Cyclotomic Hecke L-values of a totally real field

Hae-Sang Sun (UNIST)
Joint w/ Byungheup Jun and Jungyun Lee

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Main Theme

Question

Can L-values detect or recover its coefficients:

- *in an analytic way,*
- *in an algebraic way in $\overline{\mathbb{Q}}$, or*
- *in an algebraic way in $\overline{\mathbb{F}}_\ell$?*

For a Dirichlet series $L(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$, one gets

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} L(\sigma + it) n^{\sigma + it} dt = a_n \qquad n \in \mathbb{Z}_+$$

for all $\sigma > \sigma_a$. : abscissa of convergence

Question

When a_n are algebraic and $L(s)$ possesses algebraicity, can we recover the algebraic properties of a_n from the L-values?

Previous Results

- Let f be a newform of weight 2 and level N with

= Hecke eigen cusp form, "primitive", i.e. N: minimal

$$f(z) = \sum_{n\geq 1} a_f(n)\exp(2\pi i n z).$$

$$L(s,f,\psi) := \sum_{n\geq 1} \frac{a_f(n)\psi(n)}{n^s}$$

Hecke field or Coefficient Field.

- Set $\mathbb{Q}_f := \mathbb{Q}(a_f(n) \mid n \geq 1). \subseteq \overline{\mathbb{Q}}$
- Let $L(s,f,\psi)$ the modular L-function twisted by a Dirichlet character ψ.
- There exist two complex numbers Ω_f^+ and Ω_f^- such that

$$L_f(\psi) := \frac{\tau(\overline{\psi})L(1,f,\psi)}{\Omega_f^{\pm}} \in \mathbb{Q}_f(\psi)$$

Adjoing all the values of ψ to $\mathbb{Q}_f$.

for all ψ with $\psi(-1) = \pm 1$ and the Gauss sum $\tau(\psi)$ of ψ.

- Set $\Xi_p :=$ the set of Dirichlet characters of p-power conductors.

p: a prime.

Theorem (Luo-Ramakrishnan)

- $\mathbb{Q}_f(\mu_{p^\infty}) = \mathbb{Q}(\mu_{p^\infty}, L_f(\psi), \psi \in \Xi_p)$.
- $\mathbb{Q}_f = \mathbb{Q}(L_f(\chi_D) \mid \chi_D: \ \textit{quadratic})$.

μ_{p^∞} = the set of all p-power roots of 1.

Previous Results, II

"all but finitely many"

Theorem (S.)

$\mathbb{Q}_f(\psi) = \mathbb{Q}(L_f(\psi))$ for almost all ψ of p-power conductors.

To show "$\subseteq$"

- Let $\ell \neq p$ be primes. Set $\mathbb{F}_f := \mathbb{F}_\ell(a_n(f) : n \geq 1)$.
- If $\overline{\rho}_{f,\ell}$ is irreducible, then there exist $\Omega_f^\pm \in \mathbb{C}$ such that $L_f(\xi)$ are integral for all Dirichlet ξ and $\exists \xi$ s.t. $L_f(\xi) \not\equiv 0 \pmod{\mathcal{L}}$.

residual Galois rep'n assoc. to f.

$\mathcal{L}$ prime in $\overline{\mathbb{Q}}$.

$|$

ℓ

Conjecture

If $\overline{\rho}_{f,\ell}$ is irreducible, then $\mathbb{F}_f(\psi) = \mathbb{F}_\ell(L_f(\psi))$ for almost all $\psi \in \Xi_p$.

Conjecture (Variant of Greenberg's conjecture)

If $\overline{\rho}_{f,\ell}$ is irreducible, then $L_f(\chi) \not\equiv 0 \pmod{\mathfrak{L}}$ for almost all $\chi \in \Xi_p$.

Theorem (S.)

$\mathbb{F}_\ell(\psi) = \mathbb{F}_\ell(L(0,\psi))$ for almost all odd $\psi \in \Xi_p$.

Setup

- K : a totally real field with $d = [K : \mathbb{Q}]$.
- p : a rational prime unramified in K.
- $\mathrm{Cl}_K(p^n)$: ray class group of K modulo p^n. Can show $\mathrm{cond}(\chi) = \mathrm{cond}(\psi)$.
- χ : a ray class character of the form $\chi = \psi \circ N_{K/\mathbb{Q}}$ for a Dirichlet character ψ of p-power conductor. It is called a *cyclotomic* character of K of p-power modulus.
- $\chi_{\mathrm{f}} := \psi \circ N_{K/\mathbb{Q}} : (O/p^n)^\times \to \overline{\mathbb{Q}}^\times$.
- $\tau(\chi) := \chi(\mathfrak{d}_K) \sum_{\alpha \pmod{p^n}} \chi_{\mathrm{f}}(\alpha) \exp(\frac{2\pi i \mathrm{Tr}(\alpha)}{p^n})$.

Different of K ↑

Hecke L-function of χ is

$$L_K(s, \chi) = \sum_{\mathfrak{a} :\ \text{integral ideal} \neq (0).} \frac{\chi(\mathfrak{a})}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}.$$

It is well-known that

$$L_K(0, \chi) \in \mathbb{Q}(\chi) = \mathbb{Q}(\psi).$$

Main Result

ψ : odd.

$\Updownarrow$

Theorem (Jun-Lee-S.)

Let $F = \mathbb{Q}(\exp(\frac{2\pi i}{p(p-1)}))$. Then $F(L_K(0,\chi)) = F(\chi)$ for almost all totally odd. *χ.*

Remark

- The Leopoldt conjecture is equivalent to saying that all the characters on $\mathrm{Cl}_K(p^\infty)$ are the cyclotomic characters.
- One can obtain a generalization such that for a tame ray class character ξ,

$$F(L_K(0,\xi\chi)) = F(\xi,\chi)$$

for almost all totally odd χ. under some mild cond. on cond(ξ).

Proof: reduction to non-vanishing problem

For almost all cyclotomic characters χ of p-power moduli, verify inclusion

$$F(\chi) \subseteq F(L_K(0,\chi)).$$

- Evaluate the following quantity in two different ways:

$$\frac{1}{[F(\chi):F]}\mathrm{Tr}_{F(\chi)/F}\left(\chi_{\mathrm{f}}(\alpha)L_K(0,\overline{\chi})\right),$$

  where α is chosen so that $\mathbb{Q}(\chi_{\mathrm{f}}(\alpha)) = \mathbb{Q}(\chi)$.
- Assume that $F(\chi) \subsetneq F(L_K(0,\chi))$ for infinitely many χ.

$L_\chi := F(L_K(0,\chi))$.

$\mathrm{Tr}_{F(\chi)/F} = \mathrm{Tr}_{L_\chi/F}\left(L_K(0,\chi)\,\mathrm{Tr}_{F(\chi)/L_\chi}(\chi_f(\alpha))\right) = 0.$ $\in \mu_{p^\infty}$.

$\mathrm{Tr}_{F(\chi)/L_\chi}(\chi_f(\alpha)) \neq 0 \iff F(\chi) = L_\chi$

Next task is to show: the average is nonvanishing for χ with sufficiently large conductor.

More discussion

Need to study the Galois average of

$$\mathrm{Tr}\left(L_K(0,\overline{\chi})\right) = \mathrm{Tr}\left(\frac{i^{3[K:\mathbb{Q}]}\sqrt{d_K}}{\sqrt{\pi}^{[K:\mathbb{Q}]}}\tau(\overline{\chi})L_K(1,\chi).\right)$$

$$\sum_{\sigma\in\mathrm{Gal}(F(\chi)/F)} \tau(\overline{\chi}^{\sigma})L_K(1,\chi^{\sigma})$$

- $\mathrm{Tr}_{F(\chi)/F}(\tau(\overline{\chi})\chi(\mathfrak{a}))$ behaves essentially like an additive character.
- The Galois average the average "$\mathrm{Tr}_{F(\chi)/F}\left(\tau(\overline{\chi})L_K(1,\chi)\right)$" is essentially a special value of a Dirichlet series with additive twists

$$\sum_{\mathfrak{a}} \frac{\mathrm{Tr}_{F(\chi)/F}(\tau(\overline{\chi})\chi(\mathfrak{a}))}{N(\mathfrak{a})^s}.$$

Problem reduces to verify non-vanishing of L-values with additive twists.

- As verification for additive twists is quite similar to multiplicative one, let us focus on non-vanishing of $L_K(\cancel{0},\chi)$.
  1.

Approximate functional equation

Let $W(\chi) := i^{-d}\frac{\tau(\chi)}{\sqrt{N(p^n)}}$.

Fast convergent series expression for $L_K(1,\chi)$.

$$L_K(1,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N_{K/\mathbb{Q}}(\mathfrak{a})} F_1\left(\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{p^{\frac{3n}{4}[K:\mathbb{Q}]}}\right) + W(\chi)\sum_{\mathfrak{a}} \overline{\chi}(\mathfrak{a}) F_2\left(\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{d_K p^{\frac{n}{4}[K:\mathbb{Q}]}}\right),$$

F_1, F_2 : fast decaying functions.

The average $\frac{1}{[F(\chi):F]}\mathrm{Tr}_{F(\chi)/F}(\chi_{\mathfrak{f}}(\alpha)L_K(1,\chi))$ is decomposed into two parts:

- First part: For an integer $m \geq 1$, let us set $c_m := |\{\mathfrak{a} \mid N_{K/\mathbb{Q}}(\mathfrak{a}) = m\}|$.

$$\sum_{m\geq 1} \frac{\mathrm{Tr}_{F(\chi)/F}(\psi(m\overline{a}))c_m}{m} F_1\left(\frac{m}{p^{\frac{3n}{4}[K:\mathbb{Q}]}}\right),$$

  where $\chi = \psi \circ N_{K/\mathbb{Q}}$ for a Dirichlet character ψ.

- Second part:

$$\sum_{\mathfrak{a}} \mathrm{Tr}_{F(\chi)/F}(W(\chi)\overline{\chi}(\alpha\mathfrak{a}))\frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})} F_2\left(\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{d_K N(p)^{\frac{n}{4}}}\right)$$

First part: Lattice point counting

We can show :

$$\frac{1}{[F(\chi):F]}\mathrm{Tr}_{F(\chi)/F}(\psi(m\bar{a})) = \begin{cases} 1 \text{ if } m \equiv a\kappa \pmod{p^n} \text{for some } \kappa \in \mu_{p-1} \\ 0 \text{ otherwise} \end{cases}$$

$\subseteq \mathbb{Z}_p^\times$

$= (p-1)$-th roots of 1.

$$\sum_{\kappa\in\mu_{p-1}} \sum_{m\equiv a\kappa \pmod{p^n}} \frac{c_m}{m} F_1\left(\frac{m}{p^{\frac{3n}{4}[K:\mathbb{Q}]}}\right) = 1 + (\textit{small error}) + (\textit{smaller error}),$$

which correspond to three parts, respectively:

(i) 1 occurs only if $m = 1$.

(ii) Exceptional part: $m \equiv a\kappa \pmod{p^n}$ with $\kappa \neq 1$ and $m < p^n$.

(iii) Generic part: the summation is over $m \geq p^n$. (Easy estimation)

Lattice point counting

Now need to estimate the growth order of

$$\sum_{\substack{m \leq x \\ m \equiv b\,(p^n)}} c_m = \#\{\mathfrak{a} \mid N(\mathfrak{a}) \equiv b \pmod{p^n}, N(\mathfrak{a}) < x\}.$$

Let $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_h\}$ be representatives of Cl_K. Let C_K be the fundamental domain of $K_{\mathbb{R}}^{+}$ for the action of totally positive units. Then,

totally positive elements in $K_{\mathbb{R}} := K \otimes \mathbb{R}$ (Minkowski space)

$$\{\mathfrak{a} : \text{ integral}\} = \bigsqcup_{i=1}^{h} \mathfrak{b}_i (C_K \cap \mathfrak{b}_i^{-1}).$$

For $O_p = O_K \otimes \mathbb{Z}_p$, define $\mathcal{E} := \{\alpha \in O_p^\times | N(\alpha) = 1\}, \mathcal{E}_n := \mathcal{E} \cap (1 + p^n O_p)$.

$\supseteq O_K^\times$

Question

For a $\gamma \in O_p^\times$ and a nonzero ideal $\mathfrak{b}$, need to estimate the size of

$$\mathcal{C}_{n,\gamma}(x) := \{\alpha \in C_K \cap \mathfrak{b} \mid \alpha \equiv \varepsilon\gamma \pmod{p^n}, \varepsilon \in \mathcal{E}/\mathcal{E}_n, N(\alpha) < x\}$$

More on lattice point counting

Proposition

$$|\mathcal{C}_{n,\gamma}(x)| \ll \begin{cases} x^{1-\frac{1}{[K:\mathbb{Q}]}} & \text{if } 0 < x \leq N(p^n) \\ \dfrac{x}{p^n} & \text{if } x \geq N(p^n) \end{cases}.$$

Using a non-singular simplicial (closed) cone decomposition (not disjoint) with respect to $\mathfrak{b}$

$$C_K = \bigcup_i C_i,$$

the problem reduces to estimate

$$\{\alpha \in C_i \cap \mathfrak{b} \,|\, \alpha \equiv \varepsilon\gamma \pmod{p^n}, \varepsilon \in \mathcal{E}/\mathcal{E}_n, N(\alpha) \leq x\}.$$

On $C_i \cap \mathfrak{b}$, a "p-adic expansion" is well-defined.

This plays a role of Minkowski theory or Geometry of numbers.

Lattice point in non-singular simplicial cone

generated by a basis of $\mathfrak{b}$

Let $\mathfrak{b}$ be a nonzero ideal and C a nonsingular cone in $K_{\mathbb{R}}^{+}$ with respect to $\mathfrak{b}$.

Lemma

- *A $\gamma \in \mathfrak{b} \cap C$ can be written uniquely as*

$$\gamma = \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \cdots + \gamma_n p^n$$

with digits $\gamma_i \in p\mathcal{P}_C \cap \mathfrak{b}$.

$\mathcal{P}_C$: the fund. domain for C w.r.t $\mathfrak{b}$.

- *Set $\langle\gamma\rangle_{k,C} := \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \cdots + \gamma_k p^k$.*
- *If $N(\gamma) < N(p^k)$, then $\gamma = \langle\gamma\rangle_{k,C}$.*

- If $\alpha \in C \cap \mathfrak{b}$, $\alpha \equiv \gamma\varepsilon(p^n)$, and $N(\alpha) < x < N(p^n)$, then $\alpha = \langle\gamma\varepsilon\rangle_{n,C}$.

Proof of Proposition

- Hence we need to estimate

$$\#\{\langle\gamma\varepsilon\rangle_{n,C} \,|\, N(\langle\gamma\varepsilon\rangle_{n,C}) < x,\ \varepsilon \in \mathcal{E}\}.$$

- If $N(\langle\varepsilon\gamma\rangle_{n,C}) < N(p^k) \approx x$, then the last $(n-k)$ digits of $\langle\varepsilon\gamma\rangle_{n,C}$ are zeros.
- If $\varepsilon' \in \mathcal{E}_k/\mathcal{E}_n$, then the first k-digits of $\varepsilon\varepsilon'\gamma =$ the first k-digits of $\varepsilon\gamma$.
- In sum,

$$\#\{\langle\gamma\varepsilon\rangle_{n,C} \,|\, N(\langle\gamma\varepsilon\rangle_{n,C}) < x,\ \varepsilon \in \mathcal{E}\} \leq \#(\mathcal{E}/\mathcal{E}_k) \ll x^{1-1/d}. \quad \square$$

In the second part

$$\sum_{m\geq 1}\left\{\sum_{N(\mathfrak{a})=m}\mathrm{Tr}_{F(\chi)/F}(W(\chi)\overline{\chi}(\alpha\mathfrak{a}))\right\}\frac{1}{m}F_2\left(\frac{m}{d_K N(p)^{\frac{n}{4}}}\right),$$

the average $\mathrm{Tr}_{F(\chi)/F}(W(\chi)\overline{\chi}(\alpha\mathfrak{a}))$ is equal to

$$\frac{1}{\sqrt{N(p^n)}}\sum_{\beta}\mathrm{Tr}_{F(\chi)/F}(\chi_{\mathfrak{f}}(\beta)\overline{\chi}(\alpha\mathfrak{a}))\exp(\frac{2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(\beta)}{p^n})$$

This can be written as

$$\sum_{\varepsilon\in\mathcal{E}/\mathcal{E}_n}\exp(\frac{2\pi i\mathrm{Tr}(\varepsilon\gamma)}{p^n}).$$

Proposition

$$\sum_{\varepsilon\in\mathcal{E}/\mathcal{E}_n}\exp(\frac{2\pi i\mathrm{Tr}(\varepsilon\gamma)}{p^n})\ll_{p,d}p^{\frac{(d-1)n}{2}}.$$

Stationary Phase method

Let $\mathcal{V}$ be the $\mathbb{Z}_p$-submodule of pO_p corresponds to $\mathcal{E}_1$ via

$$\log_p = (\log_\wp)_{\wp | p} : 1 + pO_p = \prod_\wp (1 + \wp O_\wp) \to pO_p = \prod_\wp \wp O_\wp .$$

Lemma

Let us set $m = \left\lfloor \frac{n}{2} \right\rfloor$. For a sufficiently large integer n,

$$\mathcal{E}_1^{p^{m-1}} \bmod p^n = (1 + p^{m-1}\mathcal{V}) \bmod p^n .$$

$\mathbf{e}(x) = \exp(2\pi i x)$

$$\sum_{\varepsilon \in \mathcal{E}_1/\mathcal{E}_1^{p^{n-1}}} \mathbf{e}\left(\mathrm{Tr}\left(\frac{\varepsilon\gamma}{p^n}\right)\right) = \sum_{\varepsilon \in \mathcal{E}_1/\mathcal{E}_1^{p^{m-1}}} \mathbf{e}\left(\mathrm{Tr}\left(\frac{\varepsilon\gamma}{p^n}\right)\right) \sum_{w \in \mathcal{V}/p^{n-m}\mathcal{V}} \mathbf{e}\left(\mathrm{Tr}\left(\frac{\varepsilon\gamma w}{p^{n-m+1}}\right)\right)$$

* Only for a bounded number of $\varepsilon \in \mathcal{E}_1/\mathcal{E}_1^{p^{m-1}}$, we get

$$\sum_{\omega} \mathbf{e}\left(\frac{\mathrm{Tr}(\varepsilon\gamma\omega)}{p^{n-m+1}}\right) \neq 0. \rightsquigarrow$$ Square-root Cancellation.