let us set $c_m := |\{\mathfrak{a} \mid N_{K/\mathbb{Q}}(\mathfrak{a}) = m\}|$.

$$\sum_{m \geq 1} \frac{\text{Tr}_{F(\chi)/F}(\psi(m\bar{\alpha})) c_m}{m} F_1 \left(\frac{m}{p^{\frac{3n}{4}[K:\mathbb{Q}]}} \right),$$

where $\chi = \psi \circ N_{K/\mathbb{Q}}$ for a Dirichlet character ψ .

- Second part:

$$\sum_{\mathfrak{a}} \text{Tr}_{F(\chi)/F}(W(\chi) \bar{\chi}(\alpha \mathfrak{a})) \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})} F_2 \left(\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{d_K N(p)^{\frac{n}{4}}} \right)$$

First part: Lattice point counting

We can show :

$$\frac{1}{[F(\chi) : F]} \text{Tr}_{F(\chi)/F}(\psi(m\bar{a})) = \begin{cases} 1 & \text{if } m \equiv a\kappa \pmod{p^n} \text{ for some } \kappa \in \mu_{p-1} \\ 0 & \text{otherwise} \end{cases}$$

$\mu_{p-1} \subset \mathbb{Z}_p^\times$
#(p-1) not of 1.

$$\sum_{\kappa \in \mu_{p-1}} \sum_{m \equiv a\kappa \pmod{p^n}} \frac{c_m}{m} F_1\left(\frac{m}{p^{\frac{3n}{4}[K:\mathbb{Q}]}}\right) = \textcircled{1} + (\text{small error}) + (\text{smaller error}),$$

which correspond to three parts, respectively:

- (i) ~~m=1~~ 1 occurs only if $m = 1$.
- (ii) Exceptional part: $m \equiv a\kappa \pmod{p^n}$ with $\kappa \neq 1$ and $m < p^n$.
- (iii) Generic part: the summation is over $m \geq p^n$. (Easy estimation).

Lattice point counting

Now need to estimate the growth order of

$$\sum_{\substack{m \leq x \\ m \equiv b \pmod{p^n}}} c_m = \#\{\mathbf{a} \mid N(\mathbf{a}) \equiv b \pmod{p^n}, N(\mathbf{a}) < x\}.$$

Let $\{\mathbf{b}_1, \dots, \mathbf{b}_h\}$ be representatives of Cl_K . Let C_K be the fundamental domain of $\mathbb{K}_{\mathbb{R}}^+$ for the action of totally positive units. Then,

tot. pos. element
in $(\mathbb{K} \otimes \mathbb{R})$.

$$\{\mathbf{a} : \text{integral}\} = \bigsqcup_{i=1}^h \mathbf{b}_i (C_K \cap \mathbf{b}_i^{-1}).$$

For $O_p = O_K \otimes \mathbb{Z}_p$, define $\mathcal{E} := \{\alpha \in O_p^\times \mid N(\alpha) = 1\}$, $\mathcal{E}_n := \mathcal{E} \cap (1 + p^n O_p)$.

Question

For a $\gamma \in O_p^\times$ and a nonzero ideal \mathfrak{b} , need to estimate the size of

$$C_{n,\gamma}(x) := \{\alpha \in C_K \cap \mathfrak{b} \mid \alpha \equiv \varepsilon \gamma \pmod{p^n}, \varepsilon \in \mathcal{E}/\mathcal{E}_n, N(\alpha) \leq x\}$$

Proposition

$$|\mathcal{C}_{n,\gamma}(x)| \ll \begin{cases} x^{1-\frac{1}{[K:\mathbb{Q}]}} & \text{if } 0 < x \leq N(p^n) \\ \frac{x}{p^n} & \text{if } x \geq N(p^n) \end{cases}.$$

Using a non-singular simplicial (closed) cone decomposition (not disjoint) with respect to \mathfrak{b}

$$C_K = \bigcup_i C_i,$$

the problem reduces to estimate

$$\{\alpha \in C_i \cap \mathfrak{b} \mid \alpha \equiv \varepsilon\gamma \pmod{p^n}, \varepsilon \in \mathcal{E}/\mathcal{E}_n, N(\alpha) \leq x\}.$$

Lattice point in non-singular simplicial cone

= gene. by a basis of \mathcal{L} .

Let \mathfrak{b} be a nonzero ideal and C a nonsingular cone in $K_{\mathbb{R}}^+$ with respect to \mathfrak{b} .

Lemma

- A $\gamma \in \mathfrak{b} \cap C$ can be written uniquely as

$$\gamma = \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \cdots + \gamma_n p^n$$

with digits $\gamma_i \in p\mathcal{P}_C \cap \mathfrak{b}$.

\mathcal{P}_C : the fund. domain of

- Set $\langle \gamma \rangle_{k,C} := \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \cdots + \gamma_k p^k$.
- If $N(\gamma) < N(p^k)$, then $\gamma = \langle \gamma \rangle_{k,C}$.
- If $\alpha \in C \cap \mathfrak{b}$, $\alpha \equiv \gamma \varepsilon(p^n)$, and $N(\alpha) < x < N(p^n)$, then $\alpha = \langle \gamma \varepsilon \rangle_{n,C}$.

Proof of Proposition

- Hence we need to estimate

$$\#\{\langle \gamma \varepsilon \rangle_{n,C} \mid N(\langle \gamma \varepsilon \rangle_{n,C}) < x, \varepsilon \in \mathcal{E}\}.$$

- If $N(\langle \varepsilon \gamma \rangle_{n,C}) < N(p^k) \approx x$, then the last $(n - k)$ digits of $\langle \varepsilon \gamma \rangle_{n,C}$ are zeros.
- If $\varepsilon' \in \mathcal{E}_k / \mathcal{E}_n$, then the first k -digits of $\varepsilon \varepsilon' \gamma =$ the first k -digits of $\varepsilon \gamma$.
- In sum,

$$\#\{\langle \gamma \varepsilon \rangle_{n,C} \mid N(\langle \gamma \varepsilon \rangle_{n,C}) < x, \varepsilon \in \mathcal{E}\} \leq \#(\mathcal{E} / \mathcal{E}_k) \ll x^{1-1/d}. \quad \square$$

Second part: Exponential sum

In the second part

$$\sum_{m \geq 1} \left\{ \sum_{N(\mathfrak{a})=m} \mathrm{Tr}_{F(\chi)/F}(W(\chi)\bar{\chi}(\alpha\mathfrak{a})) \right\} \frac{1}{m} F_2 \left(\frac{m}{d_K N(p)^{\frac{n}{4}}} \right),$$

the average $\mathrm{Tr}_{F(\chi)/F}(W(\chi)\bar{\chi}(\alpha\mathfrak{a}))$ is equal to

$$\sum_{\beta} \mathrm{Tr}_{F(\chi)/F}(\chi_f(\beta)\bar{\chi}(\alpha\mathfrak{a})) \exp\left(\frac{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(\beta)}{p^n}\right)$$

This can be written as

$$\sum_{\varepsilon \in \mathcal{E}/\mathcal{E}_n} \exp\left(\frac{2\pi i \mathrm{Tr}(\varepsilon\gamma)}{p^n}\right).$$

Proposition

$$\sum_{\varepsilon \in \mathcal{E}/\mathcal{E}_n} \exp\left(\frac{2\pi i \mathrm{Tr}(\varepsilon\gamma)}{p^n}\right) \ll_{p,d} p^{\frac{(d-1)n}{2}}.$$

Stationary Phase method

Let \mathcal{V} be the \mathbb{Z}_p -submodule of $p\mathcal{O}_p$ corresponds to \mathcal{E}_1 via

$$\log_p = (\log_{\wp})_{\wp|p} : 1 + p\mathcal{O}_p = \prod_{\wp} (1 + \wp\mathcal{O}_{\wp}) \rightarrow p\mathcal{O}_p = \prod_{\wp} \wp\mathcal{O}_{\wp}.$$

Lemma

Let us set $m = \lfloor \frac{n}{2} \rfloor$. For a sufficiently large integer n ,

$$\mathcal{E}_1^{p^{m-1}} \bmod p^n = (1 + p^{m-1}\mathcal{V}) \bmod p^n.$$

$$\sum_{\varepsilon \in \mathcal{E}_1 / \mathcal{E}_1^{p^{n-1}}} \mathbf{e} \left(\text{Tr} \left(\frac{\varepsilon \gamma}{p^n} \right) \right) = \sum_{\varepsilon \in \mathcal{E}_1 / \mathcal{E}_1^{p^{m-1}}} \mathbf{e} \left(\text{Tr} \left(\frac{\varepsilon \gamma}{p^n} \right) \right) \sum_{w \in \mathcal{V} / p^{n-m}\mathcal{V}} \mathbf{e} \left(\text{Tr} \left(\frac{\varepsilon \gamma w}{p^{n-m+1}} \right) \right)$$