Asymptotic emergent dynamics of the Schrödinger-Lohe model

French-Korean IRL in Mathematics
Kinetic and Fluid Equations for Collective Behavior

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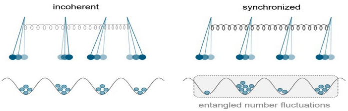
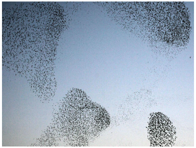
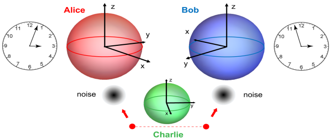
Introduction





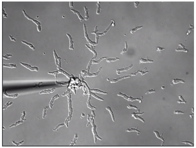
Birds flock

Fish flock

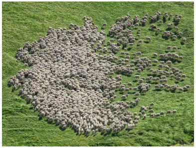


Quantum synchronization

[Witthaut et al., ’17] (top) [Okeke et al., ’18] (bottom)



Bacteria aggregation

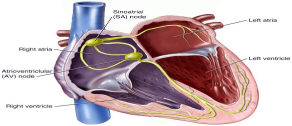


Sheep herding

[Google image]

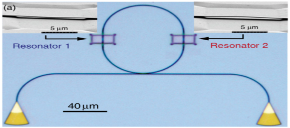
- Synchronization: adjustment of rhythm due to interaction

[Fireflies, Google image]

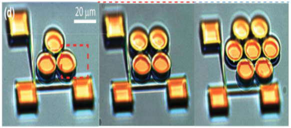


[Cardiac pacemaker, Google image]

- Quantum synchronization: synchronization in quantum systems

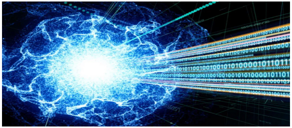


[Tang et al., PRL '13]

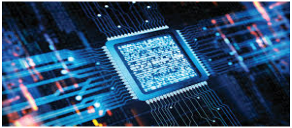


[Lipson et al., PRL '15]

- Application to quantum information and quantum computing



[Google image]



[Google image]

Similarity between classical and quantum synchronizations

- Quantum synchronization : synchronization in quantum systems.

◇ D. Witthaut, S. Wimberger, R. Burioni and M. Timme: *Classical synchronization indicates persistent entanglement in isolated quantum systems*, Nature Communications. (2017).

- Find a link between collective classical and quantum dynamics.

- Isolated quantum systems can synchronize in a very similar way to classical systems.

Schrödinger-Lohe model

- The Schrödinger-Lohe (S-L) model [*Lohe, J. Phys. A (2010)*]:

$$\begin{cases} \mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + V_j\psi_j + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N}\left(\psi_k - \frac{\langle \psi_j, \psi_k\rangle}{\langle \psi_j, \psi_j\rangle}\psi_j\right), \quad t>0, \\ \psi_j(0,x) = \psi_j^0(x), \quad (t,x)\in \mathbb{R}^d \times \mathbb{R}_+, \quad \|\psi_j^0\|_{L^2(\mathbb{R}^d)} = 1, \quad j = 1, \cdots, N. \end{cases}$$

  Here, V_j represents an external one-body potential acted on j-th node, and κ measures a coupling strength between oscillators. In addition, the inner product is defined as

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(x)\bar{g}(x)\mathrm{d}x.$$

- The S-L model enjoys L^2-conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_j\|^2_{L^2(\mathbb{R}^d)} = 0, \quad t>0.$$

  Thus, one has

$$\|\psi_j(t)\|_{L^2(\mathbb{R}^d)} = 1, \quad t>0.$$

Relation with well-known models

- (A decoupled system): if $\kappa = 0$, the S-L system reduces to the juxtaposition of N-independent linear Schrödinger equations:

$$\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta \psi_j + V_j \psi_j, \quad j = 1, \cdots, N.$$

- (A space homogeneous system): if we write

$$V_j(x) = \nu_j \quad \text{and} \quad \psi_j(x,t) = \psi_j(t) = e^{-\mathrm{i}\theta_j(t)},$$

so that the S-L system does not depend on the space variable $x \in \mathbb{R}^d$, then $\theta_j(t)$ satisfies the Kuramoto model:

$$\dot{\theta}_j = \nu_j + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j).$$

- We would say that the S-L model is a generalized Kuramoto model.
- Can we rigorously derive the Kuramoto model (ODE) from the Schrödinger-Lohe model (PDE)? *(Ongoing project)*

Global existence of a unique solution

- There have been some results on global existence of a unique solution, for instance, [Huh-Ha '17], [Antonelli-Marcati '17], [Bao-Ha-K.-Tang '19], etc.

Theorem

Suppose that initial data and external potentials satisfy

$$\psi_j^0 \in L^2(\mathbb{R}^d), \quad V_j \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \quad p > \max\{1, d/2\}, \quad j = 1, \cdots, N.$$

Then, the S-L system admits a global unique solution $\psi_j \in C(\mathbb{R}_+; L^2(\mathbb{R}^d))$. *In addition,* $\psi_j^0 \in H^1(\mathbb{R}^d)$, *then the corresponding global unique solution* $\psi_j \in C(\mathbb{R}_+; H^1(\mathbb{R}^d))$.

Sketch of the proof

- Main ingredient
  - Strichartz estimate (for local existence)
  - L^2 conservation (for extending the local solution to global one)
- Main difficulty
  - Lack of the energy conservation

For the decoupled system ($\kappa = 0$), the (total) energy is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathsf{T}}[\Psi] = 0, \quad t > 0, \quad \mathcal{E}_{\mathsf{T}}[\Psi] := \sum_{j=1}^{N} \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla \psi_j|^2 + V_j|\psi_j|^2 \right) \mathrm{d}x.$$

- However for $\kappa \neq 0$, the (total) energy would not be conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathsf{T}}[\Psi] = -\kappa \sum_{j=1}^{N} r_j \mathcal{E}_j[\Psi] + \kappa(\text{extra terms}), \quad t > 0,$$

$$r_j(t) := \frac{1}{N}\sum_{k=1}^{N} \mathrm{Re}\langle \psi_j, \psi_k \rangle(t), \quad \mathcal{E}_j[\Psi] := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla \psi_j|^2 + V_j |\psi_j|^2 \right) \mathrm{d}x.$$

- Is the system dissipative? Is the total energy uniformly bounded? *(Ongoing project)*

(Local existence): for $H_j = -\frac{1}{2}\Delta + V_j$, define $\mathcal{U}_j(t) := e^{-\mathrm{i}tH_j}$ as the Schrödinger group generated by H_j. Then, Duhamel's formula yields

$$\psi_j(t) = \mathcal{U}_j(t)\psi_j^0 + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N} \underbrace{\int_0^t \mathcal{U}_j(t-s)\left(\psi_k - \frac{\langle\psi_j,\psi_k\rangle}{\langle\psi_j,\psi_j\rangle}\psi_j\right)\mathrm{d}s}_{=:\,\mathcal{I}}, \quad t\in[0,T]. \tag{1}$$

Denote the right-hand side of (1) as $\mathcal{S}[\psi_j](t)$. For the term $\mathcal{I}$, we use the Strichartz estimate to find

$$\left\|\int_0^t \mathcal{U}_j(t-s)\left(\psi_k - \frac{\langle\psi_j,\psi_k\rangle}{\langle\psi_j,\psi_j\rangle}\psi_j\right)\mathrm{d}s\right\|_{L^{\frac{8}{d}}(\mathbb{R};L^4(\mathbb{R}^d))} \le C\left\|\psi_k - \frac{\langle\psi_j,\psi_k\rangle}{\langle\psi_j,\psi_j\rangle}\psi_j\right\|_{L^2(\mathbb{R}^d)} \le 2CT.$$

Since the second term in (1) can be also treated by the literature (e.g., Cazenave '03), we choose T sufficiently small so that the map $\mathcal{S}$ becomes a strict contraction in $\mathcal{X}_T$ and then standard fixed point theory yields the local solution.

(Global existence): it follows from the a priori energy estimate that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathsf{T}}[\Psi] &= -\kappa\sum_{j=1}^{N} r_j\mathcal{E}_j[\Psi] + \kappa(\text{extra terms}) \\ &\leq (\cdots) \\ &\leq \kappa\left(1+\frac{N}{2}\right)\mathcal{E}_{\mathsf{T}}[\Psi].\end{aligned}$$

Thus, the energy does not blow up in any finite time interval. This completes the proof.

Definition of synchronization for the S-L model

Definition (Lohe '10, Choi-Ha '14)

Let $\psi_j = \psi_j(t, x)$ be a global smooth solution to the S-L model.

1. (Complete synchronization): all relative distances between wavefunctions converge to zero:
$$\lim_{t\to\infty} \|\psi_i(t) - \psi_j(t)\| = 0.$$
2. (Locked states): all relative distances between wavefunctions tend to positive definite values:
$$\lim_{t\to\infty} \|\psi_i(t) - \psi_j(t)\| = d_{ij}.$$

- $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^d)}$.

Alternative definitions for complete synchronization

- (Huh-Ha '17): We define the two-point correlation function

$$h_{k\ell}(t) := \langle \psi_k, \psi_\ell \rangle(t).$$

Then, it follows from the mass conservation that

$$\|\psi_k - \psi_\ell\|^2_{L^2(\mathbb{R}^d)} = 2\mathrm{Re}(1 - h_{k\ell}) \quad \text{and}$$

$$\lim_{t\to\infty} \|\psi_k - \psi_\ell\|_{L^2(\mathbb{R}^d)} = 0 \iff \lim_{t\to\infty} |1 - h_{k\ell}(t)| = 0.$$

- (Antonelli-Marcati '17): We define the centroid of wave functions and the order parameter as its norm:

$$\rho(t) := \left\| \frac{1}{N}\sum_{k=1}^{N} \psi_k(t) \right\|_{L^2(\mathbb{R}^d)}, \quad \frac{1}{2N^2}\sum_{k,\ell=1}^{N} \|\psi_k - \psi_\ell\|^2_{L^2(\mathbb{R}^d)} = 1 - \rho^2.$$

Then, we observe

$$\lim_{t\to\infty} D(\Psi) = 0 \iff \lim_{t\to\infty} \rho(t) = 1.$$

Emergent dynamics for identical potentials

-based on [Huh-Ha-K. '18]

Identical potentials

- Consider the case of $V_j \equiv V$ for $j = 1, \cdots, N$.
- Since the (Schrödinger) operator $t \mapsto e^{-\mathrm{i}(-\Delta+V)t}$ denoted as $S = S(t)$ is unitary, it suffices to consider the following simplified model: for $x \in \mathbb{R}^d$,

$$\begin{cases} \dfrac{\mathrm{d}\psi_j}{\mathrm{d}t} = \dfrac{\kappa}{2N} \displaystyle\sum_{K=1}^{N} (\psi_k - \langle \psi_j, \psi_k \rangle \psi_j), \quad t > 0, \\ \psi_j(0, x) = \psi_j^0(x), \end{cases} \tag{2}$$

where the space variable x can be regarded as a parameter. If we define the solution operator $L = L(t)$ for (2), then one has

$$\Psi(t) = L(t)\Psi^0, \quad \text{or equivalently,} \quad \psi_j(t) = \Big(L(t)\Psi^0\Big)_j.$$

Consequently, the solution can be represented as the composition of $S(t)$ and $L(t)$:

$$\Psi(t, x) = S(t) \circ L(t)\Psi^0.$$

Previous results

Theorem (Choi-Ha '14)

Suppose that initial data and external potentials satisfy

$$\kappa > 0, \quad V_j \equiv V, \quad \max_{1\le i,j\le N} \|\psi_i^0 - \psi_j^0\| < \frac{1}{2},$$

and let $\{\psi_j\}$ be a global solution to the S-L model. Then, the system achieves complete synchronization with an exponential convergence rate:

$$\max_{1\le i,j\le N} \|\psi_i(t) - \psi_j(t)\| \lesssim e^{-\kappa t}, \quad t > 0.$$

(Sketch of the proof) Define the maximal diameter $\mathcal{D}(\Psi(t))$:

$$\mathcal{D}(\Psi(t)) := \max_{1\le i,j\le N} \|\psi_i(t) - \psi_j(t)\|,$$

and derive a differential inequality for $\mathcal{D}(\Psi)$.

Dynamics of the order parameter

- Our goal is to extend the initial data leading to complete synchronization.

- Define the centroid and its norm:
$$\zeta := \frac{1}{N}\sum_{k=1}^{N}\psi_k, \quad \rho(t) := \|\zeta(t)\|.$$
- The order parameter ρ satisfies
$$\frac{\mathrm{d}\rho^2}{\mathrm{d}t} = \kappa\left(\rho^2 - \frac{1}{N}\sum_{k=1}^{N}\mathrm{Re}(\langle\zeta,\psi_k\rangle^2)\right) \geq 0.$$
- Then, ρ is non-decreasing and bounded $(\rho(t) \leq 1)$. Hence, there exists $\rho_\infty \in [0,1]$ such that
$$\lim_{t\to\infty}\rho(t) = \rho_\infty.$$
- After careful analysis of the possible values ρ_∞, we can classify all possible asymptotic states.

Classification of asymptotic states

Theorem (Huh-Ha-K. '18)

Suppose that initial data and external potentials satisfy

$$\kappa > 0, \quad V_j \equiv V, \quad \psi_i^0 \neq \psi_j^0 \quad \text{for } i \neq j \quad \text{and} \quad \rho_0 := \left\| \frac{1}{N} \sum_{\ell=1}^{N} \psi_\ell^0 \right\| > 0,$$

and let $\{\psi_j\}$ be a global smooth solution to the S-L system. Then, one of the following assertion holds:

1. *Complete synchronization: the order parameter $\rho = \rho(t)$ tends to 1:*

$$\lim_{t \to \infty} \rho(t) = 1.$$

2. *Bi-polar synchronization: there exists a single index $\ell_0 \in \{1, \cdots, N\}$ such that*

$$\lim_{t \to \infty} \langle \psi_i, \psi_j \rangle = 1 \quad \text{for } i, j \neq \ell_0 \quad \text{and} \quad \lim_{t \to \infty} \langle \psi_{\ell_0}, \psi_i \rangle = -1 \quad \text{for } i \neq \ell_0.$$

Bi-polar state is unstable

(Idea) From the previous dichotomy, $N-1$ oscillators aggregate. Without loss of generality, we would assume that

$$\psi_2 = \psi_3 = \cdots = \psi_N.$$

Then, S-L system reduces to the system of two oscillators (ψ_1, ψ_2):

$$\mathrm{i}\partial_t \psi_1 = -\frac{1}{2}\Delta\psi_1 + V\psi_1 + \frac{\mathrm{i}\kappa}{2N}(\psi_2 - \langle\psi_1, \psi_2\rangle\psi_1),$$
$$\mathrm{i}\partial_t \psi_2 = -\frac{1}{2}\Delta\psi_2 + V\psi_2 + \frac{\mathrm{i}\kappa}{2N}(\psi_1 - \langle\psi_2, \psi_1\rangle\psi_2),$$

and the two-point correlation function $h := \langle\psi_1, \psi_2\rangle$ satisfies:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\kappa}{N}(1-h^2), \quad t > 0, \quad h(0) = h_0,$$

which can be explicitly solved as

$$h(t) = \frac{(1+h^0)e^{\frac{2\kappa t}{N}} - (1-h^0)}{(1-h^0) + (1+h^0)e^{\frac{2Kt}{N}}} = \begin{cases} \to -1, & h^0 = -1, \\ \to 1, & h^0 \neq -1. \end{cases}$$

Emergent dynamics for non-identical potentials

-based on [Ha-Hwang-K. In preparation]

Non-identical potentials

- Our main ingredient is two-point correlations $h_{ij} = \langle \psi_i, \psi_j \rangle$:

$$\dot{h}_{ij} = \mathrm{i} \int_{\mathbb{R}^d} (V_j(x) - V_i(x)) \psi_i \bar{\psi}_j \mathrm{d}x + \frac{\kappa}{2N} \sum_{k=1}^{N} (h_{ik} + h_{kj})(1 - h_{ij}).$$

- For a simple case, we consider the case of $V_i(x) - V_j(x)$: constant realized when

$$V_i(x) = V(x) + \omega_i, \quad \omega_i : \text{constant}.$$

In this case, the dynamics above becomes

$$\dot{h}_{ij} = \mathrm{i}(\omega_j - \omega_i) h_{ij} + \frac{\kappa}{2N} \sum_{k=1}^{N} (h_{ik} + h_{kj})(1 - h_{ij}),$$

which is a closed system with respect to $\{h_{ij}\}$.

- Recall the relation $\|\psi_i - \psi_j\| \to d_{ij} \iff \mathrm{Re} h_{ij} \to 1 - \frac{d_{ij}^2}{2}$

Two oscillators

- Consider the case of $N = 2$ as the simplest one in [Huh-Ha '17].
- If we denote $h := h_{12}$ and $\omega := \omega_1 - \omega_2$, then h satisfies

$$\dot{h} = -\mathrm{i}\omega h + \frac{\kappa}{2}(1 - h^2), \quad t > 0, \quad h(0) = h_0. \tag{3}$$

- Then depending on the relation between κ and ω, solutions are classified into three types.
- Case A ($\kappa > \omega$): In this case, (3) admits two equilibria: $h_{\infty,-}$ and $h_{\infty,+}$

$$h_{\infty,-} := -\frac{\omega}{\kappa}\mathrm{i} - \sqrt{1 - \left(\frac{\omega}{\kappa}\right)^2}, \quad h_{\infty,+} := -\frac{\omega}{\kappa}\mathrm{i} + \sqrt{1 - \left(\frac{\omega}{\kappa}\right)^2}.$$

  The following explicit formula for h is obtained by straightforward calculation:

$$h(t) = \frac{h_{\infty,+}(h_0 - h_{\infty,-}) + h_{\infty,-}(h_0 - h_{\infty,+})e^{-\sqrt{\kappa^2 - \omega}t}}{h_0 - h_{\infty,-} - (h_0 - h_{\infty,+})e^{-\sqrt{\kappa^2 - \omega^2}t}}.$$

- Then for any initial datum $h_0 \neq h_{\infty,-}$, one has

$$\lim_{t\to\infty} h(t) = h_{\infty,+}.$$

- Case B ($\kappa = \omega$): In this case, two equilibria $h_{\infty,-}$ and $h_{\infty,+}$ collapse to $-\mathrm{i}$. Thus,

$$h(t) = \frac{h_0 - \mathrm{i}(h_0 + \mathrm{i})\kappa t}{1 + (h_0 + \mathrm{i})\kappa t}, \quad t > 0,$$

which yields

$$\lim_{t\to\infty} h(t) = h_\infty.$$

- Case C ($\kappa < \omega$): In this case, $h = h(t)$ becomes a periodic orbit with period $\frac{2\pi}{\sqrt{\omega^2-\kappa^2}}$:

$$h(t) = \frac{h_0 \cos(\sqrt{\omega^2 - \kappa^2} t) - \frac{2\kappa}{\sqrt{\omega^2-\kappa^2}}(\mathrm{i}\frac{\omega}{\kappa} - 1) \sin(\sqrt{\omega^2 - \kappa^2} t)}{\cos(\sqrt{\omega^2 - \kappa^2} t) + \frac{2\kappa}{\sqrt{\omega^2-\kappa^2}}(h_0 + \mathrm{i}\frac{\omega}{\kappa}) \sin(\sqrt{\omega^2 - \kappa^2} t)}$$

- Thus for $N = 2$, the system undergoes a bifurcation at $\kappa = \omega$ from the periodic orbit to the convergence toward equilibrium.

- *In particular, slow relaxation is obtained for a critical case* $\kappa = \omega$*.*

- Our goal is to extend the result for $N = 2$ to the one for $N > 2$.

Many oscillators

- Due to the dissimilarity of nonidentical potentials, one may not expect emergence of complete synchronization where all relative distances converge to zero.
- However, we can make relative distances small as we wish by controlling the coupling strength κ.
- Define the maximal diameter for non-identical potentials $\{V_j\}$:

$$\mathcal{D}(\mathcal{V}) := \max_{1 \le i,j \le N} \|V_i - V_j\|_\infty .$$

Lemma

Suppose that initial data and external potentials satisfy

$$\kappa > 4\mathcal{D}(\mathcal{V}) > 0, \quad \mathcal{D}(\Psi^0)^2 < \frac{\kappa + \sqrt{\kappa^2 - 4\kappa\mathcal{D}(\mathcal{V})}}{\kappa},$$

and let $\{\psi_j\}$ *be a global solution to the S-L model. Then, there exists a finite entrance time* $T_* > 0$ *such that*

$$\mathcal{D}(\Psi(t)) < \frac{2\mathcal{D}(\mathcal{V})}{\kappa + \sqrt{\kappa^2 - 4\kappa\mathcal{D}(\mathcal{V})}} = \mathcal{O}\left(\frac{1}{\kappa}\right), \quad t > T_*.$$

- For the convergence of h_{ij} towards some definite values, we adopt the strategy developed in [Ha-Ryoo, '16].
- Let $\{\psi_j\}$ and $\{\tilde{\psi}_j\}$ be any two global solutions and denote

$$h_{ij}(t) = \langle \psi_i, \psi_j \rangle(t), \quad \tilde{h}_{ij}(t) = \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle(t).$$

- Define the diameter measuring the dissimilarity of two correlation functions:

$$d(\mathcal{H}, \tilde{\mathcal{H}})(t) := \max_{1 \leq i,j \leq N} |h_{ij}(t) - \tilde{h}_{ij}(t)|, \quad t > 0.$$

- As a first step, we show that the diameter $d(\mathcal{H}, \tilde{\mathcal{H}})$ converges to zero.

- As assumed for $N = 2$, we need to impose the condition on $\{V_j\}$:

$$V_j(x) = V(x) + \omega_j, \quad \omega_j \in \mathbb{R},$$

so that V_j is a (small) perturbation of a common potential V. Denote

$$\mathcal{D}(\omega) := \max_{1 \le i,j \le N} |\omega_i - \omega_j|.$$

Lemma

Let $\{\psi_j\}$ and $\{\tilde{\psi}_j\}$ be any two global solutions. Then, $d(\mathcal{H}, \tilde{\mathcal{H}})$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} d(\mathcal{H}, \tilde{\mathcal{H}}) \le -\kappa(1 - \mathcal{D}(\Psi)^2) d(\mathcal{H}, \tilde{\mathcal{H}}), \quad t > 0.$$

- Then if we find a sufficient condition leading to $\mathcal{D}(\Psi(t)) < 1$, then zero convergence of $d(\mathcal{H}, \tilde{\mathcal{H}})$ is obtained together with an exponential rate.

- Once zero convergence of $d(\mathcal{H}, \tilde{\mathcal{H}})$ is derived, since our system is autonomous, we can choose $\tilde{h}_{ij}$ as for any $T > 0$,

$$\tilde{h}_{ij}(t) = h_{ij}(t + T).$$

- By discretizing the time $t \in \mathbb{R}_+$ as $n \in \mathbb{Z}_+$ and setting $T = m \in \mathbb{Z}_+$, we deduce that $\{h_{ij}(n)\}_{n \in \mathbb{Z}_+}$ becomes a Cauchy sequence in the complete space $B_1(0) := \{z \in \mathbb{C} : |z| \le 1\}$.

- Hence for each i, j, there exists a complex number h_{ij}^{∞} such that

$$\lim_{t \to \infty} h_{ij}(t) = h_{ij}^{\infty}.$$

Theorem

Suppose that initial data and external potentials satisfy

$$\kappa > 4\mathcal{D}(\omega) > 0, \quad \mathcal{D}(\Psi^0)^2 < \frac{\kappa + \sqrt{\kappa^2 - 4\kappa\mathcal{D}(\omega)}}{\kappa},$$

and let $\{\psi_j\}$ *be a global solution to the S-L model. Then, one has*

$$\lim_{t\to\infty} d(\mathcal{H}, \tilde{\mathcal{H}}) = 0.$$

In addition, there exists a complex number h_{ij}^{∞} *with* $|h_{ij}^{\infty}| \le 1$ *such that*

$$\lim_{t\to\infty} h_{ij}(t) = h_{ij}^{\infty}.$$

Numerical simulations

-based on [Bao-Ha-K.-Tang '19]

Time splitting method to discretize the S-L system

Choose $\Delta t > 0$ as the time step size and denote time steps $t_n := n\Delta$ for $n \geq 0$. From $t = t_n$ to $t = t_{n+1}$, the S-L system can be solved in splitting steps.

- One solves first

$$\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j, \tag{4}$$

- and then solves

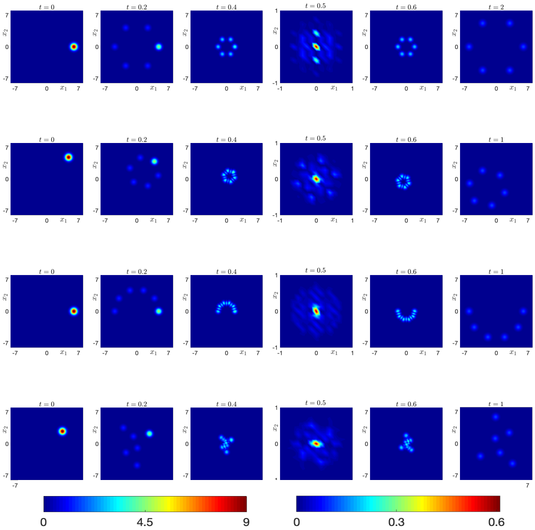
$$\mathrm{i}\partial_t \psi_j = V_j\psi_j + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N} a_{jk}\left(\psi_k - \frac{\langle\psi_j, \psi_k\rangle}{\langle\psi_j, \psi_j\rangle}\psi_j\right). \tag{5}$$

(4) will be discretized in space by the Fourier pseudospectral method and integrated in time analytically in the phase space, and (5) with $\kappa = 0$ can be explicitly integrated in time, since $|\psi_k(\cdot, t)|$ is conserved.

Main difficulty

- However, due to the presence of the communication term involving $\kappa \neq 0$, (5) cannot be explicitly (or analytically) integrated.
- Thus, the Crank-Nicolson method is adopted to discretize (5) and our method is called the Time Splitting Crank-Nicolson Fourier Pseudospectral (TSCN-FP) method.
- In addition, TSCN-FP is of spectral accuracy in space and of second-order accuracy in time.

Numerical simulation



Contour plots of $|\psi_1(t,x)|^2$ at different times t for $N = 6$ with different initial data

Summary and Future projects

Summary

- The Schrödinger-Lohe system would be classified into two types depending on the dissimilarity of external potentials:

  (i) Identical (ii) Non-identical.

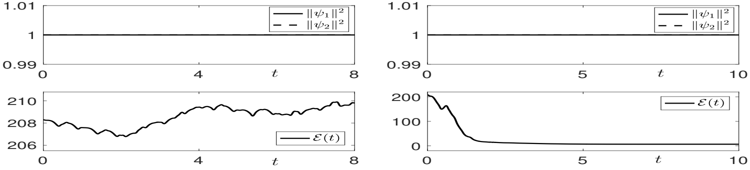
- For the identical system, complete synchronization occurs for generic initial data.

- On the other hand for the non-identical system, we may not expect the emergence of complete synchronization.

- Instead, one can find a sufficient framework with a large coupling regime under which a solution to the system tends to the locked state.

- **Theoretical aspect:** find a (suitable) structure of the convergent values $\{h_{ij}^{\infty}\}$ for the non-identical system.

- **Numerical aspect:** propose an improved asymptotic preserving numerical scheme:



Thanks for your attention!