Hydrodynamic limits of the nonlinear Schrödinger equation coupled with gauge fields

Webinar kinetic and fluid equations for collective behavior

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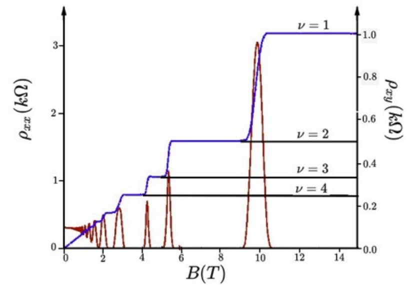
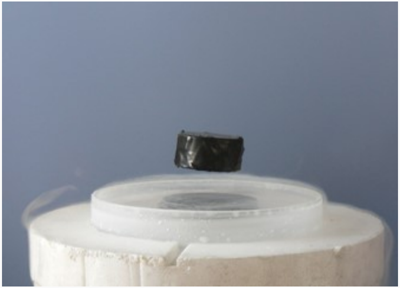
Introduction

Physical motivation





Physical phenomena induced from the interaction between particle and gauge field. Fractional quantum Hall effect (Left) High-temperature superconductivity (Right)

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Gague potentials

- The famous Maxwell equations read as

$$\partial_t E - c^2 \nabla \times B = -\frac{j}{\varepsilon_0}, \quad \nabla \cdot E = \frac{\rho}{\varepsilon_0},$$
$$\partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0,$$

- Noticing the divergence-free condition, consider the vector field A (magnetic potential) such that $B = \nabla \times A$.
- Then, equation for magnetic field becomes

$$\nabla \times (\partial_t A + E) = 0,$$

which inspire the scalar field A_0 (electric potential) such that

$$E = -\partial_t A + \nabla A_0.$$

Gauge potentials

- Substituting the gauge potential to the Maxwell equations, we obtain

$$\partial_t(-\partial_t A + \nabla A_0) - c^2(\nabla\nabla \cdot A - \Delta A) = -\frac{j}{\varepsilon_0},$$
$$\nabla \cdot (-\partial_t A + \nabla A_0) = \frac{\rho}{\varepsilon_0},$$

which can be written in as

$$\left(\Delta A - \frac{1}{c^2}\partial_t^2 A\right) = -\frac{j}{\varepsilon_0 c^2} + \nabla\left(\nabla \cdot A - \frac{1}{c^2}\partial_t A_0\right),$$
$$\left(\Delta A_0 - \frac{1}{c^2}\partial_t^2 A_0\right) = \frac{\rho}{\varepsilon_0} + \partial_t\left(\nabla \cdot A - \frac{1}{c^2}\partial_t A_0\right).$$

Particle in the electromagnetic fields

- Considering the Lorentz force $F = q(E + v \times B)$, the (classical) equation of motion is

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = q(E + v \times B),$$

which can be written in terms of the gauge potential:

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = q(-\partial_t A + \nabla A_0 + v \times (\nabla \times A)).$$

- After using several vector calculus identity, one can show that this is equivalent to

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = q\left(\nabla(A_0 + v \cdot A) - \frac{\mathrm{d}}{\mathrm{d}t}\nabla_v(A_0 + v \cdot A)\right),$$

which can be considered as an Euler-Lagrange equation for the Lagrangian

$$\mathcal{L} = \frac{1}{2}m|v|^2 + q(A_0 + A \cdot v).$$

Hamiltonian formalism

- Since the canonical momentum is

$$p = \partial_v L = mv + qA,$$

corresponding Hamiltonian $\mathcal{H}$ becomes

$$\mathcal{H} = p \cdot v - \mathcal{L} = \frac{1}{2m}|p - qA|^2 - qA_0,$$

i.e.,

$$E + qA_0 = \frac{1}{2m}|p - qA|^2.$$

- Canonical quantization implies the following Schrödinger-type equation:

$$(\mathrm{i}\hbar\partial_t + qA_0)\,\psi = \frac{(-\mathrm{i}\hbar\nabla - qA)^2}{2m}\psi,$$

or

$$\mathrm{i}\hbar\left(\partial_t - \mathrm{i}\frac{q}{\hbar}A_0\right)\psi + \frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}q}{\hbar}A\right)^2\psi = 0.$$

Gauge-involved Schrödinger equations

- By introducing the differential operator $D_\alpha := \partial_\alpha - \frac{\mathrm{i}q}{\hbar} A_\alpha$,

$$\mathrm{i}\hbar D_0 \psi + \frac{\hbar^2}{2m} D^2 \psi = 0,$$

  which is the Schrödinger equation describing the dynamics of the particle interacting with the (Maxwell) gauge fields A_α.

Chern-Simons-Schrödinger equations

Chern-Simons-Schrödinger equations

- The Chern-Simons-Schrödinger (CSS) equations describes the dynamics of two-dimensional electron system.
- The CSS equations is given by

$$
\begin{aligned}
&\mathrm{i}\hbar D_0\psi + \frac{\hbar^2}{2m}(D_1 D_1 + D_2 D_2)\psi - V'(|\psi|^2)\psi = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
&\partial_0 A_1 - \partial_1 A_0 = -\hbar \mathrm{Im}(\bar{\psi} D_2 \psi), \quad \partial_0 A_2 - \partial_2 A_0 = \hbar \mathrm{Im}(\bar{\psi} D_1 \psi), \\
&\partial_1 A_2 - \partial_2 A_1 = -m|\psi|^2.
\end{aligned}
$$

- $\partial_0 = \partial_t, \quad \partial_i = \partial_{x_i},$
- $\psi : \mathbb{R}_+ \times \Omega \to \mathbb{C}$:Complex scalar field,
- $A_\mu : \mathbb{R}_+ \times \Omega \to \mathbb{R}$: Gague field,
- $D_\mu = \partial_\mu + \frac{\mathrm{i}}{\hbar} A_\mu$: covariant derivative,
- V: Self-interacting potenital energy density.

Chern-Simons-Schrödinger equations

- The CSS equation is invariant under the gauge transform:

$$\psi \to \psi e^{i\chi}, \quad A_\mu \to A_\mu - \hbar \partial_\mu \chi.$$

- Therefore, one need to give a gauge condition. Usually, one consider the Coulomb gauge condition $\nabla \cdot A = \partial_1 A_1 + \partial_2 A_2 = 0$.
- One can also consider the other gauge condition:
  -Temporal gauge condition: $A_0 = 0$,
  -Lorentz gauge condition: $\partial_\mu A_\mu = 0$.

Chern-Simons-Schrödinger equations

- Under the Coulomb gauge condition, the CSS equation becomes

$$
\begin{aligned}
&\mathrm{i}\hbar\partial_t\psi - A_0\psi + \frac{\hbar^2}{2m}\left(\Delta\psi + \frac{2\mathrm{i}}{\hbar}A\cdot\nabla\psi - \frac{1}{\hbar^2}|A|^2\psi\right) - V'(|\psi|^2)\psi = 0,\\
&\Delta A_0 = \hbar\mathrm{Im}(Q_{12}(\bar{\psi},\psi)) + \partial_1(A_2|\psi|^2) - \partial_2(A_1|\psi|^2),\\
&\Delta A_1 = m\partial_2|\psi|^2, \quad \Delta A_2 = -m\partial_1|\psi|^2,
\end{aligned}
$$

  where $Q_{12}(\bar{\psi},\psi) := \partial_1\bar{\psi}\partial_2\psi - \partial_2\bar{\psi}\partial_1\psi$.

- Choosing $m = 1$ and $\hbar = \varepsilon$, we have the family of the scaled CSS equations:

$$
\begin{aligned}
&\mathrm{i}\varepsilon\partial_t\psi - A_0\psi + \frac{\hbar^2}{2}\left(\Delta\psi + \frac{2\mathrm{i}}{\varepsilon}A\cdot\nabla\psi - \frac{1}{\varepsilon^2}|A|^2\psi\right) - V'(|\psi|^2)\psi = 0,\\
&\Delta A_0 = \varepsilon\mathrm{Im}(Q_{12}(\bar{\psi},\psi)) + \partial_1(A_2|\psi|^2) - \partial_2(A_1|\psi|^2),\\
&\Delta A_1 = \partial_2|\psi|^2, \quad \Delta A_2 = -\partial_1|\psi|^2,
\end{aligned}
$$

Conservation laws

- The CSS system conserves the total charge and the total energy. Define

$$\mathcal{Q}(t) := \int_\Omega |\psi^\varepsilon|^2 \,\mathrm{d}x,$$

$$\mathcal{E}^\varepsilon(t) := \int_\Omega \frac{\varepsilon^2}{2} \sum_{j=1}^{2} |D_j^\varepsilon \psi^\varepsilon(t,x)|^2 + V(|\psi^\varepsilon(t,x)|^2)\,\mathrm{d}x,$$

  where $D_j^\varepsilon := \partial_j + \frac{\mathrm{i}}{\varepsilon} A_j^\varepsilon$.

- Then,

$$\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{E}^\varepsilon}{\mathrm{d}t} = 0.$$

Hydrodynamic formulation : Madelung transformation

- Considering the Madelung transformation

$$\psi^\varepsilon = \sqrt{\rho^\varepsilon} \exp\left(\frac{\mathrm{i}}{\varepsilon} S^\varepsilon\right),$$

we introduce the hydrodynamic variables

$$\rho^\varepsilon = |\psi^\varepsilon|^2, \quad \rho^\varepsilon u^\varepsilon := \rho^\varepsilon(\nabla S^\varepsilon + A^\varepsilon) = \frac{\mathrm{i}\varepsilon}{2}(\psi^\varepsilon \nabla \bar{\psi}^\varepsilon - \bar{\psi}^\varepsilon \nabla \psi^\varepsilon) + |\psi^\varepsilon|^2 A^\varepsilon.$$

- Then, the imaginary part of the Schrödinger equation becomes

$$\partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0,$$

which corresponds to the continuity equation in the classical mechanics.

Hydrodynamic formulation

- On the other hand, the real part of the Schrödinger equation becomes

$$\partial_t S^\varepsilon + A_0^\varepsilon + \frac{1}{2}|\nabla S^\varepsilon + A^\varepsilon|^2 + V'(\rho^\varepsilon) = \frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}.$$

- Taking gradient,

$$\partial_t(\nabla S^\varepsilon) + (u^\varepsilon \cdot \nabla)u^\varepsilon + \rho^\varepsilon (u^\varepsilon)^\perp + \nabla A_0^\varepsilon + \frac{\nabla p(\rho^\varepsilon)}{\rho^\varepsilon} = \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right),$$

where $p(\rho) = \rho V'(\rho) - V(\rho)$. Choosing $V = \frac{1}{\gamma}\rho^\gamma$, $p(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$.

- Using the gauge equation, one can derive

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \frac{\nabla p(\rho^\varepsilon)}{\rho^\varepsilon} = \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right).$$

Hydrodynamic formulation

- To sum up, we have the following hydrodynamic system:

$$\partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0,$$

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla p(\rho^\varepsilon) = \frac{\rho^\varepsilon \varepsilon^2}{2} \nabla \left(\frac{\Delta \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right),$$

$$\Delta A_0^\varepsilon = \nabla \times (\rho^\varepsilon u^\varepsilon), \quad \Delta A^\varepsilon = -(\nabla \rho^\varepsilon)^\perp.$$

- As $\varepsilon \to 0$, the hydrodynamic equations formally converges to the Euler-Chern-Simons equations:

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = 0,$$

$$\Delta A_0 = \nabla \times (\rho u), \quad \Delta A = -(\nabla \rho)^\perp.$$

- The main concern is to provide a rigorous analysis for this convergence.

Hydrodynamic limit of the CSS system

- Consider the well-prepared initial data condition:

$$\int_\Omega \frac{\rho_0^\varepsilon |u_0^\varepsilon - u_0|^2}{2}\,\mathrm{d}x + \int_\Omega \frac{p(\rho_0^\varepsilon|\rho_0)}{\gamma-1}\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\sqrt{\rho_0^\varepsilon}|^2\,\mathrm{d}x = \mathcal{O}(\varepsilon^\lambda),$$

where $p(n|\rho) := \frac{\gamma-1}{\gamma}(n^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(n-\rho))$.

Theorem

Suppose $\gamma > 1$. Let $(\psi^\varepsilon, A_0^\varepsilon, A^\varepsilon)$ be the global solution to the CSS equations. Moreover, let (ρ, u, A_0, A) be the unique local-in-time smooth solution to the Euler-Chern-Simons equations for $0 \le t \le T_*$.

Hydrodynamic limit of the CSS system

Theorem (continued)

Then, for any $0 \le t \le T_*$, we have

$$
\begin{aligned}
&\rho^\varepsilon(t,\cdot) \to \rho(t,\cdot), \quad \text{in} \quad L^\gamma(\Omega),\\
&(\rho^\varepsilon u^\varepsilon)(t,\cdot) \to (\rho u)(t,\cdot), \quad \text{in} \quad L^{\frac{2\gamma}{\gamma+1}}(\Omega),\\
&(\sqrt{\rho^\varepsilon} u^\varepsilon)(t,\cdot) \to (\sqrt{\rho} u)(t,\cdot), \quad \text{in} \quad L^2(\Omega),\\
&A_0^\varepsilon \to A_0, \quad \text{in} \quad L^{2\gamma}(\Omega), \quad \nabla A_0^\varepsilon \to \nabla A_0 \quad \text{in} \quad L^{\frac{2\gamma}{\gamma+1}}(\Omega),\\
&A^\varepsilon \to A, \quad \text{in} \quad L^{2\gamma}(\Omega), \quad \nabla A^\varepsilon \to \nabla A, \quad \text{in} \quad L^\gamma(\Omega),
\end{aligned}
$$

as $\varepsilon \to 0$.

Relative entropy

- To obtain a hydrodynamic limit (of the classical systems), the relative entropy method is successful.
- Consider the following general system of conservation laws:

$$\partial_t U_i + \sum_{k=1}^{d} \partial_k A_{ik}(U) = 0, \quad U \in \mathbb{R}^m, \quad A \in \mathbb{R}^{m\times d}.$$

- The compressible Euler equation can be written in this form with $U = (\rho, \rho u)$ and

$$A(U) = \frac{1}{\rho}\begin{pmatrix} \rho P_1 & \rho P_2 \\ P_1^2 + \frac{\gamma-1}{\gamma}\rho^{\gamma+1} & P_1 P_2 \\ P_2 P_1 & P_2^2 + \frac{\gamma-1}{\gamma}\rho^{\gamma+1} \end{pmatrix} = \begin{pmatrix} \rho u^\top \\ \rho u \otimes u + \frac{\gamma-1}{\gamma}\rho^\gamma I_2 \end{pmatrix}.$$

Relative entropy

- A usual entropy defined for the compressible Euler equation is

$$\eta(U) := \frac{|P|^2}{2\rho} + \frac{\rho^\gamma}{\gamma} = \frac{\rho|u|^2}{2} + \frac{\rho^\gamma}{\gamma}.$$

- Corresponding relative entropy and relative flux is given as

$$\eta(V|U) := \eta(V) - \eta(U) - D\eta(U) \cdot (V - U),$$
$$A(V|U) := A(V) - A(U) - DA(U) \cdot (V - U).$$

Here,

$$[DA(U) \cdot (V - U)]_{ij} := \sum_{k=1}^{3} \partial_{U_k} A_{ij}(U)(V_k - U_k).$$

Relative entropy method

- The relative entropy method is based on the following key estimate on the relative entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\eta(V|U)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\eta(V)\,\mathrm{d}x - \int_{\mathbb{R}^d}\nabla_x(D\eta(U)) : A(V|U)\,\mathrm{d}x \\ - \int_{\mathbb{R}^d} D\eta(U)\cdot(\partial_t V + \nabla_x\cdot A(V))\,\mathrm{d}x.$$

- We note that the energy functional $\mathcal{E}$ can be written in terms of the hydrodynamic quantities:

$$\mathcal{E}^\varepsilon = \frac{1}{2}\rho^\varepsilon|u^\varepsilon|^2 + \frac{(\rho^\varepsilon)^\gamma}{\gamma} + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho^\varepsilon}|^2 = \eta(U^\varepsilon) + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho^\varepsilon}|^2.$$

Modulated energy

- On the other hand, the hydrodynamic limit of the quantum system is based on the modulated energy estimate.
- The natural modulated energy is

$$\begin{aligned}\mathcal{H}^\varepsilon(t) &:= \int_\Omega \frac{1}{2}|(\varepsilon D^\varepsilon - \mathrm{i}u)\psi^\varepsilon|^2 + \frac{p(\rho^\varepsilon|\rho)}{\gamma-1}\,\mathrm{d}x \\ &= \int_\Omega \frac{1}{2}|(\varepsilon D^\varepsilon - \mathrm{i}u)\psi^\varepsilon|^2 + \frac{(\rho^\varepsilon)^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(\rho^\varepsilon - \rho)}{\gamma}\,\mathrm{d}x.\end{aligned}$$

  After tedious computation, we find

$$\mathcal{H}^\varepsilon = \int_\Omega \eta(U^\varepsilon|U)\,\mathrm{d}x + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho^\varepsilon}|^2\,\mathrm{d}x.$$

- Therefore, the modulated energy and the relative entropy are almost the same quantity, except for the "quantum term".

Modulated energy estimate

- Using the equivalent relation between modulated energy and the relative entropy, one can use the celebrated theory of relative entropy to modulated energy of the CSS equations.

Proposition

Let $(\psi^\varepsilon, A_0^\varepsilon, A^\varepsilon)$ be the solution to the CSS equations and (ρ, u) be the unique local-in-time smooth solution to the compressible Euler equation. Then,

$$\mathcal{H}^\varepsilon(t) \leq C\varepsilon^{\min\{\lambda,2\}}.$$

- The proof is based on the previous proposition on the relative entropy, and an appropriate estimate for the quantum correction term.
- Therefore, one can conclude that the modulated energy vanishes as $\varepsilon \to 0$.

Proof of Proposition

- We estimate $\frac{d\mathcal{H}^\varepsilon}{dt}$ as

$$\begin{aligned}\frac{d\mathcal{H}^\varepsilon}{dt} &= \frac{d}{dt}\mathcal{E}^\varepsilon - \int_\Omega \nabla_x(D\eta(U)) : A(U^\varepsilon|U)\,dx \\ &\quad - \int_\Omega D\eta(U)\cdot(\partial_t U^\varepsilon + \nabla_x \cdot A(U^\varepsilon))\,dx \\ &= 0 + I_1 + I_2.\end{aligned}$$

- Using the definition of $D\eta$ and $A(U^\varepsilon|U)$, we have

$$I_1 \le C\left(\int_\Omega \rho^\varepsilon|u^\varepsilon - u|^2\,dx + \int_\Omega p(\rho^\varepsilon|\rho)\,dx\right) \le \int_\Omega \eta(U^\varepsilon|U)\,dx.$$

- On the other hand, using the governing equation of U^ε,

$$I_2 = -\frac{\varepsilon^2}{2}\int_\Omega \rho^\varepsilon u\cdot\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right)dx \le C\varepsilon^2\int_\Omega |\nabla\sqrt{\rho^\varepsilon}|^2\,dx + C\varepsilon^2.$$

Proof of Proposition

- Combining the estimates, we have
$$\frac{\mathrm{d}\mathcal{H}^\varepsilon}{\mathrm{d}t} \le C\mathcal{H}^\varepsilon + C\varepsilon^2.$$
- Gronwall's inequality and the assumption of well-prepared initial data imply the desired estimate.
- With the modulated energy estimate in hand, one can obtain the desired convergence.

Lemma

Let $\gamma > 1$ be a constant. Then,

$$p(\rho^\varepsilon|\rho) \ge C \min\{(\rho^\varepsilon)^{\gamma-2}, \rho^{\gamma-2}\}(\rho^\varepsilon - \rho)^2,$$

and

$$p(\rho^\varepsilon|\rho) \ge C \begin{cases} (\rho^\varepsilon - \rho)^2, & \text{if} \quad \frac{\rho}{2} \le \rho^\varepsilon \le 2\rho, \\ (1 + (\rho^\varepsilon)^\gamma), & \text{o.w.} \end{cases}$$

Proof of Theorem

- Convergence of the density:

$$\int_{\mathbb{R}^3} |\rho^\varepsilon - \rho|^\gamma \, dx = \int_{\{\frac{\rho}{2} \le \rho^\varepsilon \le 2\rho\}} |\rho^\varepsilon - \rho|^\gamma \, dx + \int_{\{\frac{\rho}{2} \le \rho^\varepsilon \le 2\rho\}^c} |\rho^\varepsilon - \rho|^\gamma \, dx$$
$$= \mathcal{I}_1 + \mathcal{I}_2.$$

- If $1 < \gamma < 2$,

$$\mathcal{I}_1 \le \left(\int \min\{(\rho^\varepsilon)^{\gamma-2}, \rho^{\gamma-2}\} |\rho^\varepsilon - \rho|^2 \right)^{\gamma/2} \left(\int \max\{(\rho^\varepsilon)^\gamma, \rho^\gamma\} \right)^{\frac{2-\gamma}{2}}$$
$$\le C \, (\mathcal{H}^\varepsilon)^{\gamma/2} \to 0,$$

and if $\gamma \ge 2$,

$$\mathcal{I}_1 \le \int |\rho^\varepsilon - \rho|^2 |\rho^\varepsilon - \rho|^{\gamma-2} \le C \int |\rho^\varepsilon - \rho|^2 \le C\mathcal{H}^\varepsilon \to 0.$$

- Moreover $\mathcal{I}_2$ can be estimated as

$$\mathcal{I}_2 \le \int \|\rho\|_{L^\infty}^\gamma \left| \frac{\rho^\varepsilon}{\rho} + 1 \right|^\gamma \le C \int (1 + (\rho^\varepsilon)^\gamma) \le C\mathcal{H}^\varepsilon \to 0.$$

Proof of Theorem

- Convergence of the momentum:

$$
\begin{aligned}
\|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^{\frac{2\gamma}{\gamma+1}}} &\le \|\rho^\varepsilon (u^\varepsilon - u)\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|(\rho^\varepsilon - \rho) u\|_{L^{\frac{2\gamma}{\gamma+1}}} \\
&\le \|\sqrt{\rho^\varepsilon}\|_{L^{2\gamma}} \|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + \|\rho^\varepsilon - \rho\|_{L^\gamma} \|u\|_{L^{\frac{2\gamma}{\gamma-1}}} \\
&\le C\|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + C\|\rho^\varepsilon - \rho\|_{L^\gamma} \le C\mathcal{H}^\varepsilon \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\sqrt{\rho^\varepsilon} u^\varepsilon - \sqrt{\rho} u\|_{L^2} &\le \|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + \|(\sqrt{\rho^\varepsilon} - \sqrt{\rho})|u|\|_{L^2} \\
&\le \|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + \|u\|_{L^{\frac{2\gamma}{\gamma-1}}} \|\sqrt{\rho^\varepsilon} - \sqrt{\rho}\|_{L^{2\gamma}} \\
&\le \mathcal{H}^\varepsilon + C\|\rho^\varepsilon - \rho\|_{L^\gamma}^{1/2} \to 0.
\end{aligned}
$$

Proof of Theorem

- To prove the convergence of the gauge fields, we recall that

$$\Delta(A_0^\varepsilon - A_0) = \partial_1(\rho^\varepsilon u_2^\varepsilon - \rho u_2) - \partial_2(\rho^\varepsilon u_1^\varepsilon - \rho u_1).$$

- Using HLS inequality and CZ inequality,

$$\|A_0^\varepsilon - A_0\|_{L^{2\gamma}} \leq \|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^{\frac{2\gamma}{\gamma+1}}} \to 0,$$

and

$$\|\nabla(A_0^\varepsilon - A_0)\|_{L^{\frac{2\gamma}{\gamma+1}}} \leq \|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^{\frac{2\gamma}{\gamma+1}}} \to 0.$$

Proof of Theorem

- Similarly, the gauge difference $A^\varepsilon - A$ satisfies

$$\Delta(A^\varepsilon - A) = (\nabla(\rho - \rho^\varepsilon))^\perp,$$

which implies

$$\begin{aligned}\|A^\varepsilon - A\|_{L^{2\gamma}} &\le \|\rho^\varepsilon - \rho\|_{L^{\frac{2\gamma}{\gamma+1}}} \le (\|\sqrt{\rho^\varepsilon}\|_{L^2} + \|\sqrt{\rho}\|_{L^2})\|\sqrt{\rho^\varepsilon} - \sqrt{\rho}\|_{L^{2\gamma}} \\ &\le C\|\rho^\varepsilon - \rho\|_{L^\gamma} \to 0,\end{aligned}$$

and

$$\|\nabla(A^\varepsilon - A)\|_{L^\gamma} \le \|\rho^\varepsilon - \rho\|_{L^\gamma} \to 0.$$

Remarks

- A global-in-time well-posedness of the nonlinear CSS system is recently guaranteed.
- For the case of Lorenz gauge condition, the hydrodynamic formulation is the same as the case of Coulomb gauge condition. However, since the gauge equation becomes time-dependent equation for the Lorenz gauge condition, one might need an extra well-prepared assumption on the initial gauge fields.

Maxwell-Schrödinger equations

Maxwell-Schrödinger equations

- Unlike the CSS equations, Maxwell-Schrödinger (MS) equations describe the dynamics of the particle in three-dimensional space interacting with the electromagnetic fields.
- The MS equations is given by

$$\begin{aligned} &\mathrm{i}\hbar D_0\psi + \frac{\hbar^2}{2} D^2\psi - V'(|\psi|^2)\psi = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ &\Box A_0 - \partial_t(\partial_t A_0 - \nabla \cdot A) = -|\psi|^2, \\ &\Box A - \nabla(\partial_t A_0 - \nabla \cdot A) = \hbar \mathrm{Im}(\overline{\psi} D\psi), \end{aligned}$$

  where $\Box = \partial_t^2 - \Delta$ is d'Alembertian and $D_\alpha = \partial_\alpha - \frac{\mathrm{i}}{\hbar} A_\alpha$.
- MS system is also gauge invariant and therefore, one need to provide an appropriate gauge condition.

Maxwell-Schrödinger equations

- Under the Coulomb gauge condition $\nabla \cdot A = 0$ and $\hbar = \varepsilon$ and $m = 1$, one again obtains the following family of system:

$$\mathrm{i}\varepsilon\partial_t\psi^\varepsilon + A_0^\varepsilon\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - \mathrm{i}\varepsilon A^\varepsilon \cdot \nabla\psi^\varepsilon - \frac{1}{2}|A^\varepsilon|^2\psi^\varepsilon - V'(|\psi|^2)\psi = 0,$$
$$\Delta A_0^\varepsilon = |\psi^\varepsilon|^2, \quad \Box A^\varepsilon - \nabla\partial_t A_0^\varepsilon = \varepsilon \mathrm{Im}(\overline{\psi^\varepsilon} D^\varepsilon \psi^\varepsilon).$$

- As in the introduction, we may define the electromagnetic fields E^ε and B^ε as

$$E^\varepsilon := -\partial_t A^\varepsilon + \nabla A_0^\varepsilon, \quad B^\varepsilon = \nabla \times A^\varepsilon.$$

Then, the gauge equations become the Maxwell equation:

$$\partial_t E = \nabla \times B^\varepsilon - \rho^\varepsilon u^\varepsilon, \quad \nabla \cdot E^\varepsilon = \rho^\varepsilon,$$
$$\partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, \quad \nabla \cdot B^\varepsilon = 0,$$

where $\rho^\varepsilon := |\psi^\varepsilon|^2$ and $\rho^\varepsilon u^\varepsilon := \varepsilon \mathrm{Im}\left(\overline{\psi^\varepsilon} D^\varepsilon \psi^\varepsilon\right)$.

Conservation laws

- MS equations also have the conservation laws for the total charge and the total energy.
- Define $\mathcal{Q}^\varepsilon$ and $\mathcal{E}^\varepsilon$ as

$$\mathcal{Q}^\varepsilon(t) := \int_{\mathbb{R}^3} |\psi^\varepsilon|^2 \, \mathrm{d}x,$$

$$\mathcal{E}^\varepsilon(t) := \int_{\mathbb{R}^3} \frac{1}{2}|\varepsilon D^\varepsilon \psi^\varepsilon|^2 + V(|\psi^\varepsilon|^2) + \frac{1}{2}(|E^\varepsilon|^2 + |B^\varepsilon|^2) \, \mathrm{d}x.$$

Hydrodynamic formulation: Direct computation

- Although one can derive the hydrodynamic equations for the MS equations via Madelung transform as in the case of CSS equations, one also can derive from the system equations itself.
- For example, multiplying $\overline{\psi^\varepsilon}$ to the Schrödinger equation and taking the imaginary part yields

$$\partial_t|\psi^\varepsilon|^2 + \varepsilon \mathrm{Im}\left(\overline{\psi^\varepsilon}\Delta\psi^\varepsilon\right) - A^\varepsilon \cdot \nabla|\psi^\varepsilon|^2 = 0,$$

which is equivalent to

$$\partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0.$$

Hydrodynamic formualtion

- Also, after tedius computation, we obtain the equation for the momentum as

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \frac{\varepsilon^2}{4}\nabla\cdot\big(2(D^\varepsilon\psi^\varepsilon \otimes \overline{D^\varepsilon\psi^\varepsilon} + \overline{D^\varepsilon\psi^\varepsilon}\otimes D^\varepsilon\psi^\varepsilon) - \nabla^2|\psi^\varepsilon|^2\big)$$
$$+ \nabla p(\rho^\varepsilon) = \rho^\varepsilon(E^\varepsilon + u^\varepsilon \times B^\varepsilon).$$

- Direct computation yields

$$\frac{\varepsilon^2}{4}\nabla\cdot\big(2(D^\varepsilon\psi^\varepsilon \otimes \overline{D^\varepsilon\psi^\varepsilon} + \overline{D^\varepsilon\psi^\varepsilon}\otimes D^\varepsilon\psi^\varepsilon) - \nabla^2|\psi^\varepsilon|^2\big)$$
$$= \nabla\cdot(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \frac{\varepsilon^2\rho^\varepsilon}{2}\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right)$$

- Therefore, the momenutum equation becomes

$$\partial_t(\rho^\varepsilon u^\varepsilon)+\nabla\cdot(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon)+\nabla p(\rho^\varepsilon) = \rho^\varepsilon(E^\varepsilon+u^\varepsilon\times B^\varepsilon)+\frac{\varepsilon^2\rho^\varepsilon}{2}\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right).$$

Hydrodynamic formulation

- In conclusion, the MS equations can be written as the following form:

$$
\begin{aligned}
&\partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0, \\
&\partial_t (\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla p(\rho^\varepsilon) = \rho^\varepsilon (E^\varepsilon + u^\varepsilon \times B^\varepsilon) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{\varepsilon^2 \rho^\varepsilon}{2} \nabla \left(\frac{\Delta \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right), \\
&\partial_t E^\varepsilon = \nabla \times B^\varepsilon - \rho^\varepsilon u^\varepsilon, \quad \partial_t B^\varepsilon = -\nabla \times E^\varepsilon, \\
&\nabla \cdot E^\varepsilon = \rho^\varepsilon, \quad \nabla \cdot B^\varepsilon = 0.
\end{aligned}
$$

- Therefore, in a formal limit, the system converges to the Euler-Maxwell equations:

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
&\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = \rho (E + u \times B), \\
&\partial_t E = \nabla \times B - \rho u, \quad \partial_t B = -\nabla \times E, \\
&\nabla \cdot E = \rho, \quad \nabla \cdot B = 0.
\end{aligned}
$$

A priori assumptions

- We assume the following assumptions:
  - ($\mathcal{H}1$): The initial data is well-prepared:

$$\mathcal{H}^\varepsilon(0) = \mathcal{O}(\varepsilon^\lambda).$$

  - ($\mathcal{H}2$): The local charge is positive:

$$|\psi^\varepsilon(t,x)|^2 > 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

  - ($\mathcal{H}3$): The local charge is uniformly bounded:

$$\sup_{t \geq 0} \|\psi^\varepsilon(t)\|_{L^\infty} < C.$$

- Note that the third assumption can be deleted once one can show that ε-independent estimate for $\|\psi^\varepsilon\|_{H^{3/2+\delta}}$.

Hydrodynamic limit of the MS system

Theorem

Suppose $1 < \gamma \leq 2$. Let $(\psi^\varepsilon, A_0^\varepsilon, A^\varepsilon)$ be the global solution to the MS equations. Moreover, let (ρ, u, E, B) be the unique local-in-time smooth solution to the Euler-Maxwell equations for $0 \leq t \leq T_*$. Suppose that the *a priori* assumptions $(\mathcal{H}1)$–$(\mathcal{H}3)$ holds. Then, for any $0 \leq t \leq T_*$, we have

$$
\begin{aligned}
&\rho^\varepsilon(t,\cdot) \to \rho(t,\cdot), \quad \text{in} \quad L^2(\mathbb{R}^3),\\
&(\sqrt{\rho^\varepsilon} u^\varepsilon)(t,\cdot) \to (\sqrt{\rho} u)(t,\cdot), \quad \text{in} \quad L^2(\mathbb{R}^3),\\
&E^\varepsilon \to E, \quad \text{in} \quad L^2(\mathbb{R}^3),\\
&B^\varepsilon \to B, \quad \text{in} \quad L^2(\mathbb{R}^3),
\end{aligned}
$$

as $\varepsilon \to 0$.

Modulated energy

- Again, we use the modulated energy to obtain a desired convergence.
- The natural modulated energy is

$$
\begin{aligned}
\mathcal{H}^\varepsilon(t) :=& \int_{\mathbb{R}^3} \frac{1}{2}|(\varepsilon D^\varepsilon - \mathrm{i}u)\psi^\varepsilon|^2 + \frac{p(\rho^\varepsilon|\rho)}{\gamma-1}\,\mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}^3} (|E^\varepsilon - E|^2 + |B^\varepsilon - B|^2)\,\mathrm{d}x \\
=& \int_{\mathbb{R}^3} \frac{\rho^\varepsilon|u^\varepsilon - u|^2}{2}\,\mathrm{d}x + \int_{\mathbb{R}^3} \frac{p(\rho^\varepsilon|\rho)}{\gamma-1}\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^3} |\nabla\sqrt{\rho^\varepsilon}|^2\,\mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}^3} (|E^\varepsilon - E|^2 + |B^\varepsilon - B|^2)\,\mathrm{d}x.
\end{aligned}
$$

Modulated energy estimate

- Then, the main proposition is to estimate the modulated energy.

Proposition

Under the same assumption with the main theorem, we have,

$$\mathcal{H}^{\varepsilon}(t) \leq C\varepsilon^{\min\{\lambda,2\}}.$$

- Unlike the CSS system, we may not use the relative entropy method, since the limit system (Euler-Maxwell system) is not a conservative form, due to the presence of the electromagnetic equations.
- Instead, we directly compute the time derivative of the modulated energy to derive the desired estimate.

Proof of Proposition

- The modulated energy can be split as

$$
\begin{aligned}
\mathcal{H}^\varepsilon(t) = \mathcal{E}^\varepsilon(t) &+ \frac{\varepsilon \mathrm{i}}{2}\int_{\mathbb{R}^3} u\cdot\left(D^\varepsilon\psi^\varepsilon\overline{\psi^\varepsilon} - \overline{D^\varepsilon\psi^\varepsilon}\psi^\varepsilon\right)\mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}^3}\rho^\varepsilon|u|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}\rho^{\gamma-1}\left(\rho - \frac{\gamma}{\gamma-1}\rho^\varepsilon\right)\mathrm{d}x \\
&- \int_{\mathbb{R}^3}(E^\varepsilon\cdot E + B^\varepsilon\cdot B)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}(|E|^2+|B|^2)\,\mathrm{d}x \\
=:\ & \mathcal{E}^\varepsilon(t) + \sum_{i=1}^{5}\mathcal{J}_i
\end{aligned}
$$

- Thanks to the energy conservation law, we only need to estimate the last 5 terms.

Proof of Proposition

- Since $\frac{\varepsilon \mathrm{i}}{2}(\overline{D^\varepsilon\psi^\varepsilon}\psi^\varepsilon - D^\varepsilon\psi^\varepsilon\overline{\psi^\varepsilon}) = \rho^\varepsilon u^\varepsilon$, $\mathcal{J}_1$ can be estimated as

$$\mathcal{J}_1 = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho^\varepsilon u^\varepsilon\cdot u\,\mathrm{d}x = -\int_{\mathbb{R}^3}\partial_t(\rho^\varepsilon u^\varepsilon)\cdot u\,\mathrm{d}x - \int_{\mathbb{R}^3}\rho^\varepsilon u^\varepsilon\cdot\partial_t u\,\mathrm{d}x.$$

- Using the governing equations and several vector calculus identities, we derive

$$\begin{aligned}\mathcal{J}_1 = &\int_{\mathbb{R}^3}\rho^\varepsilon(u^\varepsilon\otimes(u-u^\varepsilon)):\nabla u\,\mathrm{d}x + \frac{\gamma}{\gamma-1}\int_{\mathbb{R}^3}\nabla(\rho^{\gamma-1})\cdot\rho^\varepsilon u^\varepsilon\,\mathrm{d}x\\ &-\int_{\mathbb{R}^3}(\rho^\varepsilon)^\gamma(\nabla\cdot u)\,\mathrm{d}x - \int_{\mathbb{R}^3}\rho^\varepsilon u^\varepsilon\cdot E\,\mathrm{d}x - \int_{\mathbb{R}^3}\rho^\varepsilon u\cdot E^\varepsilon\,\mathrm{d}x\\ &+\int_{\mathbb{R}^3}\rho^\varepsilon(u^\varepsilon-u)\cdot(u\times(B^\varepsilon-B))\,\mathrm{d}x - \frac{\varepsilon^2}{2}\int_{\mathbb{R}^3}\rho^\varepsilon\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right)\cdot u\,\mathrm{d}x.\end{aligned}$$

Proof of Proposition

- Similarly, using the governing equations,

$$\begin{aligned}\mathcal{J}_2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho^\varepsilon |u|^2\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3}(\partial_t\rho^\varepsilon)|u|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}\rho^\varepsilon u\cdot u_t\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}(\rho^\varepsilon u\otimes(u^\varepsilon - u)) : \nabla u\,\mathrm{d}x - \gamma\int_{\mathbb{R}^3}\rho^\varepsilon\rho^{\gamma-2}u\cdot\nabla u\,\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^3}\rho^\varepsilon u\cdot E\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_3 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho^{\gamma-1}\left(\rho - \frac{\gamma}{\gamma-1}\right)\mathrm{d}x\\ &= -(\gamma-1)\int_{\mathbb{R}^3}\rho^\gamma\nabla\cdot u\,\mathrm{d}x + \gamma\int_{\mathbb{R}^3}\rho^{\gamma-2}(\rho^\varepsilon u\cdot\nabla\rho + \rho\rho^\varepsilon\nabla\cdot u)\,\mathrm{d}x\\ &\quad - \frac{\gamma}{\gamma-1}\int_{\mathbb{R}^3}\nabla(\rho^{\gamma-1})\cdot\rho^\varepsilon u^\varepsilon\,\mathrm{d}x.\end{aligned}$$

Proof of Proposition

- Using the Maxwell equations to derive

$$\mathcal{J}_4 = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(E^\varepsilon \cdot E + B^\varepsilon \cdot B)\,\mathrm{d}x = \int_{\mathbb{R}^3}\rho^\varepsilon u^\varepsilon \cdot E\,\mathrm{d}x + \int_{\mathbb{R}^3}\rho u \cdot E^\varepsilon\,\mathrm{d}x,$$

and

$$\begin{aligned}\mathcal{J}_5 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|E|^2 + |B|^2)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} E \cdot (\nabla \times B - \rho u)\,\mathrm{d}x - \int_{\mathbb{R}^3} B \cdot \nabla \times E\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\rho u \cdot E\,\mathrm{d}x.\end{aligned}$$

Proof of Proposition

- Therefore, we close the estimate for the modulated energy as

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}^\varepsilon}{\mathrm{d}t} = &-\int_{\mathbb{R}^3} \rho^\varepsilon(u^\varepsilon - u)\otimes(u^\varepsilon - u) : \nabla u \,\mathrm{d}x \\ &- \int_{\mathbb{R}^3} [(\rho^\varepsilon)^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(\rho^\varepsilon - \rho)](\nabla\cdot u)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} (\rho - \rho^\varepsilon)u\cdot(E^\varepsilon - E)\,\mathrm{d}x + \int_{\mathbb{R}^3} \rho^\varepsilon(u^\varepsilon - u)\cdot(u\times(B^\varepsilon - B))\,\mathrm{d}x \\ &- \frac{\varepsilon^2}{2}\int_{\mathbb{R}^3} \rho^\varepsilon\nabla\left(\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right)\cdot u\,\mathrm{d}x =: \sum_{i=1}^5 \mathcal{K}_i.\end{aligned}$$

- Since u is smooth, we have

$$\mathcal{K}_1 \lesssim \int_{\mathbb{R}^3} \rho^\varepsilon|u^\varepsilon - u|^2\,\mathrm{d}x, \quad \mathcal{K}_2 \lesssim \int_{\mathbb{R}^3} p(\rho^\varepsilon|\rho)\,\mathrm{d}x$$

and

$$\mathcal{K}_3 \lesssim \|\rho - \rho^\varepsilon\|_{L^2}^2 + \|E^\varepsilon - E\|_{L^2}^2.$$

Proof of Proposition

- Using the *a priori* bound for ψ^ε, we control $\mathcal{K}_4$ as

$$\begin{aligned}\mathcal{K}_4 &\leq \|\sqrt{\rho^\varepsilon}\|_{L^\infty}\|u\|_{L^\infty}\int_{\mathbb{R}^3}\sqrt{\rho^\varepsilon}|u^\varepsilon - u||B^\varepsilon - B|\,\mathrm{d}x\\ &\lesssim \int_{\mathbb{R}^3}\rho^\varepsilon|u^\varepsilon - u|^2\,\mathrm{d}x + \|B^\varepsilon - B\|_{L^2}^2.\end{aligned}$$

- Finally, as in the CSS system, we estimate the last term as

$$\mathcal{K}_5 \lesssim \varepsilon^2\int_{\mathbb{R}^3}|\nabla\sqrt{\rho^\varepsilon}|^2\,\mathrm{d}x + \varepsilon^2.$$

- To sum up, we have the following estimates on the modulated energy:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}^\varepsilon}{\mathrm{d}t} \lesssim &\int_{\mathbb{R}^3}\rho^\varepsilon|u^\varepsilon - u|^2 + p(\rho^\varepsilon|\rho)\,\mathrm{d}x\\ &+ \|\rho^\varepsilon - \rho\|_{L^2}^2 + \|E^\varepsilon - E\|_{L^2}^2 + \|B^\varepsilon - B\|_{L^2}^2\\ &+ \varepsilon^2\int_{\mathbb{R}^3}|\nabla\sqrt{\rho^\varepsilon}|^2\,\mathrm{d}x + \varepsilon^2.\end{aligned}$$

Proof of Proposition

- To close the estimate, we note that the Taylor expansion implies

$$p(\rho^\varepsilon|\rho) = (\rho^\varepsilon)^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(\rho^\varepsilon - \rho) = \frac{\gamma(\gamma-1)(\zeta^\varepsilon)^{\gamma-2}}{2}(\rho^\varepsilon - \rho)^2,$$

  where $\min\{\rho^\varepsilon, \rho\} < \zeta^\varepsilon < \max\{\rho^\varepsilon, \rho\}$.
- Since we assume that the local charge density is bounded, and $\gamma \leq 2$, we have

$$p(\rho^\varepsilon|\rho) \geq C(\rho^\varepsilon - \rho)^2.$$

- Therefore, we conclude that

$$\frac{\mathrm{d}\mathcal{H}^\varepsilon}{\mathrm{d}t} \lesssim \mathcal{H}^\varepsilon + \varepsilon^2,$$

  which, together with the well-prepared initial data implies

$$\mathcal{H}^\varepsilon(t) \leq C\varepsilon^{\min\{\lambda,2\}}.$$

Proof of Theorem

- The proof of convergence of the macroscopic variables directly follows from the estimate of modulated energy:

$$
\begin{aligned}
&\|\rho^\varepsilon - \rho\|_{L^2}^2 \le C \int_{\mathbb{R}^3} p(\rho^\varepsilon|\rho)\,\mathrm{d}x \le C\mathcal{H}^\varepsilon(t) \to 0,\\
&\|\sqrt{\rho^\varepsilon}u^\varepsilon - \sqrt{\rho}u\|_{L^2} \le \|\sqrt{\rho^\varepsilon}|u^\varepsilon - u|\|_{L^2} + \|(\sqrt{\rho^\varepsilon} - \sqrt{\rho})|u|\|_{L^2}\\
&\qquad \le C\mathcal{H}^\varepsilon + C\|\rho^\varepsilon - \rho\|_{L^2}^{1/2} \to 0\\
&\|E^\varepsilon - E\|_{L^2}^2 \le C\mathcal{H}^\varepsilon(t) \to 0,\\
&\|B^\varepsilon - B\|_{L^2}^2 \le C\mathcal{H}^\varepsilon(t) \to 0,
\end{aligned}
$$

  as $\varepsilon \to 0$.

Remarks

- It is easy to observe (indeed, physically obvious) that the same hydrodynamic formulation holds for the case of Lorenz gauge condition. Therefore, the same hydrodynamic limit estimates also hold for the case of Lorenz gauge condition.
- The (global-in-time) well-posedness of the non-linear MS system is recently known, under the condition on $1 < \gamma \leq 2$. Therefore, the assumption on the adiabatic constant is needed not only for the technical estimate on the modulated energy, but also for the existence of the system.

Thank you very much for attention.