Non–Archimedean analytic curves and the local–global principle

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Overview

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A modern variant: Geometric LGP

F-the function field of a curve, $(F_i)_i$ interpreted locally on a model of said curve (*e.g.* discrete completions of F)

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- LGP-$\mathscr{M}_x$ $\Rightarrow$ LGP-HHK
- LGP-HHK $\Rightarrow$ LGP-$\mathscr{M}_x$ if k is discrete and other hypotheses

Analytic functions and complete ultrametric fields

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Ways to avoid the problem:

1. Tate's rigid geometry;
2. Raynaud's approach using formal schemes and models;
3. Berkovich's analytic geometry;
4. Huber's adic spaces.

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GAGA theorem for $\mathscr{M}$

If X/k-normal irreducible projective algebraic curve, then $\kappa(X) = \mathscr{M}(X^{\mathrm{an}})$.

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- analytic functions: formal power series over k convergent somewhere.

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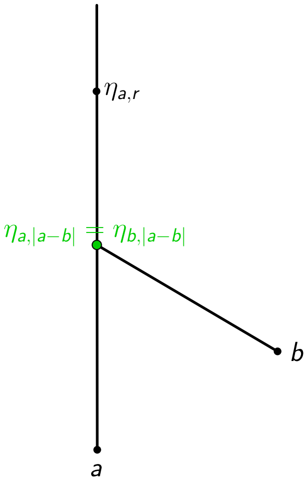

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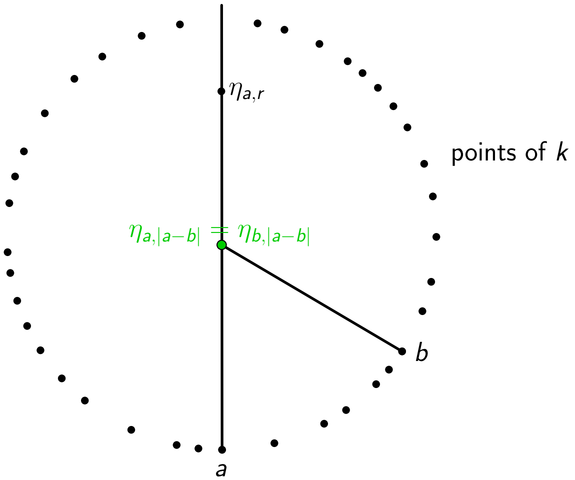

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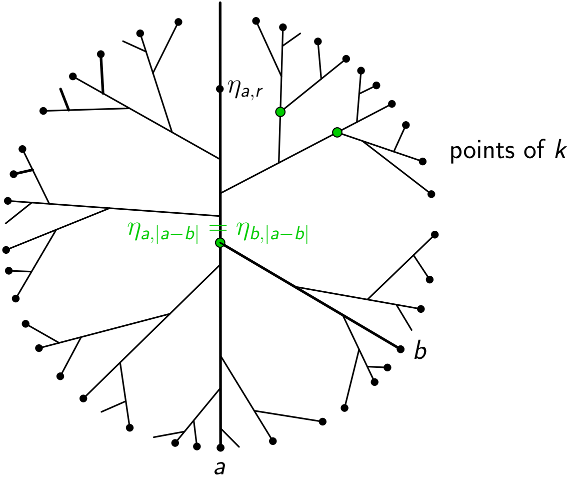

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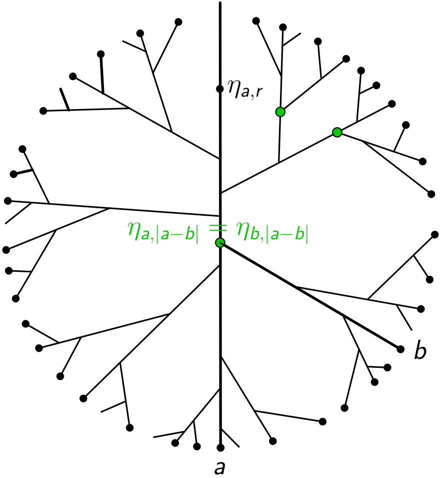

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  - All analytic curves have a graph-like structure with infinite branching.

LGP-$\mathscr{M}_x$ and consequences

Theorem (LGP-$\mathscr{M}_x$)

Let k be a complete ultrametric field. Let C/k be a normal irreducible projective curve. Let F denote its function field. Suppose V/F is a "homogeneous" variety over a rational linear algebraic group G/F. Then $F = \mathscr{M}(C^{\mathrm{an}})$, where C^{an} - Berkovich analytification of C, and

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Corollary (Parimala-Suresh '09, HHK '09, M. '19)

Any quadratic form of dimension $\geqslant 9$ defined over $\mathbb{Q}_p(T)$, $p \neq 2$, has a non-trivial zero over $\mathbb{Q}_p(T)$.

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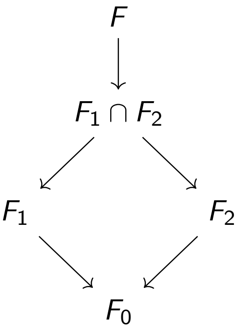

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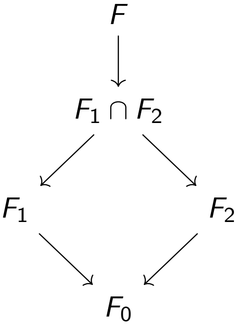

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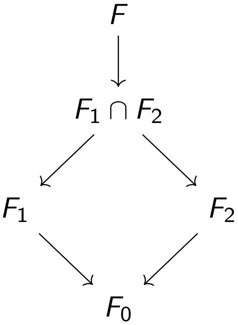

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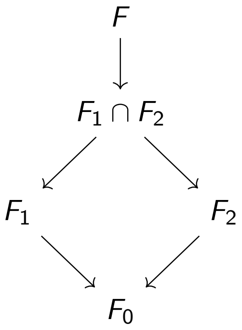

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Question

Under what conditions on $F, F_i, i = 0, 1, 2,$ and G is (PP) satisfied?

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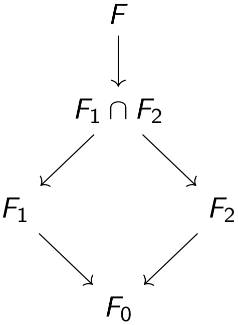

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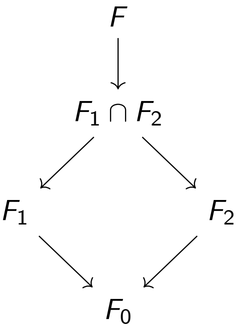

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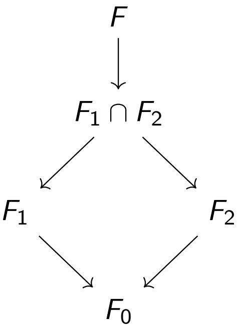

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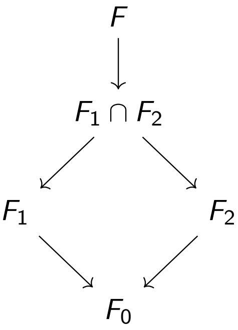

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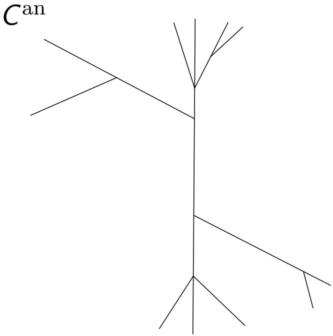

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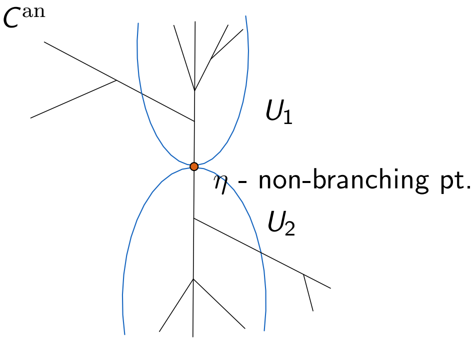

U_1, U_2 – compact analytic domains in C^{an} (building blocks of the analytic structure)

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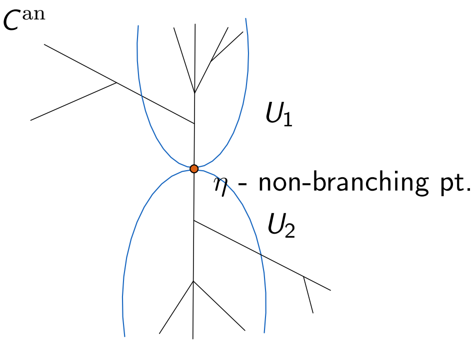

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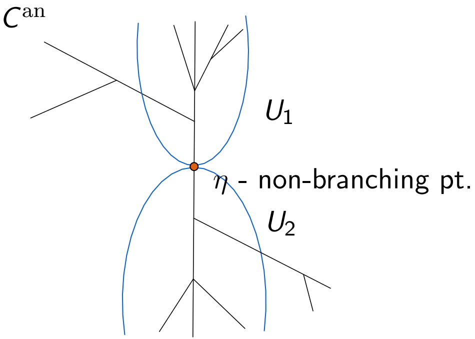

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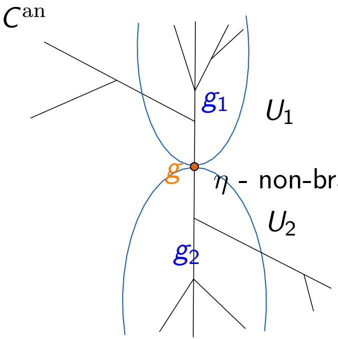

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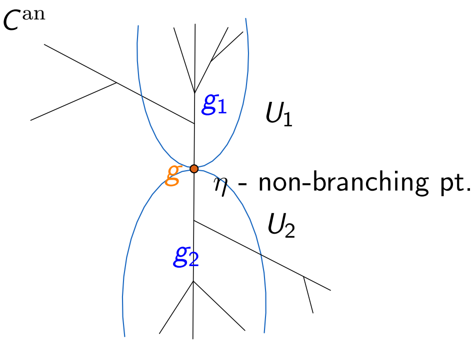

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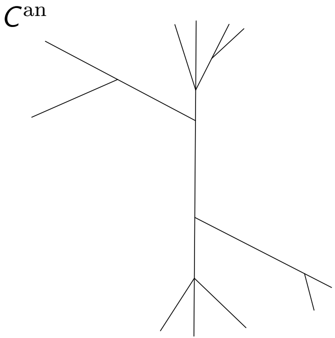
$\forall g \in G(\mathscr{M}(\{\eta\}))$, $\exists g_i \in G(\mathscr{M}(U_i)), i = 1, 2,$ *such that* $g = g_1 \cdot g_2$

Patching and proof of LGP-$\mathscr{M}_x$



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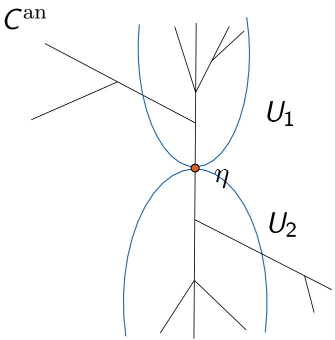

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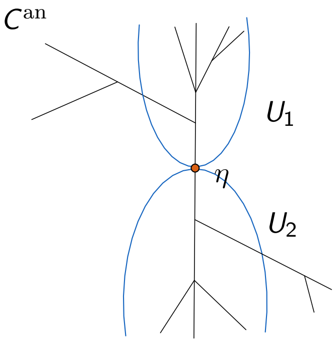

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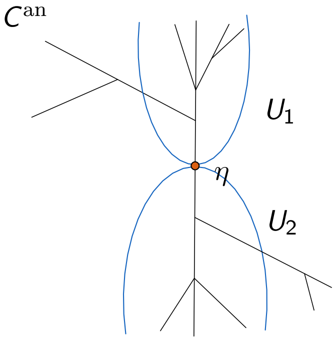

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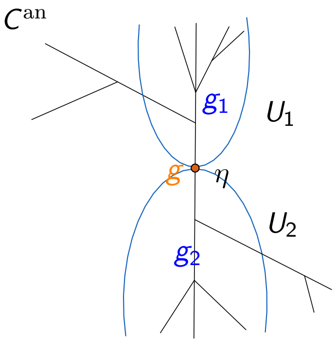

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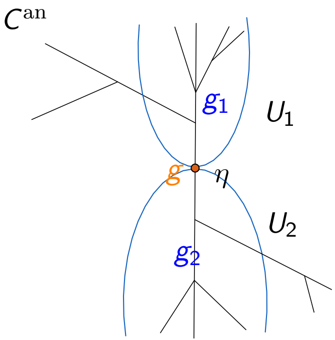

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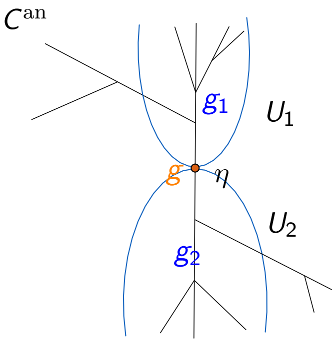

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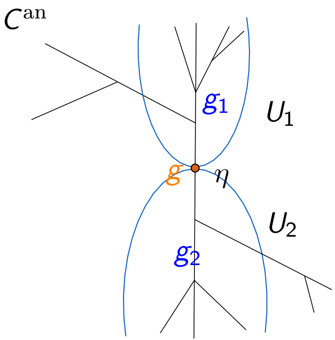

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Goal: Generalize Proposition $(\star)$ to more complicated covers.

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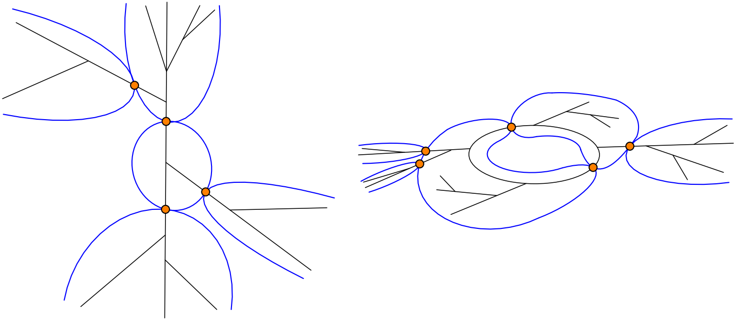

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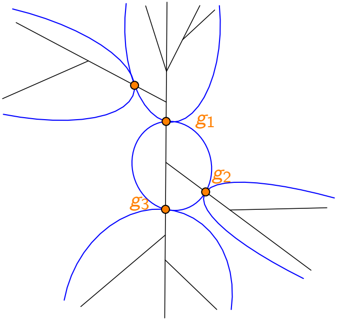

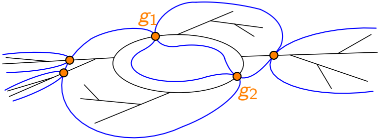

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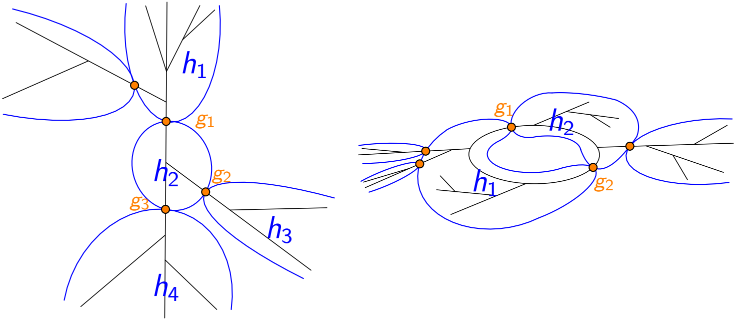

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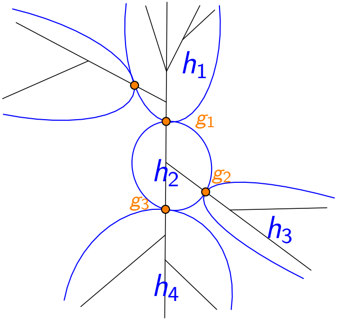

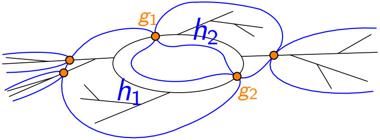

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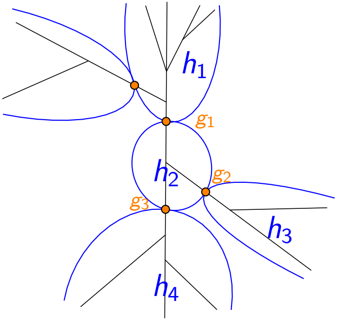

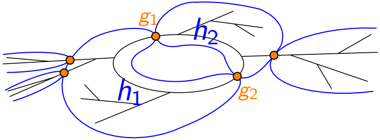

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There exists a bijection $C^{\mathrm{an}} \longleftrightarrow \mathcal{P}_F$, s.t. if $x \mapsto v_x$, then $\widehat{\mathscr{M}_x} = F_{v_x}$, where F_{v_x} is the completion of F w.r.t. v_x.

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If $\mathrm{char}\ k \neq 2$, LGP-val applies to quadratic forms.

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Conjecture CTPS (Colliot-Thélène, Parimala, Suresh '09)

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k-complete discretely valued, C/k a normal irreducible projective curve, $F = k(C)$, V a projective homogeneous variety over a connected lin. alg. group; then

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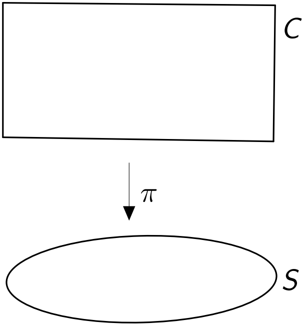
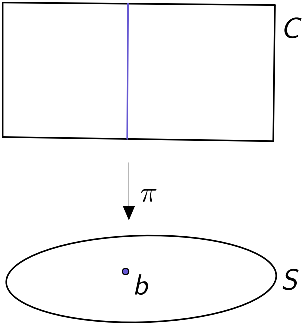
- LGP-disc can be applied to reprove Conjecture CTPS for quadratic forms.

Higher–dimensional patching



- $\pi : C \to S$ a proper relative analytic curve

Higher–dimensional patching



- $\pi : C \to S$ a proper relative analytic curve
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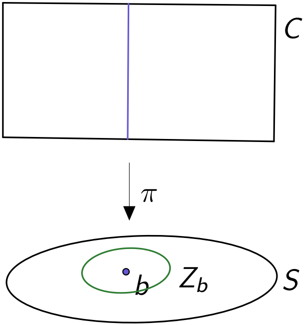

- $\pi : C \to S$ a proper relative analytic curve
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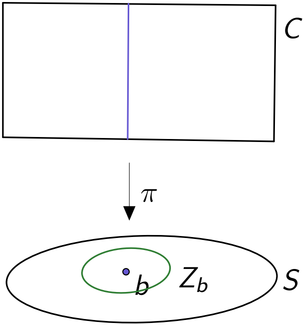

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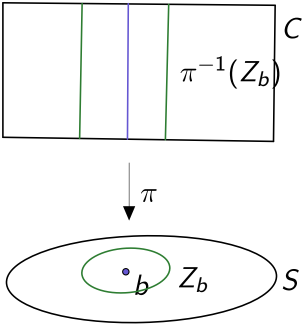

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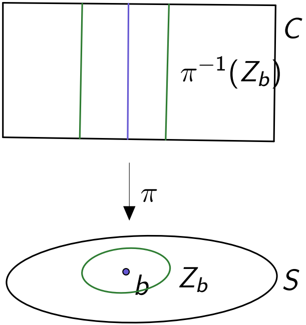

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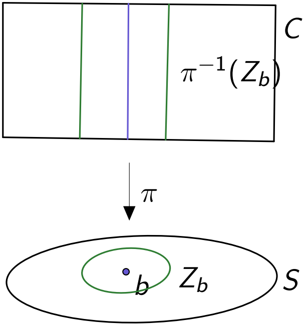

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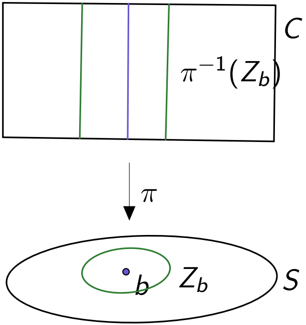

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- both these LGP–hd can be applied to quadratic forms.